RECURSIVE IDENTIFICATION OF TIME-VARYING SYSTEMS

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Preface

The present thesis is concerned with recursive identification of time–varying systems. Methods designed for estimating slowly drifting system parameters are studied. A general algorithm containing most existing methods as special cases is formulated, and a unified treatment of its convergence properties in an ideal deterministic environment is presented. An algorithm which discounts information non–uniformly and in accordance with the distribution of the information flow is developed and examined. Finally, the discussed techniques are applied to a practical problem with adaptive control of a non–linear process.

The work leading to this thesis has been carried out at the Institute of Mathematical Statistics and Operational Research (IMSOR), the Technical University of Denmark. The thesis is part of the requirements for the Ph. D. degree.

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Summary

In the present thesis methods for recursive identification of time–varying systems are considered.

In chapter 2 some of the facts, methods and assumptions forming the background of the presentation are described. The model structure is formulated and the connected assumptions are discussed. A review of the recursive least squares estimation method and its properties is given. The methods which are studied in the later chapters are obtained by modifying the least squares scheme, but the basic principles and variable interpretations remain the same. The chapter also contains a description of the well known adaptive controller obtained by combining least squares estimation with minimum variance control. Modified versions of this algorithm are applied and discussed throughout the thesis.

Chapter 3 gives a survey of the so–called forgetting methods designed for tracking slowly drifting system parameters. The main result of the chapter is the formulation of a general forgetting algorithm containing the existing methods as special cases.

The analytical properties of the general forgetting algorithm are studied in chapter 4. The attention is focused on an ideal environment without disturbances and with a time–invariant plant description. It is of fundamental importance that the estimator has good performance in this simple case. The analysis leads to quite general results which can readily be applied to any forgetting method belonging to the general family.

In chapter 5 a method based on a principle of selective forgetting is formulated. The theoretical properties of the method are studied and its practical performance is examined via simulation experiments.

The last chapter, chapter 6, contains development and examination of an adaptive controller for the nitrification process. The controller has to be based on a strongly simplified model, since the true process dynamics is complicated and only partly understood. However, the model captures the basic features of

the variation, and the ability of the adaptive controller to adjust itself to the current conditions compensates for the simplification. This is verified via simulation studies by application of an extended model for representation of the true system. The non–linear structure of the model and the restrictions on the control signal make it impossible directly to apply the results known from the theory of linear systems. However, closely related results are established by using slightly modified proving techniques.

Resume

I nærværende afhandling studeres metoder til rekursiv identifikation af tidsvarierende systemer.

I kapitel 2 beskrives nogle af de metoder og antagelser, som danner baggrund for fremstillingen. Modelstrukturen fastlægges, og de tilknyttede antagelser diskuteres. En gennemgang af den rekursive mindste kvadraters metode og dens egenskaber gives. De metoder, som studeres i de følgende kapitler, er udviklet ved modifikation af mindste kvadraters teknikken, men de grundlæggende principper og variabelfortolkninger forbliver uændrede. Kapitlet indeholder også en beskrivelse af den velkendte adaptive regulator opnået ved kombination af en mindste kvadraters estimator med en minimal varians regulator. Modificerede versioner af denne algoritme anvendes og diskuteres i det følgende.

Kapitel 3 giver en oversigt over de såkaldte glemselsmetoder, som er udviklet til estimation af langsomt varierende systemparametre. Hovedresultatet i kapitlet er formuleringen af en generel glemselsalgoritme, der indeholder de eksisterende metoder som specialtilfælde.

De analytiske egenskaber af den generelle glemselsalgoritme studeres i kapitel 4. Opmærksomheden fokuseres på et ideelt miljø uden forstyrrelser og med en tidsinvariant systembeskrivelse. Det er af afgørende vigtighed, at estimatoren har gode egenskaber i dette simple tilfælde. Analysen fører til ganske generelle resultater, som kan anvendes på enhver glemselsmetode, der tilhører den generelle familie.

I kapitel 5 formuleres en metode baseret på et princip om selektiv glemsel. De teoretiske egenskaber for metoden studeres, og dens praktiske virkemåde undersøges via simulationseksperimenter.

Det sidste kapitel, kapitel 6, indeholder udvikling og undersøgelse af en adaptiv regulator for nitrifikationsprocessen. Regulatoren må baseres på en stærkt simplificeret model, idet den virkelige procesdynamik er kompliceret og kun delvist forstået. Modellen er dog i stand til at beskrive hovedparten af

variationen, og den adaptive regulators evne til at tilpasse sig de øjeblikkelige forhold kompenserer for simplifikationen. Dette verificeres via simulationstudier ved anvendelse af en udvidet model til repræsentation af det sande system. Den ulineære struktur af modellen og begrænsningerne på styresignalet gør det umuligt direkte at anvende de resultater, som er kendt fra teorien om lineære systemer. Nært beslægtede resultater udledes dog ved anvendelse af lettere modificerede bevisteknikker.

Table of contents

1

Introduction

Mathematical models play an important role in modern science. They offer a way to express relevant process knowledge in an exact and concentrated form. Their ability to imitate the behaviour of systems and processes makes them excellent tools for design, simulation, forecasting, control, monitoring and fault detection.

There are two fundamentally different ways in which the problem of formulating a mathematical model can be attacked. One possibility is to apply an analytical approach and to construct the model solely on the basis of detailed understanding of the internal mechanisms, using physical laws and relations. This is the classical *modeling* procedure. Alternatively, an empirical approach can be chosen. Experiments can be performed, and the model can be constructed using the resulting data material. This procedure is known as *system identification*, and it consists of the following steps:

1 experiment design
2 choice of model structure
3 parameter estimation
4 model validation

The steps $1 - 4$ have to be repeated iteratively until a satisfactory model has been found.

In many cases a mixture of the two approaches is preferable. For example, insight into the process dynamics may make it possible to determine a good and reliable model up to, but not including, a set of model constants (parameters). Physical insight may even make it possible to specify intervals wherein the unknown parameters can be expected to lie. The final calibration of the model can then be based on experiments and parameter estimation.

The present thesis mainly concerns the parameter estimation problem, and for this discussion it is immaterial whether the model structure has been obtained in one way or another. Furthermore, the attention will be focused on the case where the estimation has to be performed on–line, i. e. while the system is operating. On–line estimation methods are key instruments in adaptive procedures, i.e. procedures which are able to tune themselves automatically in

accordance with the measured behaviour and the desired performance. Adaptive procedures are applied for example in connection with filtering, prediction, control and signal processing.

The on–line parameter estimation problem calls for *recursive* techniques which are able to process the information as it is being received. The communication with the plant is normally handled by a computer, and the data acquisition most often takes place in discrete time. The estimates, and in some cases also their uncertainties, are updated repeatedly in accordance with the persistent flow of incoming information. Therefore the algorithm has to be able to concentrate and use the data within the time interval separating the measurements.

Generally the recursive methods are less accurate than their off–line counterparts. In some cases this difference disappears as the length of the data record increases. This depends on the asymptotic properties of the recursive identification method, and consequently this is an issue of central importance.

An important advantage of the on–line identification techniques is that they are able to handle the case where the process parameters are time–varying. Traditionally, many parametric procedures are based on the assumption that the process can be described by a sufficiently detailed model with time–invariant parameters. However, in many cases such an assumption is either not realistic or impractical. The available process knowledge may be insufficient, the dynamics may actually be time–varying or perhaps the end use requires a special model structure. For example, many algorithms for process control are based on a linearity assumption. If the underlying physical process is nonlinear, a linearized description will only be locally valid. As an alternative to comprehensive and exact modeling one might consider to employ a simplified description, for example a linear model of low order, and then to accommodate the mismatch between model and process by considering the parameters as time–varying quantities. In order to enable the parametric procedure to cope with this situation, a recursive estimator designed for the time–varying case has to be employed. This class of estimators will be discussed in the present thesis.

The perhaps most widely used recursive identification method produces estimates that minimize the sum of squared output prediction errors. Unfortunately, this method is not directly applicable when the unknown parameters vary with time. Due to the persistent flow of information, the algorithm gain tends towards zero, and the ability to track the variations is gradually lost. Consequently it becomes necessary to modify the algorithm in order to solve the problem. One feasible approach is to formulate a model of the parameter variation, and to handle the resulting estimation problem by use of one of the filtering techniques developed in the area of state estimation. Another approach, and this is the one which will mainly be considered in this thesis, is to gradually discount old measurements as they become obsolete. The corresponding class of estimators is called forgetting methods. By storing only a limited amount of information, these algorithms avoid the situation with vanishing gain and loss of tracking ability.

A very popular method is based on the principle of exponential forgetting. A scalar forgetting factor is introduced, and old information is gradually discounted. The discounting takes place uniformly in all directions of the parameter space. Unfortunately, this method turns out to have some serious draw backs. For example, if the information flow turns off, certain elements updated in the algorithm start to grow unboundedly, and this may eventually lead to large estimation errors. Several alternative forgetting methods have been suggested in the literature, in most cases as an attempt to improve the robustness of the estimator in difficult situations.

The performance of the complete adaptive system often depends critically on the choice of estimation method. Generally it is reasonable to require that a good estimator should possess certain fundamental theoretical properties which guarantee good performance in the most simple and ideal cases. However, typically the forgetting methods have been developed using ad hoc argumentation, and their theoretical properties are not always completely analyzed.

There is a need not only for continued development of better and more robust recursive estimation techniques, but also for unification and analysis of the existing ones. It is also important to persist in seeking new fields, where the

methods might be successfully applied. The goal in this thesis is to give contributions in these areas. The main results of the work can be summarized as follows:

Unification: A general recursive algorithm containing most existing forgetting methods as special cases is formulated. It turns out that the methods differ only by the choice of a single update equation.

Development of new methods: A new method based on a principle of selective forgetting is suggested. The idea is to assign individual forgetting factors to stochastically independent parameters. By choosing the forgetting factors as functions of the amount of information received in the corresponding directions, certain problems encountered when using the standard techniques can be completely avoided.

Analysis: Analytical results for the general recursive algorithm are presented. It turns out that fundamental deterministic convergence properties (including exponential convergence) can be established by simply verifying that a particular matrix updated in the algorithm is bounded. The results are applied to some of the existing methods including the new selective forgetting algorithm.

Application: The thesis contains a number of simulation examples which aim at illustrating the practical behaviour of the studied methods. Special attention is devoted to the development of an adaptive controller for the nitrification process. The basic convergence properties of the controller are established and the performance is tested in simulation experiments. This illustrates that in many cases it is possible to generalize the discussed techniques and results to realistic nonlinear processes.

The thesis has the following outline:

Chapter 2 contains a description of the class of models which will be considered. The attention is focused on the linear regression structure. Generally it will be assumed that the model structure is correct, and that

only the parameters have to be identified. The measurements may be corrupted by uncorrelated disturbances. Only models of the single input–single output type are studied. The chapter also reviews the well known recursive least squares estimation method, and discusses the basic properties of this method. Finally, a description of a simple algorithm for adaptive control of linear plants is given.

Chapter 3 surveys some of the forgetting methods which have been suggested in the literature. Emphasis is on clarifying the underlying ideas. Methods which enable handling of problematic situations where the distribution of information is non–uniform in time or in parameter space, or where the different parameters have different types of variation, are treated. The presentation is supported by several examples. Finally, the general forgetting algorithm is formulated within a Bayesian framework, and it is pointed out how each of the discussed methods appear as special cases.

Chapter 4 deals with the analytical examination of forgetting algorithms. A set of basic asymptotic properties related to the deterministic case is formulated, and the importance in connection with parameter tracking and adaptive control is explained. Next, the general forgetting algorithm is studied. Simple sufficient conditions which ensure existence of the fundamental properties are given. The results are applied to the existing methods.

The selective forgetting method is introduced in chapter 5. Different interpretations of the method are given, and it is shown how it can be implemented without requiring excessive computational time. A particular choice of forgetting factors is suggested, and the algorithm is examined analytically as well as practically. For example, it is shown that the algorithm completely avoids the covariance wind up problems encountered by most standard methods, that it is exponentially convergent provided that the data are sufficiently exciting and that it smoothly resets itself if the information flow turns off.

Chapter 6 is devoted to an application study. Adaptive control of a realistic nonlinear process is considered, and it is shown how the discussed techniques can be modified and used also in this case. Again the examination of the

algorithm is divided into a theoretical part and a more practical part based on simulation experiments. A similar approach will be applicable in connection with a large number of other processes, even if the methods and proof techniques may have to be modified in each case.

2

Fundamental methods and assumptions

Before entering the discussion of estimation methods designed for the time–varying case, it is necessary to establish a suitable framework for the presentation. This is the objective of the present chapter, where the basic time–invariant model, a standard method for recursive parameter estimation and a simple adaptive controller are described.

A rather strict set of assumptions will be employed. For example, it is assumed that the system equation has a known structure, it is linear in the parameters and it is corrupted only by uncorrelated noise. The estimates are determined by minimization of the sum of squared prediction errors. This leads to the recursive least squares (RLS) method.

RLS can be regarded as an archetypical algorithm for on–line parameter estimation. Different choices of design objectives (model structure, estimation criterion, optimization technique etc.) typically lead to methods with very similar structures. Consequently it is straight forward to generalize most ideas developed in the RLS framework to more general classes of methods.

Ideally, a recursive estimation algorithm should not be studied only as an isolated algorithm, but also as part of the complete system in which it is applied. For example, it is well known that experiments performed in closed loop (i.e. with output feedback) often lead to data which are not sufficiently informative to allow a unique solution of the parameter estimation problem. The discussion in this thesis will focus on the use of the estimators in adaptive control. The purpose is to illustrate the performance in connection with a realistic application, and in particular to explain why certain asymptotic properties are considered important. Again a very simple algorithm is considered, and several extensions are possible.

The chapter is organized as follows: In section 2.1 the time–invariant plant description is discussed. Section 2.2 addresses the problem of recursive parameter estimation in the time–invariant case. Finally, in section 2.3 the adaptive controller is formulated by combining the estimator with a simple control law.

