Vehicle System Dynamics
Vol. 00, No. 00, August 2011, 1–6

On multiple Attractors and critical Parameters and how to find them numerically: The Right, the Wrong and the American Way*

Hans True**
The Technical University of Denmark, IMM,
Richard Petersens Plads 321, DK-2800 Kgs. Lyngby, Denmark
(Received 00 Month 200x; In final form 00 Month 200x)

In recent years several authors have proposed 'easier numerical methods' to find multiple attractors and the critical speed in railway dynamical problems. Actually, the methods do function in some cases, but they are not safe in the sense that you will calculate the relevant critical parameter values with a reasonable accuracy. In some cases the 'easier numerical methods' are really just a gamble. In this presentation the methods will be discussed. In this relation the linearisations of the nonlinear dynamical problem are made. A linearisation of the nonlinear dynamical problem simplifies the calculations and may give relevant answers to important questions such as the possibility of resonance phenomena in the designs, but a linearisation is not always allowed.
We shall also address the curious fact that the hunting motion is more robust than the ideal stationary state motion in the track.

1 Introduction

The calculation of critical parameters leads to the mathematical problem of finding multiple solutions to a nonlinear initial value problem. The mathematical problem is an *existence problem* and *not* a stability problem. The asymptotically stable solutions attract the solutions in a certain domain in the state space, and they are therefore called *attractors* and the domain is called *its domain of attraction.* The lowest critical parameter needs not be the value at which the fundamental stationary solution loses its stability. It has been known now for decades, and the problem of finding the critical speed through the solution of a problem of existence has been described by the author in (1), (2), (3), (4), (5), and (6). The multiplicity of solutions arises through bifurcations. Such bifurcations occur in railway dynamical problems often as a combination of a subcritical bifurcation of a periodic solution from

*Quotation from Nick Wilson, TTCI
**Corresponding author. Email: ht@imm.dtu.dk

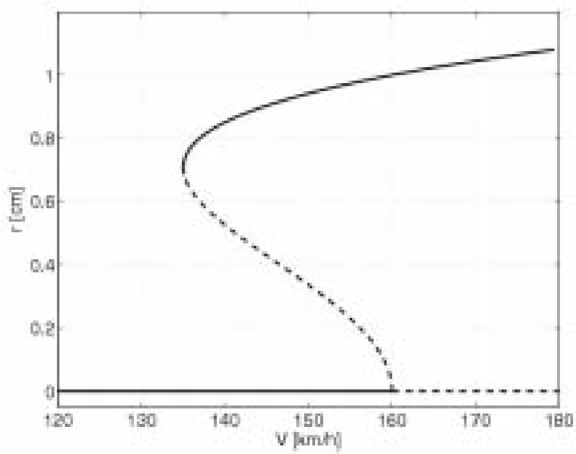

Figure 1. A bifurcation diagram with a subcritical bifurcation from the stationary solution and a tangent bifurcation between two periodic solutions that are characterized by their amplitudes. The stationary solution is the x-axis. The stable solutions are drawn by a full line and the unstable solutions by a dotted line. The critical speed is 134 km/h

the stable stationary solution and a tangent or saddle-node bifurcation at a lower speed, where the unstable periodic solution meets a stable periodic solution (see figure 1). The stationary solution on figure 1 is an attractor, and it is stable against sufficiently small initial perturbations in the speed interval $0 < V < 160$ km/h. It is therefore impossible to calculate the critical speed through a conventional stability analysis of the stationary solution. In the speed interval $0 < V < 134$ km/h the stationary solution is a globally unique attractor, meaning that it is the only attractor in the entire parameter-state space. The uniqueness is lost when the speed is larger than 134 km/h, and in the interval $134 < V < 160$ km/h three equilibrium solutions exist, out of which two are stable and one unstable. From figure 1 it is easily seen that it is the loss of uniqueness of the stationary attractor that determines the critical speed, because above $V = 134$ km/h finite disturbances of the stationary solution exist, which will abruptly change the stationary solution into a periodic motion - hunting.

2 The right Way

The stationary solution is known. In our example it is the trivial solution - all state variables equal zero. It exists for all speeds, V, up to a sufficiently high value. Its first bifurcation point can be calculated by a conventional stability analysis. It may not be the smallest bifurcation point in the parameter space (see figure 1). The mathematically optimal way to determine the smallest bifurcation point is to apply the method known as *path following* along the new bifurcating solution to the next bifurcation point. In the railway dynamical problems, such as figure 1, that will be the tangent bifurcation, which determines the critical speed. Unfortunately no numerical routines have until now been developed that can follow an unstable periodic solution of a differential-algebraic problem such as the railway dynamical problems usually are. Schupp (7) has, however, developed a routine, which can follow *stable periodic* solutions to railway vehicle dynamical problems in the parameter-state space, but since the bifurcating solution on figure 1 is unstable, we must use another strategy.