2.1 Description of the plant

It will be assumed that a single input–single output (SISO) system is studied. The input signal $u(\tau)$ and the output signal $y(\tau)$ are measured at

(2.1.1) $\tau = T,\ \tau = 2\ T,\ \ldots$

T is the sampling period. $u(\tau)$ is constant between the measurement instants. For notational convenience the time scale is normalized by introduction of

(2.1.2) $t = \tau\ /\ T$

It will be assumed that a discrete time mathematical model of the system is available in terms of a prediction function

(2.1.3) $\hat{y}(t,\theta)$

$\theta \in R^p$ is the parameter vector containing the constants needed in the model. For a fixed value of θ, $\hat{y}(t,\theta)$ yields a guess (prediction) of the output at time t, based on the observations $y(t-1),\ \ldots\ ,\ y(1),\ u(t-1),\ \ldots\ ,\ u(1)$. It will be assumed that $\hat{y}(t,\theta)$ is a linear function of θ:

(2.1.4) $\hat{y}(t,\theta) = \varphi^T(t)\ \theta$

This is known as a linear regression. $\varphi(t) \in R^p$ is the regression vector. The elements of φ are known, perhaps nonlinear functions of lagged inputs and outputs.

The prediction error function at time t is

(2.1.5) $\epsilon(t,\theta) = y(t) - \hat{y}(t,\theta)$

A true description of the system is assumed to be included in the model structure in the sense that for some $\theta^0 \in R^p$

(2.1.6) $$y(t) = \varphi^T(t)\ \theta^0 + e(t)$$

(2.1.7) $$e(t) = \epsilon(t,\theta^0)$$

(2.1.8) $$E\ \{\ e(t)\ \} = 0$$

$$E\ \{\ e(s)\ e(t)\ \} = \sigma^2 \qquad t = s$$

(2.1.9) $$= 0 \qquad t \neq s$$

E { . } denotes expectation. { e(t) } is a sequence of uncorrelated zero mean random variables (white noise). e(t) represents the part of y(t) that cannot be predicted and is therefore sometimes called the innovation at time t.

The exact value of θ^0 is unknown, but with this exception the plant description is assumed to be known. Let $\hat{\theta}(t-1)$ be an estimate of θ based on the information available at time t–1. The corresponding prediction is denoted

$$\hat{y}(t) = \hat{y}(t,\ \hat{\theta}(t-1))$$

(2.1.10) $$= \varphi^T(t)\ \hat{\theta}(t-1)$$

and the prediction error becomes

$$\epsilon(t) = \epsilon(t,\hat{\theta}(t-1))$$

(2.1.11) $$= y(t) - \varphi^T(t)\ \hat{\theta}(t-1)$$

To conclude this section some examples are given which illustrate that the linear regression is useful in connection with a broad class of models.

Example 2.1.1

The discrete time formulation is convenient when the communication with the plant is handled via a digital computer. However, most systems are more naturally described by a continuous time equation. Consider the following linear continuous time model

(2.1.12) $$\dot{x}(\tau) = \mathcal{A}\, x(\tau) + \mathcal{B}\, u(\tau)$$

(2.1.13) $$y(\tau) = \mathcal{C}\, x(\tau)$$

$x \in R^n$ is the state (not directly measurable), $y \in R^m$ is the output and $u \in R^r$ is the input. $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are time–invariant matrices of appropriate order. It is well known from the elementary control theory, see for example Poulsen and Holst (1990), that zero–order–hold sampling leads to the discrete time representation

(2.1.14) $$x(t+1) = \Phi\, x(t) + \Gamma\, u(t)$$

(2.1.15) $$y(t) = \mathcal{C}\, x(t)$$

where

(2.1.16) $$t = \tau \,/\, T$$

(2.1.17) $$\Phi = e^{\mathcal{A}T}$$

(2.1.18) $$\Gamma = \int_0^T e^{\mathcal{A}s} ds\ \mathcal{B}$$

According to another standard result (Poulsen and Holst, 1990) this model can be rewritten in the form

$$(2.1.19) \qquad A(q^{-1})\ y(t) = \begin{bmatrix} q^{-k_{11}}\ B_{11}(q^{-1}) & \dots & q^{-k_{1r}}\ B_{1r}(q^{-1}) \\ \vdots & & \vdots \\ q^{-k_{m1}}\ B_{m1}(q^{-1}) & \dots & q^{-k_{mr}}\ B_{mr}(q^{-1}) \end{bmatrix} u(t)$$

with appropriate initial conditions. $A(q^{-1})$ and $B_{ij}(q^{-1})$ represent scalar polynomials in the unit delay operator q^{-1} defined by

$$(2.1.20) \qquad q^{-1}\ f(t) = f(t-1)$$

The factors $q^{-k_{ij}}$ represent pure time delays.

In the single input–single output case ($r = m = 1$) the model reduces to

$$(2.1.21) \qquad A(q^{-1})\ y(t) = q^{-k}\ B(q^{-1})\ u(t)$$

$$(2.1.22) \qquad A(q^{-1}) = 1 + a_1\ q^{-1} + \dots + a_{n_a}\ q^{-n_a}$$

$$(2.1.23) \qquad B(q^{-1}) = b_0 + b_1\ q^{-1} + \dots + b_{n_b}\ q^{-n_b}$$

Introducing

$$(2.1.24) \qquad \varphi(t) = (-y(t-1), \dots , -y(t-n_a), u(t-k), \dots , u(t-n_b-k))^T$$

$$(2.1.25) \qquad \theta = (a_1, \dots , a_{n_a}, b_0, \dots , b_{n_b})^T$$

(2.1.21) can be rewritten as a linear regression.

Example 2.1.2

A large class of nonlinear systems can be represented by the following difference equation model (Leontaritis and Billings, 1985)

$$y(t) = F[\, y(t-1),\ \dots\ ,\ y(t-n_y),\ u(t-k),\ \dots\ ,\ u(t-k-n_u+1)\ ] \tag{2.1.26}$$

Important special cases are the Hammerstein, Wiener and bilinear structures. Also, discretization of nonlinear differential equations typically leads to discrete time models of this type.

If (2.1.26) is nonlinear in the parameters, polynomial expansion can be applied:

$$y(t) \approx \sum_{i=1}^{n_s} \theta_i\, V_i + \sum_{i=1}^{n_s}\sum_{j=1}^{n_s} \theta_{ij}\, V_i\, V_j + \dots$$

$$+ \sum_{i=1}^{n_s}\sum_{j=1}^{n_s} \dots \sum_{m=1}^{n_s}\sum_{n=1}^{n_s} \theta_{ij\dots mn}\, V_i\, V_j \cdots V_m\, V_n \tag{2.1.27}$$

where

$$V_1 = y(t-1),\ \dots\ ,\ V_{n_y} = y(t-n_y)$$

$$V_{n_y+1} = u(t-k),\ \dots\ ,\ V_{n_y+n_u} = u(t-k-n_u+1)$$

$$n_s = n_u + n_y$$

(2.1.27) is linear in the parameters and can be written as a linear regression.

∇

Example 2.1.3

A basic restriction in (2.1.4) is that the prediction function is assumed to be linear in the parameters. This makes it possible to obtain an exact solution to the least squares estimation problem. However, consider a more general process

$$y(t) = \hat{y}(t, \theta^0) + e(t) \tag{2.1.28}$$

where the predictor $\hat{y}(t, \theta^0)$ may be a non–linear function of θ^0.

According to the mean value theorem

$$\hat{y}(t, \theta^0) = \hat{y}(t, \hat{\theta}) + (\theta^0 - \hat{\theta})^T \psi(t, \theta^*) \tag{2.1.29}$$

$$\psi(t, \theta) = \left(\frac{d \hat{y}(t, \theta)}{d \theta} \right)^T \tag{2.1.30}$$

where θ^* is a value " between " θ^0 and $\hat{\theta}$. Assume that

$$\psi(t) = \psi(t, \hat{\theta}) \tag{2.1.31}$$

is a reasonable approximation for $\psi(t, \theta^*)$. Introduce

$$z(t) = y(t) - \hat{y}(t, \hat{\theta}) + \psi^T(t) \hat{\theta} \tag{2.1.32}$$

Notice that the value of z can be calculated when $y(t)$ has been measured. Then (2.1.28) can be rewritten

$$z(t) = \psi^T(t) \theta^0 + e(t) \tag{2.1.33}$$

Hence, the approximation leads us back to the linear regression structure. The approximation becomes more accurate the closer $\hat{\theta}$ gets to θ^0. This indicates that the analysis based on linear regressions is locally valid even for more general models.

∇

2.2 The recursive least squares method

The following section contains a brief discussion of recursive estimation in the time–invariant case. This subject is treated in several standard text books, see for example Ljung and Söderström (1983). The goal is to introduce the basic estimation procedure of which the methods in the following chapters can be regarded as modifications. This leads to interpretations of the variables that will support the intuitive understanding of the methods.

A standard approach is to determine the estimate of the parameter vector in (2.1.6) by minimization of the sum of squared prediction errors. This leads to the least squares (LS) estimate

$$\hat{\theta}_{LS}(t) = \arg\min_{\theta} \sum_{k=1}^{t} (y(k) - \varphi^T(k)\ \theta)^2$$

$$= \left[\sum_{k=1}^{t} \varphi(k)\ \varphi^T(k) \right]^{-1} \left[\sum_{k=1}^{t} \varphi(k)\ y(k) \right]$$

$$= \theta^0 + R^{-1}(t)\ h(t) \tag{2.2.1}$$

where

$$R(t) = \frac{1}{t} \sum_{k=1}^{t} \varphi(k)\ \varphi^T(k) \tag{2.2.2}$$

$$h(t) = \frac{1}{t} \sum_{k=1}^{t} \varphi(k)\ e(k) \tag{2.2.3}$$

Here and in the following discussion it is assumed that the matrix $R(t)$ is invertible. From the law of large numbers it follows that under weak conditions $h(t)$ will converge to $E\ h(t)$ as t approaches infinity. Consequently it can be seen from (2.2.1) – (2.2.3) that in the stochastic

case $\hat{\theta}_{LS}(t)$ will converge to θ^0 with probability 1, provided that $\varphi(t)$ and $e(t)$ are uncorrelated. This will e.g. be the case when $\{ e(t) \}$ is white noise, as was assumed in section 2.1. It is also possible to prove that the asymptotic distribution of the random variable $\sqrt{t}\ (\theta^0 - \hat{\theta}_{LS}(t))$ is Gaussian, i.e. $N(0, \sigma^2 (R^*)^{-1})$, where

$$R^* = \lim_{t \to \infty} \frac{1}{t} \sum_{k=1}^{t} E\ \varphi(t)\ \varphi^T(t) \tag{2.2.4}$$

Hence, subject to weak assumptions $\hat{\theta}_{LS}(t)$ is consistent and asymptotically normal. Furthermore, if the innovations are Gaussian, the LS estimator is asymptotically efficient, i.e. the estimation error covariance approaches the Cramér–Rao bound.

In some situations, e.g. when the estimator is part of an adaptive controller, the observations are obtained sequentially in time. Thus, a new estimate can be computed each time a new measurement becomes available. In this case it is advantageous to rewrite the equations for the LS estimate in the recursive form:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + P(t)\ \varphi(t) \left[y(t) - \varphi^T(t)\ \hat{\theta}(t-1) \right] \tag{2.2.5}$$

$$P^{-1}(t) = P^{-1}(t-1) + \varphi(t)\ \varphi^T(t) \tag{2.2.6}$$

where $P(t) = (t\ R(t))^{-1}$. This version of the method is known as the recursive least squares (RLS) method. No approximations are needed when deriving the recursive version of the algorithm. If the RLS algorithm is initialized with $\hat{\theta}(0)$ and $P(0) > 0$, the resulting estimate can be written

$$(2.2.7) \qquad \hat{\theta}_{RLS} = \left[P^{-1}(0) + \sum_{k=1}^{t} \varphi(k)\, \varphi^{T}(k) \right]^{-1} \times \left[P^{-1}(0)\, \hat{\theta}(0) + \sum_{k=1}^{t} \varphi(k)\, y(k) \right]$$

Comparing with (2.2.1) it can be seen that the RLS estimates differ only from the LS estimates due to the effect of the initial conditions. A richness condition on $\{ \varphi \}$ will ensure that $P(0)$ and $\hat{\theta}(0)$ gradually lose their importance, and in this case the asymptotic properties are unchanged.

The initial value $\hat{\theta}(0)$ represents the a priori parameter guess and $P(0)$ reflects the corresponding uncertainty. When the parameters are completely unknown, a common choice is

$$(2.2.8) \qquad \hat{\theta}(0) = 0$$

$$(2.2.9) \qquad P(0) = c\, I$$

I is the (p x p) identity matrix and c is a large positive constant. By choosing c sufficiently large, the RLS estimate can be made arbitrarily close to the LS estimate also for finite t.

For RLS the following approximate interpretations are obtained: $\hat{\theta}(t)$ is the least squares estimate of θ based on the information available at time t. $\sigma^2 P(t)$ is the covariance of the estimation error, and $(\sigma^2 P(t))^{-1}$ is Fisher's information matrix. Furthermore, from (2.2.6) it can be seen that $\varphi(t)\, \varphi^{T}(t)$ is the new piece of information incorporated by the algorithm at time t. $\varphi(t)$ determines the direction of the parameter space in which new information is added. The covariance of the prediction error is $\sigma^2 (1 + \varphi^{T}(t)\, P(t-1)\, \varphi(t))$. Though it is not strictly correct, these interpretations will also be used also for the modified versions of RLS studied later.

Direct implementation of (2.2.5) – (2.2.6) would require matrix inversion

after each step of the algorithm. To avoid this (2.2.6) can be rewritten as

$$P(t) = P(t-1) - \frac{P(t-1)\,\varphi(t)\,\varphi^T(t)\,P(t-1)}{1 + \varphi^T(t)\,P(t-1)\,\varphi(t)} \tag{2.2.10}$$

using the matrix inversion lemma

$$(A + B\,C\,D)^{-1} = A^{-1} - A^{-1}\,B\,(D\,A^{-1}\,B + C^{-1})^{-1}\,D\,A^{-1}$$

However, the recursion (2.2.10) for P can lead to accumulation of numerical errors, and P may not remain positive definite. This problem is likely to occur if P becomes ill conditioned and especially if the dimension of P is high. It can be effectively avoided by implementation of e.g. a square root algorithm or by use of U–D factorization (Bierman, 1977).

It is interesting to note that several alternative approaches lead to the RLS method. For example:

1) If the innovations are white and Gaussian the least squares estimate is identical with the maximum likelihood estimate.

2) Based on the stochastic description it would be natural to seek the estimate minimizing the criterion function

$$E\,\{\,\frac{1}{2}\,(\,y(t) - \varphi^T(t)\,\theta\,)^2\,\} \tag{2.2.11}$$

Application of the iterative Gauss–Newton method leads to an algorithm identical with RLS. $-\varphi(t)\,(\,y(t) - \varphi^T(t)\,\hat{\theta}(t-1)\,)$ becomes an estimate of the first derivative of the criterion (the gradient), and $R(t) = \frac{1}{t}\,P^{-1}(t)$ is an estimate of the second derivative (the Hessian) (Ljung and Söderström, 1983).

3) The Kalman filter (Jazwinsky, 1970) can be applied to the following state space model based on the constant parameter assumption:

(2.2.12) $$\theta(t+1) = \theta(t)$$

(2.2.13) $$y(t) = \varphi^T(t)\ \theta(t) + e(t)$$

This gives the following algorithm:

Measurement update:

(2.2.14) $$\hat{\theta}(t|t) = \hat{\theta}(t|t-1) + P(t|t)\ \varphi(t) \left[y(t) - \varphi^T(t)\ \hat{\theta}(t|t-1) \right]$$

(2.2.15) $$P^{-1}(t|t) = P^{-1}(t|t-1) + \varphi(t)\ \varphi^T(t)$$

Time update:

(2.2.16) $$\hat{\theta}(t+1|t) = \hat{\theta}(t|t)$$

(2.2.17) $$P(t+1|t) = P(t|t)$$

The Kalman filter was developed within a Bayesian framework. In this case the parameter vector θ is considered to be a random variable. The filter minimizes a large class of criteria including the quadratic

(2.2.18) $$E\ \{\ (\ \hat{\theta} - \theta_0)^T\ (\ \hat{\theta} - \theta_0)\ \}$$

The measurement update represents the change of the estimates resulting from the latest observation. The time update represents the development of the parameter distribution (according to the parameter model) during the sampling period where no new information is received.

Let $I(t)$ denote the amount of information available at time t, i.e.

(2.2.19) $$I(t) = (y(t),\ y(t-1),...,u(t),\ u(t-1),\ ...)^T$$

Assuming that all random variables are Gaussian, the following interpretation is obtained

(2.2.20) $\theta(t+i) \mid I(t) \in N (\hat{\theta}(t+i|t), \sigma^2 P(t+i|t))$ $i = 0, 1$

By defining $\hat{\theta}(t) = \hat{\theta}(t|t)$ and $P(t) = P(t|t)$ and eliminating $\hat{\theta}(t|t-1)$ and $P(t|t-1)$ from the equations, it can easily be seen that (2.2.14) – (2.2.17) is in fact identical to the RLS–algorithm. The division of the algorithm into a time update and a measurement update will turn out to be useful in connection with the modified RLS schemes discussed in later chapters.

As was hinted in section 2.1 the specialization to models of the linear regression type does not lead to a serious loss of generality. Typically more general choices of model structure, estimation criterion etc. lead to estimation schemes with update structures very similar to RLS. The extension to models with multiple inputs and outputs is also straight forward, and again the basic structure of the equations remains unchanged. A large part of the discussion in the next chapters will deal with modifications of the recursion for P in case of time–varying parameters. By sticking to the simple linear regression model and the simple least squares criterion, some approximations and unnecessary complications of the presentation are avoided. However, most of the discussion will apply to more general frameworks than the one employed here.

2.3 An adaptive controller

The main goal in the following chapters will be to study recursive estimation algorithms designed for the time–varying case. One of the most important applications is in adaptive control, where it is often essential that the identification part remains able to update the model according to any changes in the measured behavior. Consequently it is natural to examine the effect of modifications in the estimator on the performance of the total adaptive control system. In this section the algorithm to be considered in the sequel is formulated.

Standard control techniques are based on the assumption that a time–invariant model giving perfect imitation the of plant is available. However, for some processes this is not a realistic assumption. The dynamics may be very complicated and perhaps not completely understood. Also, the behavior of the plant may vary in time, making an exact time–invariant description impossible. Such considerations have motivated the development of adaptive controllers, capable of tuning themselves automatically.

Historically the development of adaptive controllers has taken place via several different, but closely connected lines. The two dominating approaches have been model reference adaptive control (MRAC) and self–tuning regulators (STR).

In model reference control the goal is to make the system act like a given reference model. See Landau (1979) for a comprehensive treatment. Stability of the closed loop system is considered for example in Egardt (1979) and in Goodwin et al. (1980).

The self–tuning regulator was originally developed by combining a least squares estimator with a minimum variance controller, see Åström and Wittenmark (1973). Generalizations of the control law are described for example in Clarke and Gawtrop (1975). The self–tuning controller is normally considered in a stochastic environment. The stability and convergence question is treated for example in Goodwin et al. (1981).

The books by Goodwin and Sin (1984) and Åström and Wittenmark (1989) give comprehensive discussions of different approaches, and they include analytical results, application examples, historical facts and relevant references.

A fundamental principle in adaptive control is to combine the control law with an on–line estimation algorithm. By replacing the unknown parameters with their current estimates (the certainty equivalence approach), a control signal can be computed at any time instant. The essential steps repeated at t = 1, 2, ... after measurement of the relevant signals are

Estimation: Update the parameter estimates

Design: Transform the estimates into a set of control parameters

Control: Calculate the control signal

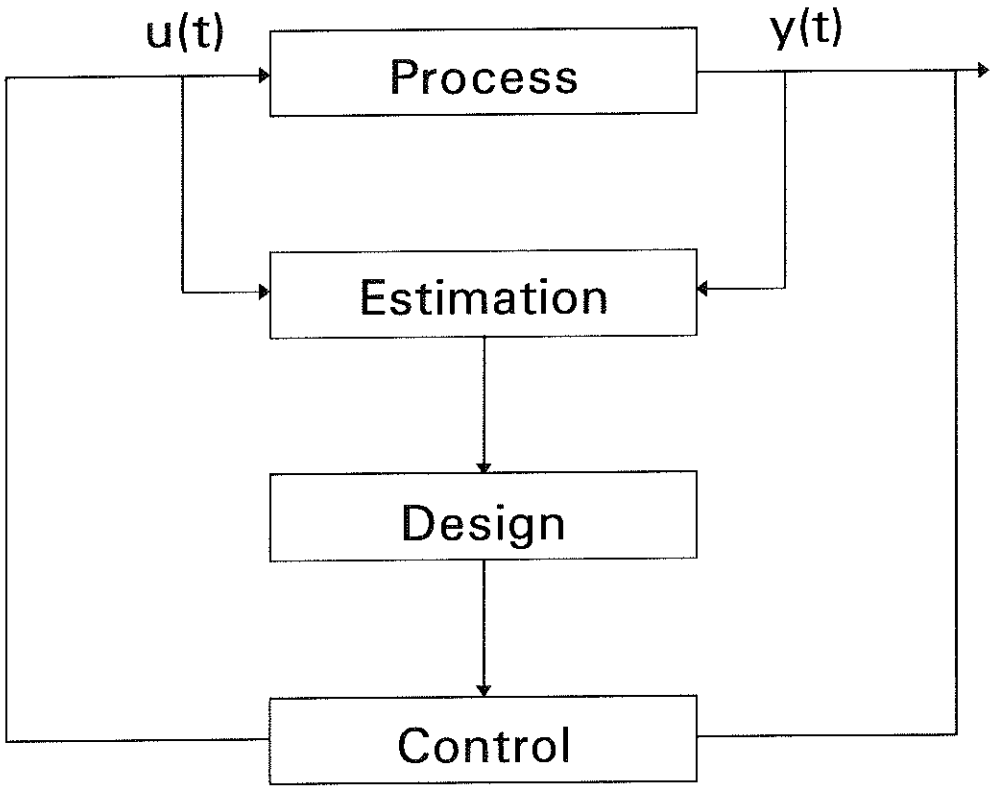

Figure 2.3.1 The adaptive control principle

A large number of algorithms can be developed by combining different estimators with different control laws. It is customary to distinguish between direct (implicit) and indirect (explicit) algorithms. In the indirect algorithm the parameters of the model are updated and inserted in the design equations. Hence, the control parameters are obtained indirectly. In the direct algorithm the model is reformulated, and the control parameters are updated directly. This simplifies the procedure since the design calculations are eliminated.

Now turn to the formulation of the adaptive controller considered in this thesis. A very simple control law will be applied for several reasons:

* plants with slowly drifting parameters can be handled by proper choice of the identification part of the adaptive controller. No modification is required for the control law

* the analytical properties of the simple adaptive controller will be of a quite general nature. Very similar results hold when more complicated control laws are applied (see Goodwin and Sin, 1984)

* unnecessary complication of the presentation is avoided

Let the plant be described by

$$A(q^{-1})\ y(t) = q^{-k}\ B(q^{-1})\ u(t) \tag{2.3.1}$$

$$A(q^{-1}) = 1 + a_1\ q^{-1} + \ldots + a_{n_a}\ q^{-n_a} \tag{2.3.2}$$

$$B(q^{-1}) = b_0 + b_1\ q^{-1} + \ldots + b_{n_b}\ q^{-n_b} \tag{2.3.3}$$

Assume that

A1 the time delay k is known and identical with 1

A2 the polynomial orders n_a and n_b are known

A3 $B(q^{-1})$ has all zeros strictly inside the unit circle

A4 $A(q^{-1})$ and $B(q^{-1})$ are relatively prime, i.e. they have no common zeros

Define

$$\varphi(t) = (-y(t-1), \ldots , -y(t-n_a), u(t-1), \ldots , u(t-n_b-1))^T \tag{2.3.4}$$

$$\theta^0 = (a_1, \ldots , a_{n_a}, b_0, \ldots , b_{n_b})^T \tag{2.3.5}$$

Let the control objective be to track a reference sequence $\{ y^*(t) \}$. Assume that $\{ y^*(t) \}$ is known a priori, and that there exists a positive constant $K < \infty$ such that

$$\| y^*(t) \| < K \quad \text{for all} \quad t \tag{2.3.6}$$

The goal can be achieved by choosing $u(t)$ as the solution to

$$\varphi^T(t+1)\ \theta^0 = y^*(t+1) \tag{2.3.7}$$

Hence, the output is forced to track y^* in a single step. Since θ^0 is unknown the current parameter estimate is used instead

$$\varphi^T(t+1)\ \hat{\theta}(t+1|t) = y^*(t+1) \tag{2.3.8}$$

This will generally lead to a tracking error

$$y(t+1) - y^*(t+1) \neq 0 \tag{2.3.9}$$

Clearly a desirable property is that this error is small and that it vanishes asymptotically.

There is a small risk that $\hat{b}_0(t+1|t) = 0$, making it impossible to solve (2.3.8) with respect to $u(t)$. If the sign of b_0 is known, this can be avoided by constraining the estimate as described in appendix 1.

(2.3.7) is the so–called minimum variance control law designed to minimize the criterion

$$E\{ (y(t+k) - y^*(t+k))^2 \mid I(t) \} \tag{2.3.10}$$

for a stochastic plant description (Åström and Wittenmark, 1973). $I(t)$ is the available information at time t, see (2.2.19). The minimum variance controller has several limitations. For example, non–minimum phase systems (systems with an unstable B–polynomial) cannot be handled since the transfer function from y^* to u contains B in the denominator. For minimum phase systems very large control signals may be required if the zeros of B are close to the unit circle. A survey of extensions including the linear quadratic controllers, the generalized minimum variance controller and the general stochastic pole placement controller can be found in Poulsen and Holst (1990). These methods obtain more robust performance and increased flexibility at the expense of more complicated design calculations.

Example 2.3.1

This example illustrates the problems encountered by the adaptive controller with RLS identification when the plant is actually time–varying.

Study the system described by the linear first order equation

$$y(t) - 0.8\ y(t-1) = 1.0\ u(t-1) + e(t)$$

for $t \leq 250$. $\{ e(t) \}$ is $N (0, 0.1^2)$ white noise. At $t = 251$ a set point change occurs, and the correct description becomes

$$y(t) - 0.4\ y(t-1) = 2.0\ u(t-1) + e(t)$$

The adaptive controller outlined above is applied in order to enable tracking of a square wave output reference signal.

Based on a constant parameter assumption the estimates are updated using RLS with the initial values

$$\hat{\theta}(1|0) = (\ \hat{a}(1|0),\ \hat{b}(1|0)\ )^T = (\ -0.5,\ 0.5\ )^T$$

$$P(1|0) = 1000\ I$$

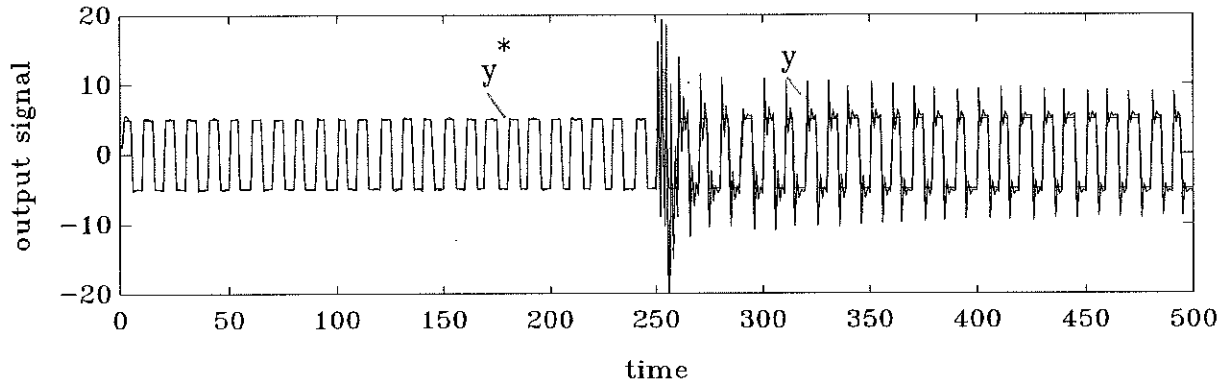

Figure 2.3.2 Output signal (——) and reference signal (– – –) for the adaptive controller in example 2.3.1. RLS identification. Set point change at t = 251.