The 'True strategy' (1), (2), (3), (4), (5) and (6) is based on - first - a calculation of the speed where the trivial solution loses its stability. That speed is 160 km/h in figure 1. Then solve the full nonlinear dynamical problem for a slightly higher speed with a small initial perturbation. The solution will tend to the nearest attractor in the state space. In our case (figure 1) it will be the periodic hunting motion. When the transient has disappeared, the equilibrium solution has been found and the end vector of the calculation must be stored. Finally Schupp's method (7) with the end vector as the initial condition can be applied to find the next bifurcation point. In figure 1 it will be the bifurcation point 134 km/h. It is the critical speed! Alternatively True's method (1), (2), (3), (4), (5) and (6) can be applied, where the speed is changed manually in small steps. In every step the end values of the preceding solution is used as initial values for the subsequent calculation. The series of calculations end, when the following bifurcation point has been found.

A fast estimate can be made by *ramping*. Instead of the stepwise discrete changes of the speed, a slowly changing speed is used in the solution of the dynamical problem. This method yields a fast but inaccurate determination of the critical speed, because the method overshoots the bifurcation point. A more accurate value can subsequently be found by using discrete steps but now only in a much shorter speed interval.

3 The wrong Way

The application of a conventional stability analysis is the wrong way to find the critical speed. The problem to be solved is *not a stability problem, it is a problem of existence of solutions*. It is clear from figure 1 that the stability analysis of the stationary solution only yields a bifurcation point, but there is no guarantee that it is the smallest one. It is necessary to follow the bifurcating periodic solution in order to find the smallest bifurcation point. There exist bifurcations in railway vehicle dynamics that are supercritical. It means that the bifurcating periodic solution turns to the right on figure 1 instead of to the left. Also in such a case it is necessary to follow the periodic solution - in that case it will be an attractor - to the next bifurcation point, because there may be a secondary bifurcation point on that branch, from where a third solution will bifurcate to the left. That third solution might reach down to a smaller speed than the supercritical bifurcation point found on the stationary solution. An example can be found in Xia (8) and (9).

Linearisations, such as those made in a conventional stability analysis of a nonlinear dynamical problem, are not always permitted. The first derivative of the function to be linearised must exist in the point of linearisation. For $\sqrt{x}$ it does not exist in $x = 0$. The railway dynamical problems are - practically all - *non-smooth*, and in the points of non-smoothness the first derivative of the force and torque functions is not defined. The Jacobian is needed when implicit routines are applied for the numerical solution, but the Jacobian does not exist in the points of non-smoothness. Problems may also occur with application of solvers of higher orders to problems with a discontinuity of the second derivative, which occurs in the description of a rail surface. The numerical procedure must therefore be modified in order to obtain reliable results. Appropriate modifications can be found in the works by Xia (10) and Hoffmann (11), and True has discussed the modification in (5).

4 The American Way

Some railway dynamicists feel that 'the right method' for the calculation of the critical speed is either too complicated or too slow. They therefore present and argue for modifications of 'the right method'. Stichel (12) proposes to use a finite disturbance of the stationary motion for growing speeds to find the periodic attractor. When it is found, then the periodic attractor is followed backwards for decreasing speed - using ramping - until the hunting motion stops. Thereby Stichel saves computer time for the path following of the stationary solution. This method is also 'right', because Stichel *follows the hunting motion for decreasing speed*. Most authors, however, make the mistake *not* to follow the hunting motion backwards in the parameter-state space and call the speed at which the hunting motion first occurs, when the speed is increased, the critical speed. It is pure luck if the found speed is anywhere near the critical speed of the vehicle, because the found value of the speed depends on the size of the initial disturbance and also on the choice of the disturbance. Usually a lateral displacement of the leading axle is chosen, but it is only one of the many states that are excited in a real situation on the railway line.

During a visit in Chengdu the author had the opportunity to test the validity of 'the American way' together with one of their scientists. We started ramp-

ing, using commercial software, at the critical speed found in 'the American way' in an earlier work. The hunting motion was followed for decreasing speed, but the calculations were stopped when the speed had decreased by 10%, and the vehicle model was still hunting. Obviously the critical speed in the model was lower than the one that had been claimed to be the critical speed in the original report. The conclusion is that *the critical speeds that have been found in 'the American way' and published are not reliable* - with the exception of Stichel's work (12), which is also 'the right way'.
In the test it also became clear why the authors find 'the right method' cumbersome. It is not due to the 'right way' but to the deficiency of the commercial routine!

5 The Robustness of the Hunting Motion

We have seen that in railway vehicle dynamics there very often exists a speed interval in which there exist co-excisting attractors - namely the stationary motion and the hunting motion. It is also a fact that the hunting motion is more robust in the sense that it is easy to start hunting, but virtually impossible to stop hunting without slowing down the vehicle. Due to the great complexity of the attractors in the high-dimensional state space it is impossible to explain this fact exactly, but a simple picture may make it plausible.