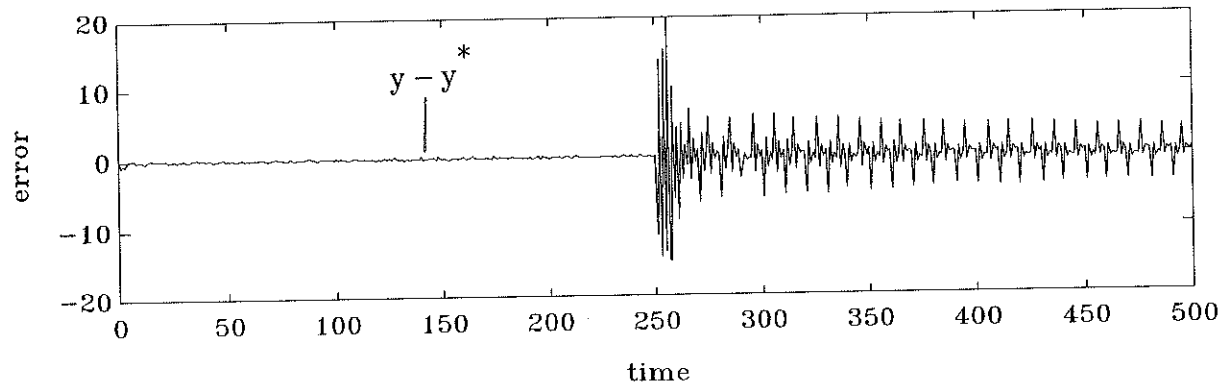

Figure 2.3.3 Output tracking error for the adaptive controller in example 2.3.1. RLS identification. Set point change at t = 251.

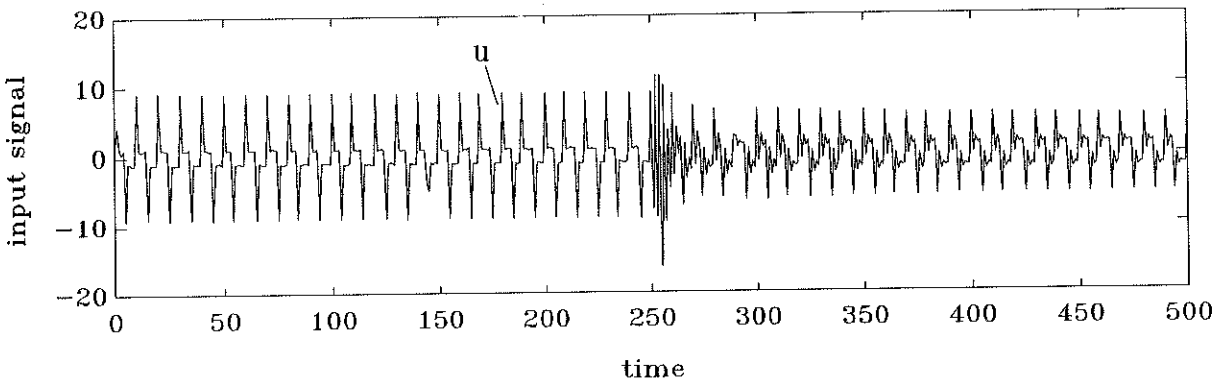

Figure 2.3.4 Input signal for the adaptive controller in example 2.3.1. RLS identification. Set point change at $t = 251$.

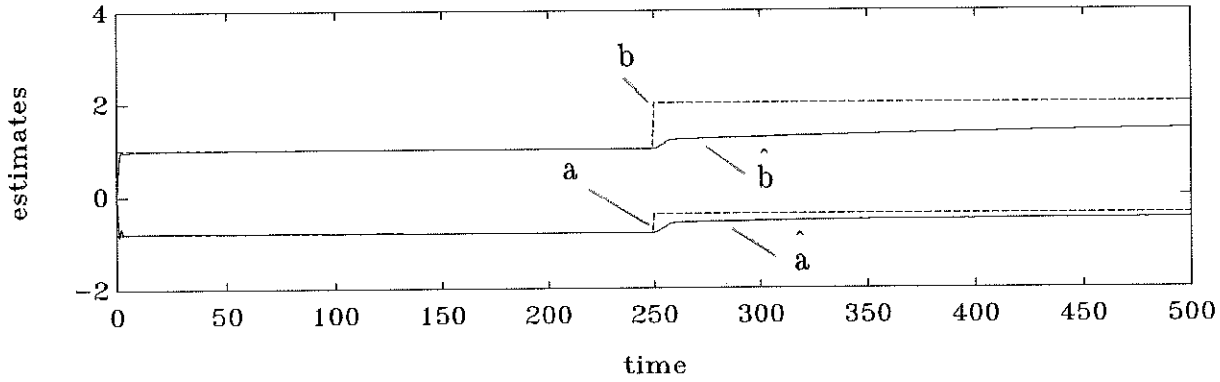

Figure 2.3.5 Parameters (——) and RLS estimates (− − −) for example 2.3.1.

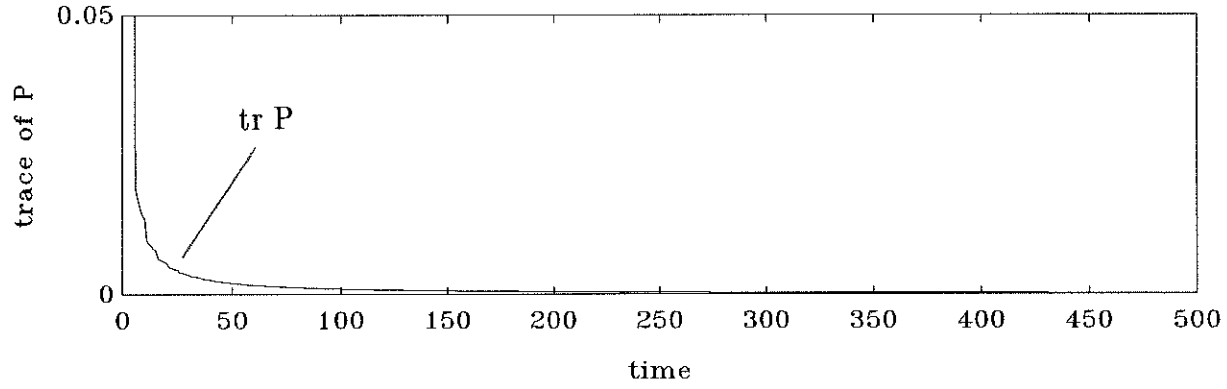

Figure 2.3.6 Trace of P for the RLS estimator in example 2.3.1.

From the figures 2.3.2 – 2.3.6 the following observations can be made: The initial estimates are rapidly corrected, and the controller performs well for $t \leq 250$. P approaches zero as the confidence in the estimates is gradually increased. However, at $t = 251$ the constant parameter assumption is broken. Due to the vanishing algorithm gain the estimation error remains large, i.e. the ability of self tuning is almost lost. This leads to large output tracking errors for $t \geq 251$.

∇

2.4 Summary

In this chapter some methods, facts and assumptions which are relevant for the discussion in the remainder of the thesis have been described.

Throughout the thesis it will be assumed that the plant which is being identified can be described by a linear regression equation. This implies that the output signal depends linearly on the set of unknown parameters. However, the model may be non–linear in the signals. The assumption about parameter linearity makes it possible to solve the least squares estimation problem without approximations. It is important to note that for more general model structures recursive estimation algorithms typically consist of quite similar update equations, and consequently the discussion in the following chapters can readily be generalized.

The on–line estimation algorithms which will be discussed later in this thesis can be regarded as modified versions of the recursive least squares (RLS) method. This chapter has given a thorough description of RLS estimation. The theoretical properties of the algorithm have been reviewed and a number of interpretations of the variables occurring in the equations have been given.

An important application of recursive parameter estimation is as the identification part of adaptive controllers. A simple adaptive control algorithm has been described in section 2.3. The control part of the algorithm calculates the control signal which will set the output equal to the desired reference value in a single step. This is known as one–step ahead control or minimum variance control. The identification part of the algorithm may be based on RLS.

In the following chapters modified versions of the adaptive controller will be considered. Since RLS relies on a constant parameter assumption, certain changes are necessary in the time–varying case. One possibility is to replace the RLS method with an estimator designed to track time–variations. How this affects the theoretical and practical properties of the algorithm is one of the issues that will be discussed. Furthermore, the basic principles of the algorithm will prove to be applicable in connection with control of the

non–linear nitrification process.

3

Unification of recursive estimation methods designed for the time–varying case

The RLS method discussed in chapter 2 is the standard recursive estimation procedure when the model parameters are assumed to be time–invariant. However, it is not directly applicable when the parameters vary with time. When there is a persistent flow of information, the algorithm gain tends to zero, making it impossible to track the variations.

In this situation it is necessary to modify the algorithm to avoid that the updating turns off. One way to handle the time–variations is by gradually discounting old information. This is the idea behind the so–called forgetting methods. In the remainder of this thesis the attention will mainly be focused on this class of estimators. Different variants of the methods will be discussed and their theoretical properties and practical performance will be examined. Special attention will be directed towards their application to adaptive control.

The present chapter has two main goals. The first is to review the most important forgetting techniques suggested in the literature. The second is to formulate a general forgetting algorithm containing most existing methods as special cases.

In the popular and widely used exponential forgetting method, a time–invariant scalar forgetting factor is used, and old information is discounted uniformly in time and in parameter space. This prevents the gain from becoming zero, and the algorithm is kept alive. Unfortunately, this simple forgetting principle has some serious drawbacks. For example, in situations where the information received about the parameter evolution is limited to certain directions of the parameter space, the information related to non–excited directions will gradually be lost. This results in unlimited growth of the algorithm gain and can lead to large estimation errors. Furthermore, the use of a scalar forgetting factor makes it impossible to tune the forgetting individually to each of the parameters. This is inappropriate in cases where the parameters are known to have different rates of variation. These characteristic features of the simple exponential forgetting method have resulted in suggestion of several alternative techniques.

More advanced forgetting schemes take into account the nature of the current information pattern and in some cases also the variation of the individual

parameters. Typically their implementation requires a more detailed prior knowledge about the approximations associated with the process model, the experimental conditions, etc. This knowledge is necessary in order to choose the tuning parameters of the method correctly. Sometimes, however, the selection of tuning parameters can be handled automatically, and a high level of adaptivity can be maintained. For example, the tuning parameters may be determined as suitable functions of variables updated in the algorithm.

The literature on forgetting methods may seem a little confusing to readers who are not familiar with the subject. A large number of methods exist, and very often different authors use different notational conventions. This chapter gives a unified survey of the subject. It turns out that most of the methods are based on a few basic ideas. The use of a consistent notation reveals that in fact they differ only by the choice of a particular update equation representing the forgetting principle employed.

In section 3.1 the exponential forgetting method is described. Section 3.2 outlines the ideas and principles behind some more advanced techniques suggested in the literature. Finally in section 3.3 it is shown that all of the discussed techniques are special cases of one *general* forgetting algorithm. This observation makes it easier to survey the subject, to point out similarities and dissimilarities between the methods, to get ideas for their improvement and, perhaps the most important bonus, to perform a unified analysis of their theoretical properties. This analysis will be presented in the next chapter.

3.1 Time–varying parameters and exponential forgetting

The RLS method is based on the assumption that the process can be described by a constant parameter model. However, in many practical situations this assumption has to be modified, for example if the operating point is drifting or if some of the model parameters are actually slowly varying variables. In such cases it may be necessary to consider the parameters as time–varying quantities.

The estimation of time–varying parameters requires use of modified algorithms. Roughly speaking they can be divided into two categories designed with the following tasks in mind:

1. detection and estimation of infrequent, large parameter jumps

2. tracking of slowly drifting parameters

The discussion in this thesis will focus on the latter category. However, most of the methods are actually able to handle parameter jumps as well. This will be illustrated in the examples.

The parameters will be assumed to exhibit little or no variation within a time horizon N, $N >> 1$.

First consider the RLS method. From the update equation

$$P^{-1}(t) = P^{-1}(t-1) + \varphi(t)\ \varphi^{T}(t) \tag{3.1.1}$$

it can be seen that if the sequence $\{\ \varphi(t)\ \}$ provides sufficiently rich excitation, the P matrix will converge to zero. Consequently $\hat{\theta}(t)$ becomes constant, meaning that the RLS algorithm eventually loses its ability to track time–variations in the parameters. This can be explained by noting that the RLS criterion

$$(3.1.2) \qquad J(t,\theta) = \sum_{k=1}^{t} (y(k) - \varphi^T(k)\ \theta)^2$$

gives all residuals equal weight. As $t \to \infty$ the influence of any finite data sequence will become insignificant. However, if the plant parameters drift in time, the new observations contain more relevant information about the current parameter values than the old measurements. One way to pay respect to this fact is by introducing a forgetting factor λ ($0 < \lambda < 1$) gradually discounting old observations. This idea leads to the modified criterion

$$(3.1.3) \qquad J(t,\theta) = \sum_{k=1}^{t} \lambda^{t-k} (y(k) - \varphi^T(k)\ \theta)^2$$

Minimization of (3.1.3) gives the so called exponential forgetting (EF) method

$$(3.1.4) \qquad \hat{\theta}(t) = \hat{\theta}(t-1) + K(t)\ (\ y(t) - \varphi^T(t)\ \hat{\theta}(t-1)\ )$$

$$(3.1.5) \qquad K(t) = \frac{P(t-1)\ \varphi(t)}{\lambda + \varphi^T(t)\ P(t-1)\ \varphi(t)} = P(t)\ \varphi(t)$$

$$(3.1.6) \qquad P^{-1}(t) = \lambda\ P^{-1}(t-1) + \varphi(t)\ \varphi^T(t)$$

or

$$(3.1.7) \qquad P(t) = [\ P(t-1) - \frac{P(t-1)\ \varphi(t)\ \varphi^T(t)\ P(t-1)}{\lambda + \varphi^T(t)\ P(t-1)\ \varphi(t)}\ ]\ \frac{1}{\lambda}$$

For $\lambda = 1$ the RLS method is obtained as a special case. The effect of choosing $\lambda < 1$ is that $P(t)$ is kept away from zero. Consequently, the algorithm retains its ability to track time–varying parameters. At time t the residual $(\ y(k) - \varphi^T(k)\ \hat{\theta}(t)\ )$ is associated with the weight

$$(3.1.8) \qquad \lambda^{t-k} = \exp\ \{\ (t-k)\ \ln\lambda\ \}$$

If the equivalent horizon N is defined to be the corresponding time constant then

$$N = -\frac{1}{\ln \lambda}$$

$$\approx \frac{1}{1-\lambda} \tag{3.1.9}$$

or

$$\lambda \approx 1 - \frac{1}{N} \tag{3.1.10}$$

This expression makes it possible to regard N as the tuning parameter of the method.

Example 3.1.1

In this example the choice of forgetting factor in the EF method is considered.

Consider the system from example 2.3.1. Recall that the RLS method was unable to cope with the parameter jump at $t = 251$. An open loop experiment is performed, and the input–output signals are as shown in fig. 3.1.1 – 3.1.2.

The initial estimates are identical with the correct parameter values and

$$P(0)=1000\ I$$

Figure 3.1.3 shows the estimates obtained with different values of λ

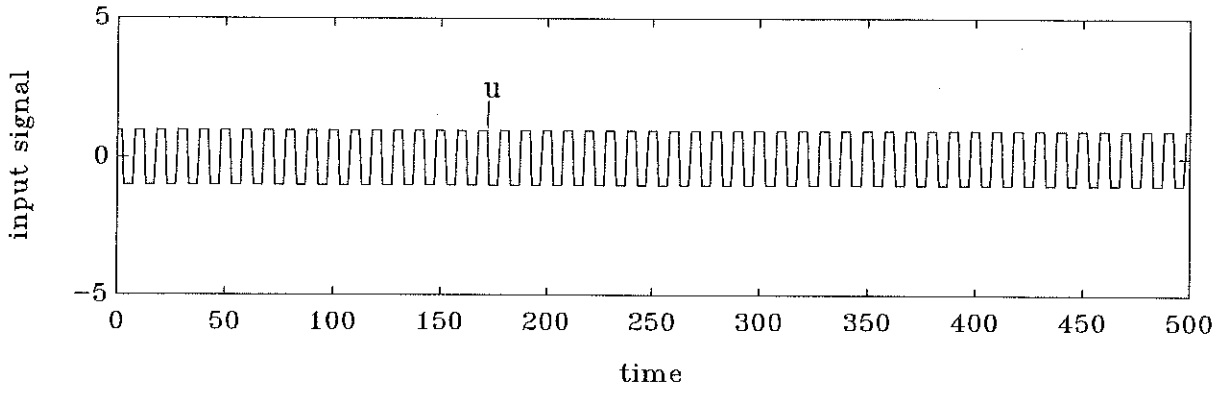

Figure 3.1.1 Input signal for example 3.1.1.

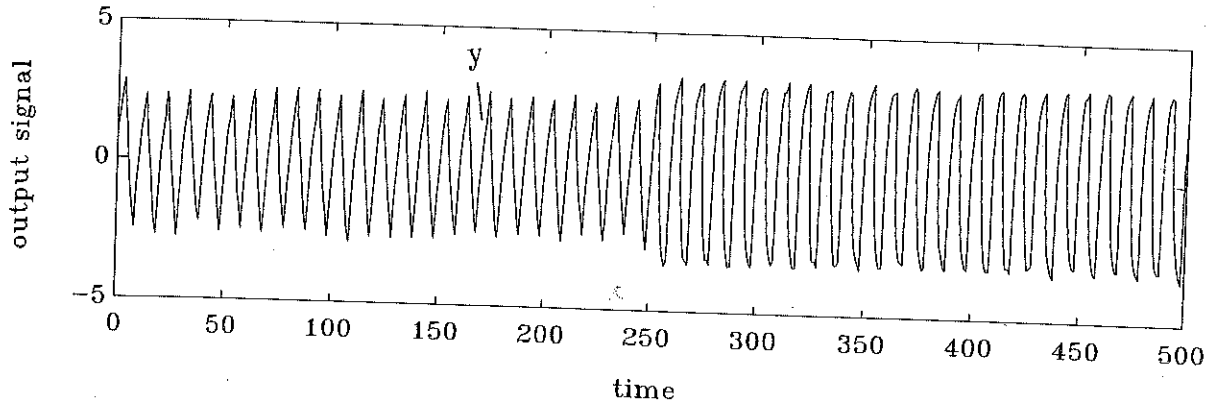

Figure 3.1.2 Output signal for example 3.1.1. Parameter jump at $t = 201$.

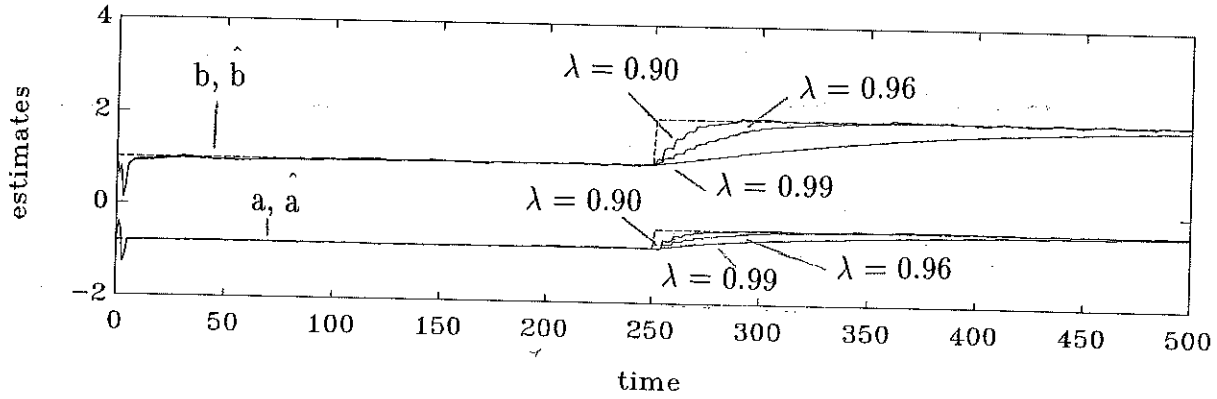

Figure 3.1.3 Parameters (– – –) and EF – estimates (——) for example 3.1.1 using different values of λ.

It can be seen that a small value of λ gives fast adaption. Unfortunately, a small forgetting factor also leads to stronger variation of the estimates. The situation is opposite when λ is close to 1. To illustrate this define

$$S_\theta = \sum_{t=50}^{250} \| \hat{\theta}(t|t) - \frac{1}{201} \sum_{k=50}^{250} \hat{\theta}(k|k) \|^2$$

S_θ is a measure of the variation of the parameter estimates. Note that only values of $\hat{\theta}(t|t)$ calculated for $50 \leq t \leq 250$ are included. Furthermore, to measure the tracking ability define T_θ to be the length of the time interval from the jump takes place until 90% of the change has been estimated, i.e. until

$$\frac{\| \tilde{\theta}(t) \|^2}{\| \Delta \theta(251) \|^2} = 0.1$$

Here $\tilde{\theta}(t)$ is the estimation error and $\Delta \theta(251)$ is the parameter change at $t = 251$. Figure 3.1.4 and figure 3.1.5 clearly illustrate that the choice of λ is a compromise between the ability to track the time–variations on one hand and the sensibility of the estimates on the other.

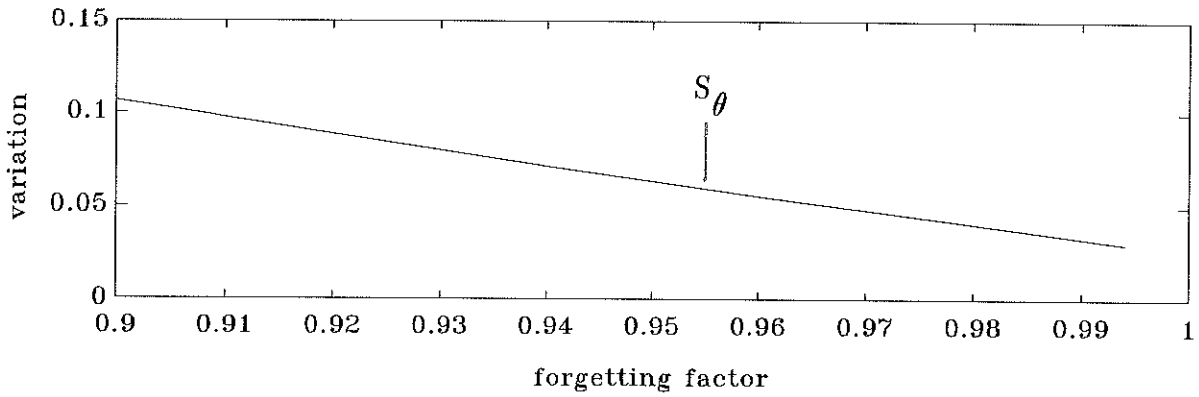

Figure 3.1.4 Variation of the EF – estimates versus choice of forgetting factor

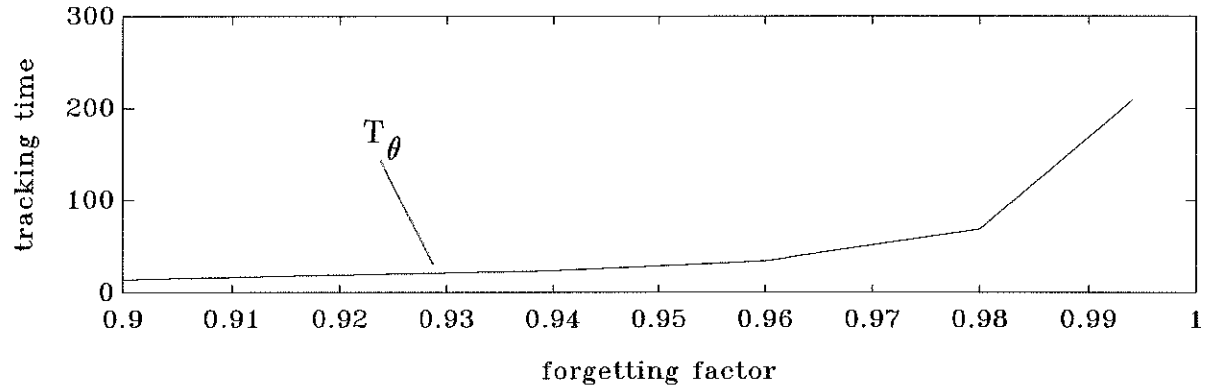

Figure 3.1.5 Time spent by the EF − method to track 90 % of the parameter change plotted as a function of the forgetting factor.

∇

The EF method performs well in many situations. However, the structure of the update equation for the covariance matrix P has a couple of serious drawbacks.

First consider a situation where the regression vector $\varphi(t)$ is identical to zero for a certain period of time. In this case the update of P becomes

(3.1.11) $$P(t+1) = P(t) \;/\; \lambda$$

and consequently all elements of P grow exponentially fast. This phenomenon is known as *covariance wind up* or *blow up*. It occurs more generally when the φ–vectors are restricted to a part of the parameter space. In this case eigenvalues for P corresponding to non–excited directions will grow exponentially fast. The adjustment of $\hat{\theta}$ is proportional to P and to $(y - \varphi^T \hat{\theta})$. If P becomes very large, then even a small measurement error can lead to a very large and perhaps erroneous change of the estimates.

Example 3.1.2

Consider again the system in example 3.1.1. This time let the parameters remain constant

$$a = -0.8$$

$$b = 1.0$$

The input signal is identical to 1 for $t > 100$. All other experimental conditions are as in example 3.1.1. The EF method is applied with $\lambda = 0.95$

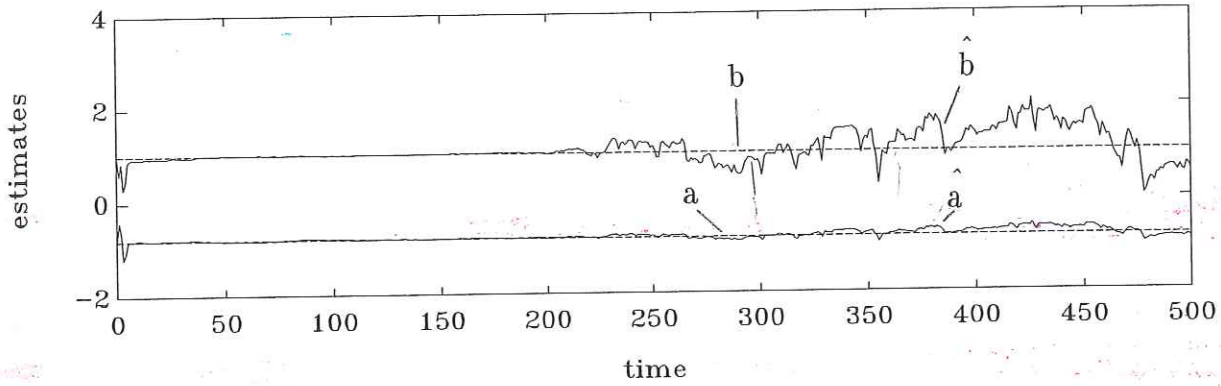

Figure 3.1.6 Parameters (– – –) and EF – estimates (——) for example 3.1.2. Constant input signal for $t > 100$.