On figure 2 the small ellipse, which is an unstable limit cycle, splits the domains of attraction of the stable equilibrium point (0,0) on one side and the

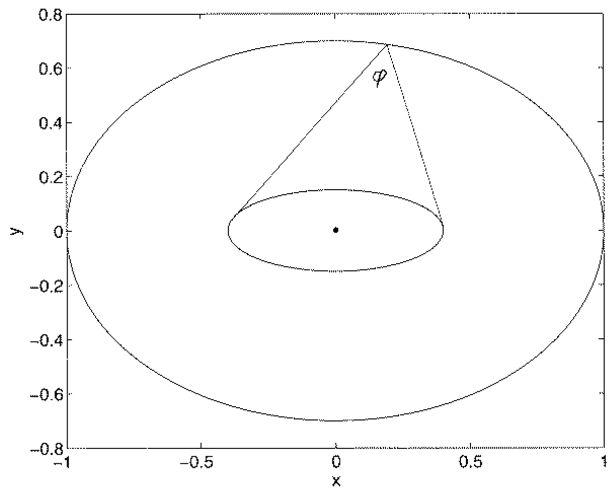

Figure 2. A picture of an unstable limit cyle - the small ellipse - with a stable equilibrium point in the centre, surrounded by a stable limit cycle - the large ellipse

domain of attraction of the stable limit cycle, the large ellipse, on the other side. Here - in two dimensions - it is easily seen that a sufficiently large initial disturbance of (0,0) will start the hunting independently of its direction in the plane. In contrast an initial disturbance on the 'hunting ellipse' must not only be sufficienly large - and not too large - but also lie inside the angle ϕ in order to reach the domain of attraction of (0,0).

6 Conclusion

The only safe computation of the theoretical critical speed of a railway vehicle uses *the right way* described in section 2, possibly using the modification by Stichel described in section 4.

REFERENCES

[1] True, H., 1994, Does a Critical Speed for Railroad Vehicles exist?, RTD-Vol. 7, Proc. of the 1994 ASME/IEEE Joint Railroad Conference, Chicago Ill., March 22-24, 1994, 125-131, American Society of Mechanical Engineers, United Engineering Center, 345 East 47th Street, New York, New York 10017, USA

[2] True, H., 1999, On the Theory of Nonlinear Dynamics and its Applications in Vehicle Systems Dynamics, Vehicle System Dynamics, 31, 5-6, 393-421

[3] True, H., 2004, Nichtlineare Schienenfahrzeugdynamik, neue Grundlagen, Methoden und Ergebnisse, ZEVrail Glasers Annalen, 128, 11-12, 526-537

[4] True, H., 2006, Recent advances in the fundamental understanding of railway vehicle dynamics, International Journal of Vehicle Design, 40, 1-3, 251-264

[5] True, H., 2007, Dynamics of Railway Vehicles and Rail/Wheel Contact, Dynamic Analysis of Vehicle Systems, Theoretical Foundations and Advanced Applications, 75-128, CISM Courses and Lectures - No. 497, Springer Wien NewYork, url http://www2.imm.dtu.dk/pubdb/p.php?5617

[6] True, H., 2008, On the critical Speed of High-speed Railway Vehicles, Workshop: Noise and Vibration on High-Speed Railways, 2. and 3. October 2008, FEUP Porto, Portugal, 149-166, Civil Engineering Department, Faculty of Engineering of the University of Porto

[7] Schupp, G., 2004, Computational Bifurcation Analysis of Mechanical Systems with Applications to Railway Vehicles, Proc. 18th IAVSD Symposium on Vehicle System Dynamics, The Dynamics of Vehicles on Roads and on Tracks, 458-467, Taylor & Francis, London, UK, 458-467

[8] Xia, F., True, H., 2003, On the Dynamics of the Three-Piece-Freight Truck, RTD-Vol. 25, IEEE/ASME Joint Rail Conference, Chicago Ill., April 22-24 2003, 149-159, American Society of Mechanical Engineers, United Engineering Center, 345 East 47th Street, New York, New York 10017, USA

[9] Xia, F., True, H., 2004, The Dynamics of the Three-Piece-Freight Truck, Proc. 18th IAVSD Symposium on Vehicle System Dynamics, The Dynamics of Vehicles on Roads and on Tracks, 212-221, Taylor & Francis, London, UK

[10] Xia, F., 2002, The Dynamics of The Three-Piece-Freight Truck, Ph.d. thesis, IMM, The Technical University of Denmark, Kgs.Lyngby, Denmark, http://www2.imm.dtu.dk/pubdb/public/search.php?searchstr=Fujie+Xia&n=5&searchtype=strict

[11] Hoffmann, M., 2006, Dynamics of European two-axle Freight Wagons, Ph.d. thesis IMM, The Technical University of Denmark, Kgs.Lyngby, Denmark, http://www2.imm.dtu.dk/pubdb/views/publication_details.php?id=4853

[12] Stichel, S.,1999, Modellierung und Parameterstudien zum Fahrverhalten von Güterwagen mit UIC-Fahrwerken, ZEV Glasers Annalen, 123, 7/8, 289-296