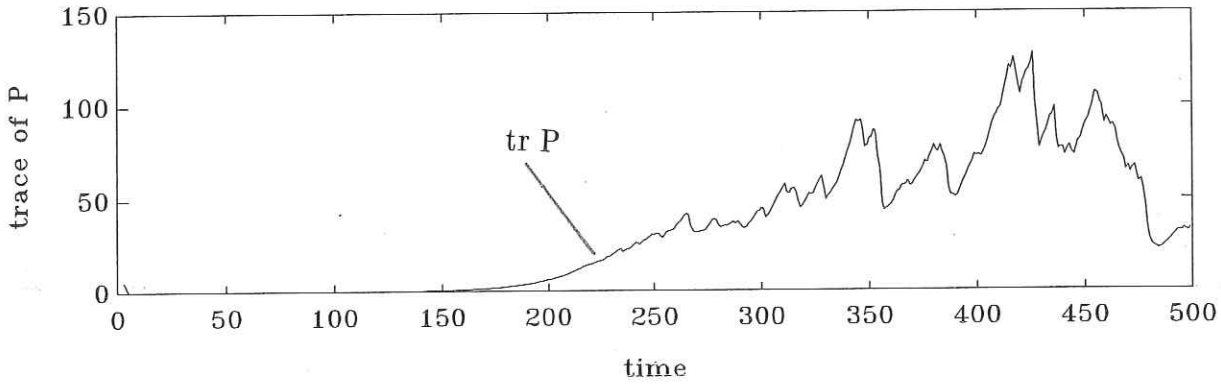

Figure 3.1.7 Trace of P for the EF – method in example 3.1.2. Constant input signal for $t > 100$.

Fig. 3.1.6 – 3.1.7 show that after $t = 100$ the trace of $\hat{P}$ starts growing exponentially. This tendency is reduced somewhat due to the excitation caused by the measurement noise. The variation of the corresponding estimates becomes large and the estimation error increases.

∇

Another limitation of the EF method is that it is impossible to utilize a priori information that some parameters vary faster than others. If for example λ is chosen in order to enable tracking of fast parameters, the estimates of parameters with a slow variation become unnecessarily sensitive to the noise variation.

Example 3.1.3

Study again the system in example 3.1.1. The noise level is raised to

$$\sigma^2 = 0.5^2$$

Let the parameters be time–varying as

$$a(t) = -0.8$$

$$b(t) = 1.0 + 0.5 \sin\left(\frac{2\pi t}{500}\right)$$

Hence, $a(t)$ is constant while $b(t)$ has a relatively fast variation. First a small forgetting factor $\lambda = 0.9$ is applied. Figure 3.1.8 illustrates that $b(t)$ can be tracked, but variation of $\hat{a}(t)$ is very large.

Next the identification experiment is repeated with $\lambda = 0.99$. Now the estimate of $a(t)$ is more steady, but $b(t)$ cannot be tracked properly, as can be seen from figure 3.1.9.

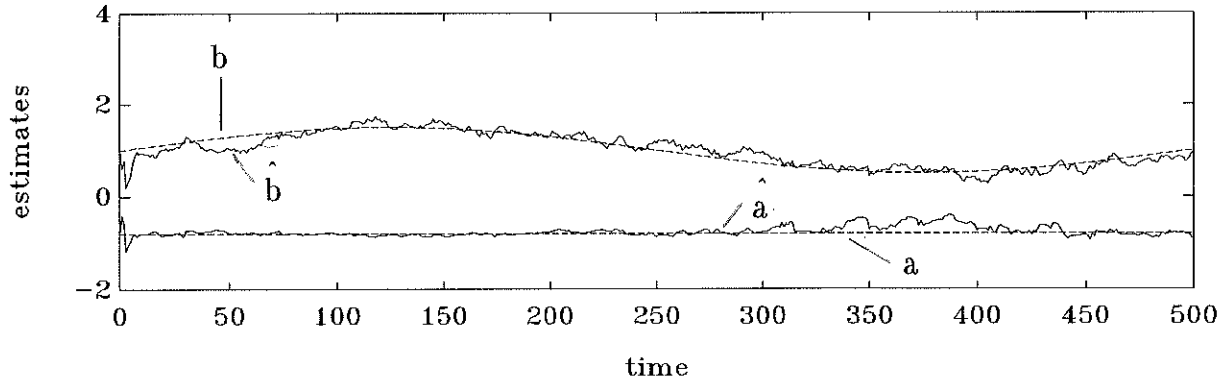

Figure 3.1.8 Parameters (− − −) and estimates (——) for example 3.1.3 using EF − estimation and a small forgetting factor ($\lambda = 0.9$).

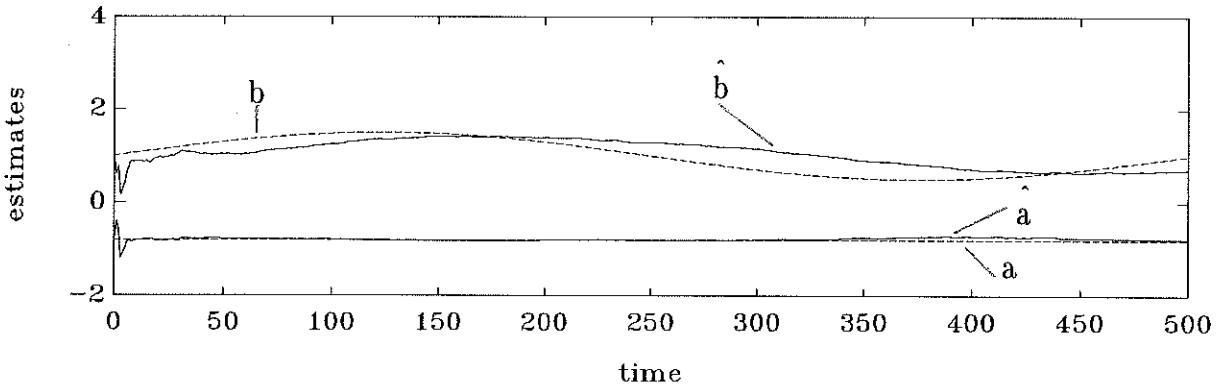

Figure 3.1.9 Parameters (− − −) and estimates (——) for example 3.1.3 using EF − estimation and a forgetting factor close to 1 ($\lambda = 0.99$).

∇

3.2 The ideas behind more advanced forgetting schemes

The limitations of the EF technique are related to the fact that a time–invariant and scalar forgetting factor is used, and old information is discounted uniformly in time and in parameter space. Consequently, several authors have proposed alternative procedures where the forgetting is non–uniform in time or in parameter space. In this section some of the most important ideas are reviewed.

Sections 3.2.1 and 3.2.2 address two different situations both typically leading to covariance wind up. Section 3.2.3 gives a discussion of methods aiming at improving the estimation when the parameters have different rates of variation.

Needless to say, the problems discussed in sections 3.2.1 – 3.2.3 may occur simultaneously. Combination of the principles leads to hybrid versions of the methods outlined here.

3.2.1 Non–uniform information flow in time

Consider a situation where the information flow is uniformly distributed in parameter space, but not in time. Such a situation can be handled without giving up the basic structure of the EF update for P by simply reducing the degree of forgetting in periods of poor excitation. This makes it possible to prevent the information content of the estimator from becoming too high or too low. An elegant solution to the problem is

* Choose a measure INF(t) of the information content of the estimator

* Choose a time–varying forgetting factor $\lambda(t)$ keeping the information content constant, i. e. $INF(t) = INF(t-1) = INF_0$

The version of EF with a time–varying forgetting factor can formally be derived by minimization of

$$(3.2.1) \qquad J(t,\theta) = \sum_{k=1}^{t} \gamma(t;k)\ (y(k) - \varphi^T(k)\ \theta)^2$$

$$\gamma(t;k) = \lambda(t)\ \gamma(t-1;k) \qquad 1 \le k \le t-1$$

$$(3.2.2) \qquad = 1 \qquad k = t$$

One possible choice of information criterion is

$$(3.2.3) \qquad INF(t) = J(t,\hat{\theta}(t))$$

A small value of J indicates that the parameters are well tuned, and in typical adaptive control applications this leads to poor excitation, i.e. a low information content of the data. Conversely, large prediction errors provide good excitation and a high level of information.

Solving the equation

$$(3.2.4) \qquad INF(t) = INF(t-1)$$

leads to the Fortescue forgetting factor

$$(3.2.5) \qquad \lambda(t) = 1 - \frac{\epsilon^2(t)}{J_0\ (\ 1 + \varphi^T(t)\ P(t-1)\ \varphi(t)\ )}$$

(Fortescue et al., 1981). Notice that application of the interpretation

$$(3.2.6) \qquad E\ \{\ \epsilon^2(t)\ |\ I(t-1)\ \} = \sigma^2\ (\ 1 + \varphi^T(t)\ P(t-1)\ \varphi(t)\ )$$

yields

$$(3.2.7) \qquad E\ \{\ \lambda(t)\ |\ I(t-1)\ \} = 1 - \frac{\sigma^2}{J_0}$$

(Recall here and in the following discussion that I(t) represents the

measurements collected at t, t–1, ..., see (2.2.19). This quantity should not be confused with INF(t)). Therefore it is common practice to define

(3.2.8) $$J_0 = \sigma^2 N$$

and regard N as the tuning parameter, compare with (3.1.10). $\lambda(t)$ is only smaller than one in periods where a non–zero prediction error actually occurs. This obviously tends to reduce the risk of blow up. However, continuously drifting parameters or a high noise level will force λ to be smaller than one, and in these cases blow up may still occur.

Example 3.2.1

Consider the experiment described in example 3.1.1. Results obtained using the Fortescue forgetting factor with $N = 25$ are shown below. For $t \leq 250$ the parameters are constant, the prediction error is small and $\lambda(t)$ is close to 1. At $t = 251$ a parameters jump occurs. This results in a large prediction error and a small forgetting factor. The gain of the method is increased, and the estimator becomes able to track the jump. For $t > 251$ $\lambda(t)$ again gradually approaches 1 as the prediction error is eliminated.

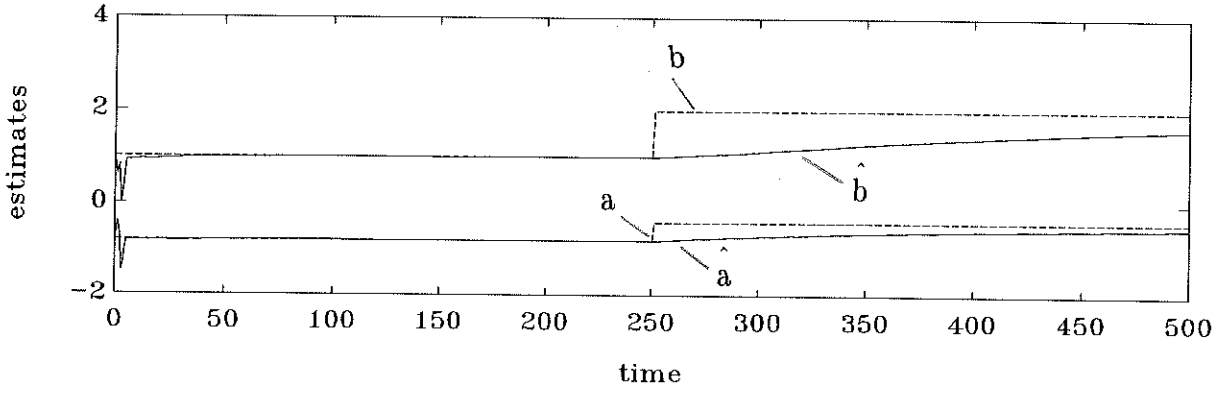

Figure 3.2.1 Parameters and estimates obtained for example 3.2.1 using the exponential forgetting method with Fortescue's forgetting factor.

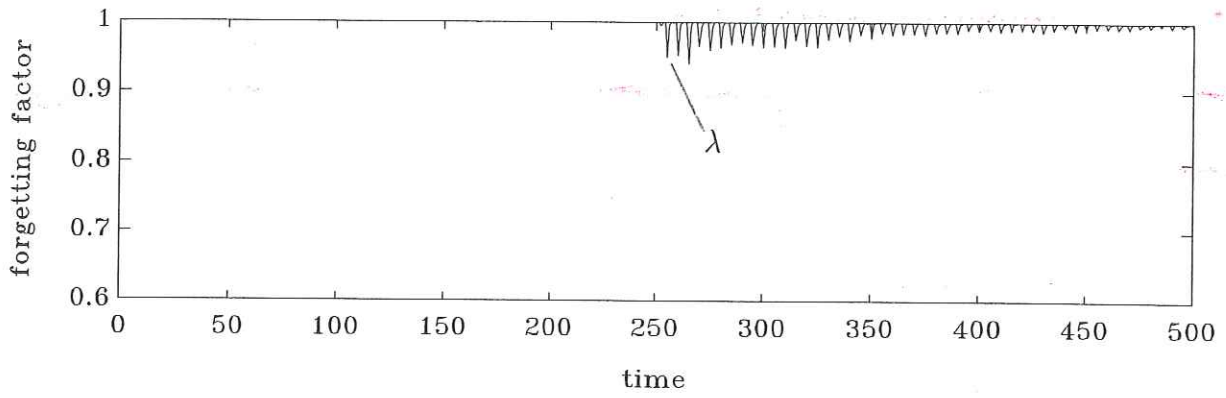

Figure 3.2.2 Fortescue forgetting factor, example 3.2.1. Parameter jump at t = 251.

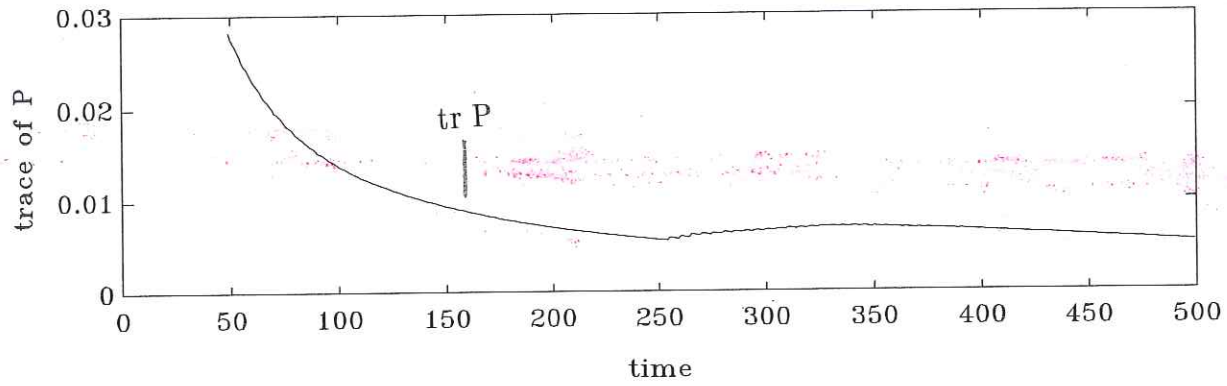

Figure 3.2.3 Trace of the P − matrix obtained for example 3.2.1 using the exponential forgetting method with Fortescue's forgetting factor.

∇

In connection with practical implementation it can be recommended to restrict the forgetting factor as follows:

(3.2.9) $$\lambda_{min} \leq \lambda(t) \leq \lambda_{max}$$

(3.2.10) $$0 < \lambda_{min} < \lambda_{max} < 1$$

Consider the special case where the parameters are constant and $\sigma^2 = 0$. From the discussion in chapter 4 it will follow that if $\{ \varphi(t) \}$ is persistently exciting and bounded then this version of the algorithm will have the following property:

$$\lim_{t \to \infty} \frac{\epsilon^2(t)}{1 + \varphi^T(t)\, P(t-1)\, \varphi(t)} = 0 \tag{3.2.11}$$

Consequently

$$\lim_{t \to \infty} \left(1 - \frac{\epsilon^2(t)}{N(1 + \varphi^T(t)\, P(t-1)\, \varphi(t))} \right) = 1 \tag{3.2.12}$$

and by the restriction on λ the algorithm asymptotically becomes identical to the standard EF method with $\lambda = \lambda_{max}$ when the information level is high.

Another possibility is to apply a scalar function of P or P^{-1} as a measure of the information content, for example

$$INF(t) = \text{tr } P(t) \tag{3.2.13}$$

The recursion for $\text{tr } P$ corresponding to (3.1.7) is

$$\text{tr } P(t) = \left[\text{tr } P(t-1) - \frac{\varphi^T(t)\, P^2(t-1)\, \varphi(t)}{\lambda(t) + \varphi^T(t)\, P(t-1)\, \varphi(t)} \right] \frac{1}{\lambda(t)} \tag{3.2.14}$$

The demand

$$\text{tr } P(t) = \text{tr } P(t-1) \tag{3.2.15}$$

leads to the equation

$$\lambda^2(t) + [f_1(t) - 1]\, \lambda(t) + \left[\frac{f_2(t)}{\text{tr } P(t-1)} - f_1(t) \right] = 0 \tag{3.2.16}$$

where

$$f_j(t) = \varphi^T(t)\, P^j(t-1)\, \varphi(t), \quad j = 1, 2 \tag{3.2.17}$$

The usable solution is

$$\lambda(t) = \frac{1}{2} \left[1 - f_1(t) - \sqrt{ (1 + f_1(t))^2 - 4 f_2(t) / \text{tr } P(t-1) } \right] \tag{3.2.18}$$

Apply the interpretation

$$E \{ \tilde{\theta}(t) \tilde{\theta}^T(t) \mid I(t) \} = \sigma^2 P(t) \tag{3.2.19}$$

Then

$$E \{ \| \tilde{\theta}(t) \|^2 \mid I(t) \} = \sigma^2 \text{ tr } P(t) \tag{3.2.20}$$

Hence, if tr P is freezed at a large value in order to make tracking of the variations possible, the price will be a larger variance of the estimation error.

Keeping tr P(t) constant corresponds to keeping the sum of P(t)'s eigenvalues constant. This gives an upper bound for the eigenvalues, but not a positive lower bound.

The information measure

$$INF(t) = \text{tr } P^{-1}(t) \tag{3.2.21}$$

leads to

$$\lambda(t) = 1 - \frac{1}{\text{tr } P^{-1}(t-1)} \varphi^T(t) \varphi(t) \tag{3.2.22}$$

This method bounds the eigenvalues of P(t) from below, but not from above.

Methods based on keeping tr P or tr P^{-1} constant have been discussed in several papers, see for example Landau and Lozano (1981).

In general it can be argued that scalar information criteria reveal nothing about the distribution of information in parameter space. For example, even if tr P is kept constant, some eigenvalues of P may converge to zero, leading to a complete loss of tracking ability in the corresponding directions.

3.2.2 Non–uniform information flow in parameter space

In this section it will be assumed that the information is uniformly distributed in time, but not in parameter space. In other words, a situation where the vector sequence $\{ \varphi(t) \}$ does not excite all directions of the parameter space equally well is considered. Here the EF update equation for P is unsuitable since it implies uniform forgetting in all directions.

A natural idea is to discount information only in the direction where new information is received, i.e the direction determined by $\varphi(t)$. This is the idea behind the directional forgetting (DF) method:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + K(t) \left(y(t) - \varphi^T(t) \hat{\theta}(t-1) \right) \tag{3.2.23}$$

$$K(t) = \frac{P(t-1) \varphi(t)}{1 + \varphi^T(t) P(t-1) \varphi(t)} \tag{3.2.24}$$

$$P^{-1}(t) = P^{-1}(t-1) + \left(1 - \kappa(t) \right) \varphi(t) \varphi^T(t) \tag{3.2.25}$$

or

$$P(t) = P(t-1) - \frac{P(t-1) \varphi(t) \varphi^T(t) P(t-1)}{\left(1 - \kappa(t) \right)^{-1} + \varphi^T(t) P(t-1) \varphi(t)} \tag{3.2.26}$$

(Hägglund, 1983, Kulhavy and Karny, 1984). Kulhavy and Karny proposed the following choice of $\kappa(t)$:

$$\kappa(t) = \left(1 - \lambda \right) \left[1 + \frac{1}{\varphi^T(t) P(t|t-1) \varphi(t)} \right] \tag{3.2.27}$$

if $\varphi^T(t) P(t-1) \varphi(t) > 0$. In the case $\varphi^T(t) P(t-1) \varphi(t) = 0$ $\kappa(t)$ is set

equal to 0.

λ is the forgetting factor, $0 < \lambda \leq 1$. λ is only applied to the part of the P–matrix which is proportional to $\varphi(t)\ \varphi^T(t)$. Notice that $(1 - \kappa(t))$ may be positive or negative, corresponding to an increase or decrease in information content in the direction determined by $\varphi(t)$. If the parameter uncertainty in this direction is large ($\varphi^T(t)\ P(t-1)\ \varphi(t)$ large) then information is added. On the other hand, if the uncertainty is small, information is subtracted.

3.2.3 Different rates of parameter variation.

Now assume that a priori information is available about the individual parameter variation rates. The exponential forgetting update offers no possibility to use such information, since the forgetting factor λ affects all parameters equally. Two approaches able to tackle this problem will be considered.

The linear forgetting method.

One possibility is to formulate a parameter model. For example, a random walk model for the parameter variation can be applied:

$$\theta(t+1) = \theta(t) + w(t) \tag{3.2.28}$$

where $\{ w(t) \}$ is $N(0, \sigma^2 R)$ white noise. The Kalman filtering approach leads to the following estimation method

Measurement update

$$\hat{\theta}(t|t) = \hat{\theta}(t|t-1) + P(t|t)\ \varphi(t) \left[y(t) - \varphi^T(t)\ \hat{\theta}(t|t-1) \right] \tag{3.2.29}$$

$$P^{-1}(t|t) = P^{-1}(t|t-1) + \varphi(t)\ \varphi^T(t) \tag{3.2.30}$$

Time update:

(3.2.31) $\hat{\theta}(t+1|t) = \hat{\theta}(t|t)$

(3.2.32) $P(t+1|t) = P(t|t) + R$

Normally R is unknown. Methods for obtaining estimates of σ^2 and R have traditionally been developed in the area of state estimation, see for example Mehra (1974), Isaksson (1986).

If the formal assumptions connected with the parameter model cannot be justified, the update can still be used for ad hoc estimation. In this case the elements of R can be regarded as tuning parameters, adjusted with reference to some performance measure. A natural choice is

(3.2.33) $R = \mathrm{diag}(r_{kk})$

The size of r_{kk} then reflects the expected rate of variation for $\theta_k(t)$. This ad hoc version of the method will be called the linear forgetting (LF) method.

Notice that in general no upper bound can be guaranteed to exist for the P–matrix.

The matrix forgetting method

In the following discussion ad hoc arguments are applied to develop a forgetting method assigning an individual forgetting factor to each parameter.

Assume that measures of the rate of variation for the elements of θ, θ_1, ..., θ_p, are available. Based on this information p forgetting factors λ_1, ..., λ_p can be chosen (λ_k corresponds to θ_k). Now the matrix

(3.2.34) $\Phi = \mathrm{diag}\ (\ 1\ /\ \sqrt{\lambda_k}\ )$

can be formed. An interesting possibility is to replace

(3.2.35) $P(t-1) \ / \ \lambda$

by

(3.2.36) $\Phi \ P(t-1) \ \Phi^T$

in the update equations for P. This leads to

(3.2.37) $P(t)^{-1} = (\ \Phi^{-1} \ )^T P^{-1}(t-1) \ \Phi^{-1} + \varphi(t) \ \varphi^T(t)$

Loosely speaking, the scalar forgetting factor is replaced by a matrix of forgetting factors. The special case

(3.2.38) $\lambda_1 = ... = \lambda_p = \lambda$

corresponds to EF. The diagonal elements of P^{-1} (denoted by η_{kk}) can be seen as measures of the available information related to θ_k. These elements are updated according to

(3.2.39) $\eta_{kk}(t) = \lambda_k \ \eta_{kk}(t-1) + \varphi_k^2(t)$

This justifies the interpretation of λ_k as the forgetting factor corresponding to θ_k. However, it should be noted that in the general case where the parameters are correlated (P non–diagonal) λ_k will also affect the parameter θ_m, $k \neq m$, via the element η_{km} updated according to

(3.2.40) $\eta_{km}(t) = (\ \lambda_k \ \lambda_m \ )^{\frac{1}{2}} \ \eta_{km}(t-1) + \varphi_k(t) \ \varphi_m(t)$

The matrix forgetting (MF) method was first presented in Saelid and Foss (1983). (There a slightly different argumentation was used and furthermore,

λ_k was chosen as a time–varying quantity following ideas closely related to the ones outlined in section 3.2.1).

The method performs well when $\{ \varphi(t) \}$ is persistently exciting and bounded, but it suffers from the blow up weakness when the information content of the data is low.

Example 3.2.3

This example illustrates the effect achieved with the MF method when different forgetting factors are assigned to different parameters.

First consider the experiment described in example 3.1.1, where both parameters jump at $t = 251$. The matrix forgetting method is applied for estimation first with

$$\Phi = \begin{bmatrix} 1 / \sqrt{0.99} & 0 \\ 0 & 1 / \sqrt{0.9} \end{bmatrix}$$

then with

$$\Phi = \begin{bmatrix} 1 / \sqrt{0.9} & 0 \\ 0 & 1 / \sqrt{0.99} \end{bmatrix}$$

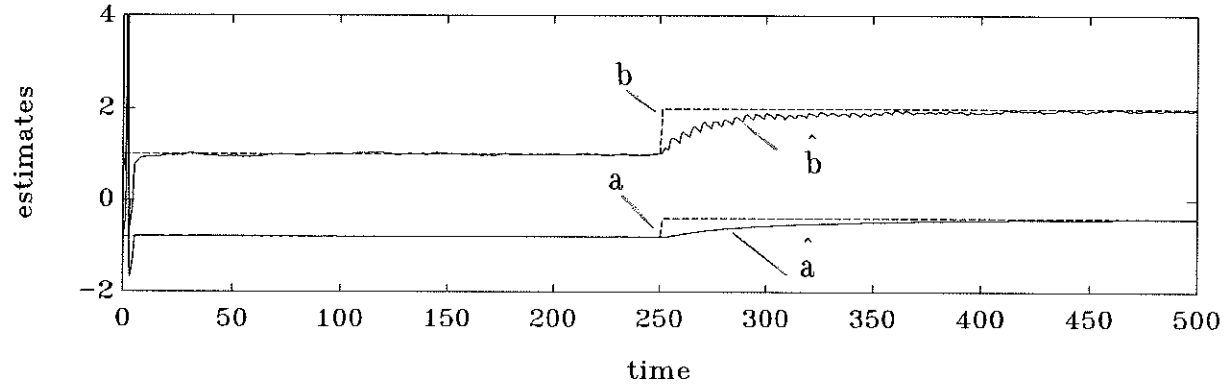

Figure 3.2.4 Parameters (– – –) and MF – estimates (——) for example 3.2.3. The forgetting factor corresponding to $\hat{a}$ is identical to 0.99, the forgetting factor corresponding to $\hat{b}$ is identical to 0.90.

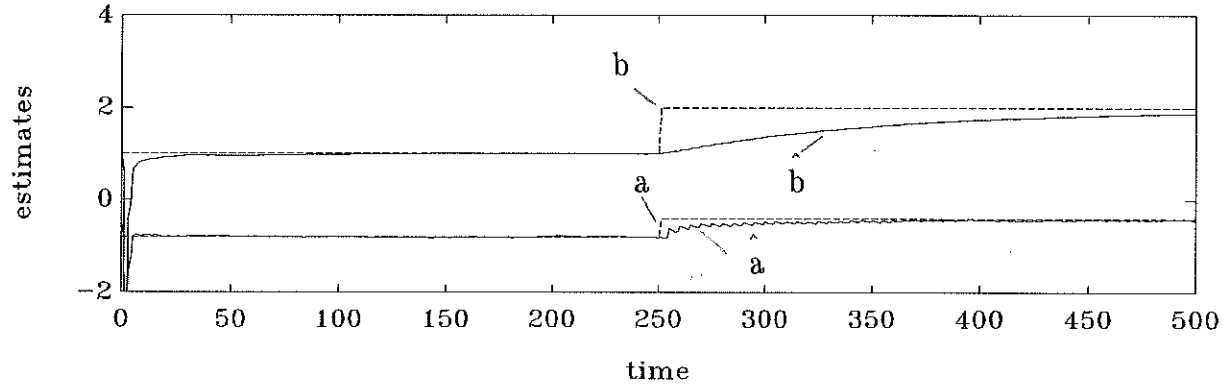

Figure 3.2.5 Parameters (– – –) and MF – estimates (——) for example 3.2.3. The forgetting factor corresponding to $\hat{a}$ is identical to 0.90, the forgetting factor corresponding to $\hat{b}$ is identical to 0.99.

The results in fig. 3.2.4 – 3.2.5 clearly demonstrate that the method has the desired effect. In the case $\lambda_1 < \lambda_2$, $\hat{\theta}_1$ obtains faster adaption than $\hat{\theta}_2$, but $\hat{\theta}_1$ also exhibits larger variation. When $\lambda_1 > \lambda_2$ the opposite effect is observed.

Now turn to the situation from example 3.1.3. Here $\theta_1(t) = a$ is a constant while $\theta_2(t) = b(t)$ has a relatively fast variation. Consequently let

$$\Phi = \begin{bmatrix} 1.0 & 0 \\ 0 & 1 \,/\, \sqrt{0.9} \end{bmatrix}$$

Results from the estimation are shown in figure 3.2.6. Compare with example 3.1.3. It can be concluded that the MF method is able to track b(t) while at the same time the yielding a low–variation estimate for the constant parameter a.

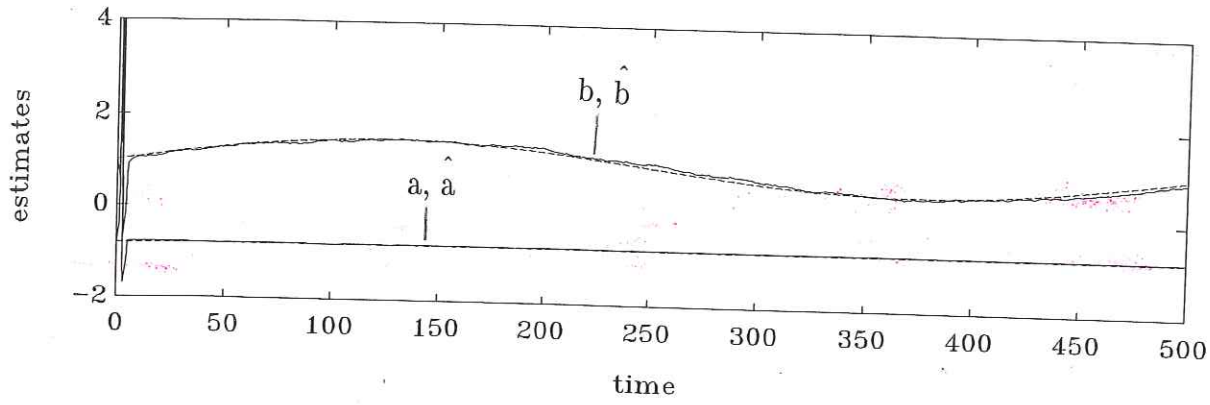

Figure 3.2.6 Parameters (– – –) and MF – estimates (——). The forgetting factor corresponding to $\hat{a}$ is identical to 1.00, the forgetting factor corresponding to $\hat{b}$ is identical to 0.90.

∇

3.3 A general forgetting algorithm.

The list of forgetting methods suggested in the literature is very long, and only a few examples were given in the previous sections. These methods can all be seen as modifications of RLS, typically introducing some kind of data discounting in order to prevent the estimator gain from becoming zero. In this section a general forgetting algorithm containing most existing schemes as special cases is formulated. The algorithm is separated into a measurement update part and a time update part. In fact, the only degree of freedom is the choice of time update for the parameter covariance matrix. This observation highlights the close resemblance between the methods and facilitates their interpretation, analysis and further development.

If a Bayesian point of view is adopted, the parameter vector $\theta \in R^p$ is considered to be a multivariate random variable. The goal is to determine the conditional distribution of θ given the information $I(t)$ available at time t. Let $p(t+i|t)$ denote the probability density function for θ at time $t + i$ ($i = 0, 1$), conditioned on $I(t)$. As pointed out in Peterka (1980), a complete recursive algorithm for estimation of time–varying parameters takes the following form:

Initial condition:

(3.3.1) $\quad p(1|0)$

Measurement update:

(3.3.2) $\quad p(t|t-1) \rightarrow p(t|t)$

Time update:

(3.3.3) $\quad p(t|t) \rightarrow p(t+1|t)$

The measurement update represents the change of the conditional probability density function after a new measurement has been obtained.

The time update represents the development of the distribution between measurements.

In order to formally derive the time update, it is necessary to specify a model describing the time development of the unknown parameter vector $\theta(t)$. This is for example done in the Kalman filter method. In the forgetting methods a different approach is taken. Here it is assumed that $\theta(t)$ is approximately constant within a time horizon $N >> 1$. Hence, a parameter model is not explicitly formulated. Instead the time update is chosen more or less in an ad hoc manner.

The following general forgetting algorithm updating the first and the second moments of the distribution is suggested:

Initial condition:

(3.3.4) $$\hat{\theta}(1|0),\ P(1|0)$$

Measurement update:

(3.3.5) $$\hat{\theta}(t|t) = \hat{\theta}(t|t-1) + P(t|t)\ \varphi(t)\ [\ y(t) - \varphi^T(t)\ \hat{\theta}(t|t-1)\ ]$$

(3.3.6) $$P^{-1}(t|t) = P^{-1}(t|t-1) + \varphi(t)\ \varphi^T(t)$$

Time update:

(3.3.7) $$\hat{\theta}(t+1|t) = \hat{\theta}(t|t)$$

(3.3.8) $$P(t+1|t) = F\ \{P(t|t),\ \gamma(t)\}$$

$F\{\ .\ ,\ .\ \}$ is a function representing the chosen forgetting principle. $\gamma(t)$ is a vector containing the tuning parameters of the method. $\gamma(t)$ may be constant or alternatively a function of measured or updated variables. $\hat{\theta}(t+i|t)$ can be interpreted as an estimate of the conditional mean $E[\ \theta(t+i)|I(t)\ ]$, and $\sigma^2\ P(t+i|t)$ as an estimate of the covariance matrix $E[\ \tilde{\theta}(t+i)\ \tilde{\theta}^T(t+i)|I(t)\ ]$ $(\ i = 0,\ 1\ )$.

The algorithm is identical to the RLS–scheme except for the modification in (3.3.8). This modification reflects the fact that during a sampling period, where no new information becomes available, the parameter uncertainty increases due to the possible drift of the parameters.

The choices of $F\{ . , . \}$ and $\gamma(t)$ can be regarded as design options, and this becomes the central point when selecting or developing a forgetting method. Different applications may call for different solutions, but some general guiding lines are listed below

* $P(t+1|t)$ must be a positive definite and symmetrical (p x p) matrix for all t

* it is reasonable to require $P(t+1|t) \geq P(t|t)$ for all t (the amount of information cannot be increased at the time update)

* avoid loss of tracking ability

* avoid blow up

* the basic asymptotic RLS properties should be retained. (This will be further discussed in chapter 4)

* choose tuning parameters in accordance with the rate of variation of the parameters, the acceptable estimation error variance and the amount of incoming information

As the discussion in the previous sections has shown, most forgetting methods have been developed using an argumentation different from the Bayesian one. This has lead to the use of notations different from the one known from the Kalman filter. However, in most cases it is very simple to rewrite the methods in the form (3.3.4) – (3.3.8). Defining $P(t) = P(t|t)$, $\hat{\theta}(t) = \hat{\theta}(t|t)$ it is also possible to rewrite any method belonging to the general family (3.3.4) – (3.3.8) in a *filter version* by eliminating $\hat{\theta}(t+1|t)$ and $P(t+1|t)$ from the equations. An alternative *predictive version* can be obtained by defining $P(t) = P(t+1|t)$, $\hat{\theta}(t) = \hat{\theta}(t+1|t)$ and eliminating $\hat{\theta}(t|t)$ and $P(t|t)$.

Two examples illustrating this are given below.

Example 3.3.1

Using the technique outlined above it is easy to verify that the EF method corresponds to the choice

$$P(t+1|t) = P(t|t) \;/\; \lambda(t) \tag{3.3.9}$$

The version derived in section 3.1 by criterion minimization is the filter version. The predictive version becomes:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + K(t)\,(\,y(t) - \varphi^T(t)\,\hat{\theta}(t-1)\,) \tag{3.3.10}$$

$$K(t) = \frac{P(t-1)\;\varphi(t)}{1 + \varphi^T(t)\,P(t-1)\;\varphi(t)} = \lambda(t)\;P(t)\;\varphi(t) \tag{3.3.11}$$

$$P^{-1}(t) = \lambda(t)\,[\,P^{-1}(t-1) + \varphi(t)\;\varphi^T(t)\,] \tag{3.3.12}$$

or

$$P(t) = [\,P(t-1) - \frac{P(t-1)\;\varphi(t)\;\varphi^T(t)\;P(t-1)}{1 + \varphi^T(t)\,P(t-1)\;\varphi(t)}\,]\;\frac{1}{\lambda(t)} \tag{3.3.13}$$

Both versions are frequently encountered in the literature (compare e.g. Åström and Wittenmark, 1989 with Saelid and Foss, 1983). They are, of course, different versions of the same method, and in a given situation they will produce identical estimates. Nevertheless, sometimes it is more practical to use one version than the other. For example, the equation (3.2.18) for the constant trace forgetting factor was rather complicated. For the predictive version tr P can be kept constant by choosing

$$\lambda(t) = 1 - \frac{1}{\mathrm{tr}\;P(t-1)}\;\frac{\varphi^T(t)\;P^2(t-1)\;\varphi(t)}{1 + \varphi^T(t)\,P(t-1)\;\varphi(t)} \tag{3.3.14}$$

This expression is much more simple than (3.2.18). While (3.2.18) keeps

tr $P(t|t)$ constant, (3.3.14) acts on tr $P(t+1|t)$.

∇

Example 3.3.2

The DF method suggested by Kulhavy and Karny (1984) corresponds to the following choice of time update in the general forgetting algorithm:

$$P^{-1}(t+1|t) = P^{-1}(t|t) - \kappa(t)\ \varphi(t)\ \varphi^{T}(t) \tag{3.3.15}$$

where

$$\kappa(t) = (\ 1 - \lambda\ )\ [\ 1 + \frac{1}{\varphi^{T}(t)\ P(t|t-1)\ \varphi(t)}\ ] \tag{3.3.16}$$

The predictive version of this method was given in section 3.2.2. To verify those statements define

$$P(t) = P(t+1|t) \tag{3.3.17}$$

and insert (3.3.17) and the measurement update equation (3.3.6) into (3.3.15). This gives

$$P^{-1}(t) = P^{-1}(t-1) + (\ 1 - \kappa(t)\ )\ \varphi(t)\ \varphi^{T}(t) \tag{3.3.18}$$

This equation is identical with (3.2.25), and the statements have been verified.

∇

The following scheme illustrates the unification obtained via the general algorithm.

Method	$F\{P(t\|t), \gamma(t)\}$	$\gamma(t)$
Recursive Least squares	$P(t\|t)$	
Exponential Forgetting	$P(t\|t) \;/\; \lambda(t)$	$\lambda(t)$
Directional Forgetting	$(P^{-1}(t\|t) - \kappa(t)\, \varphi(t)\, \varphi^{T}(t))^{-1}$ *)	$\lambda(t)$
Linear Forgetting	$P(t\|t) + R(t)$	$r_{kk}(t)$
Matrix Forgetting	$\Phi(t)\, P(t\|t)\, \Phi^{T}(t)$	$\lambda_k(t)$

*) $\kappa(t) = (1 - \lambda(t)) \left[1 + \dfrac{1}{\varphi^{T}(t)\;\; P(t|t-1)\;\; \varphi(t)} \right]$

3.4 Summary

This chapter has given a survey of some of the forgetting algorithms which have been suggested in the literature.

When the unknown model parameters drift slowly in time, it becomes necessary to modify the recursive least squares algorithm in order to make tracking possible. The basic principle of the forgetting algorithms which have been discussed here is to regard the parameters as almost constant within a certain time horizon. Old information is gradually discounted, and consequently only a limited amount of information is stored by the estimator.

In the exponential forgetting method a scalar forgetting forgetting factor is used and information is discounted uniformly in all directions. This principle works well when the data provide good excitation. The algorithm turn off which occurs for RLS is avoided, and the tracking ability is maintained. The drawback of the method is that the same forgetting factor has to be used in all directions. In situations where the distribution of the information flow is spatially uniform the update remains applicable even when the excitation level periodically becomes low. This requires use of a time–varying forgetting factor which approaches the value 1 when the excitation becomes poor. However, if the information flow is non–uniformly distributed in the different directions of the parameter space, then the exponential forgetting update is no longer suitable. Instead it can be recommended to apply a method with a forgetting profile which adapts itself to the current information pattern.

A number of methods which have been designed to avoid the problems connected with the exponential forgetting technique have been discussed. For example, one method applies the forgetting factor only in the direction where new information has been received. This gives robustness when the parameter space is only partly excited. Another method assigns different forgetting factors to different parameters. This makes it possible to utilize a priori knowledge that some parameters have a faster variation than others.

It has been shown that the forgetting methods all have very similar structures. A general algorithm has been formulated using time and measurement update

equations as in the Kalman filter. It contains all the discussed forgetting methods as special cases. This result will facilitate the further discussion and analysis in this thesis significantly.

4

Analytical examination of forgetting methods

In the previous chapter it was pointed out that only the choice of time update for the parameter covariance matrix separates the different forgetting algorithms suggested in the literature. A question that now arises is how this choice affects the performance of the estimator. In order to find some formal answers to this question, the present chapter has been devoted to an analytical study of the forgetting methods.

In the literature techniques for proving convergence for algorithms with a gain tending to zero have been developed, see for example Goodwin and Sin (1984). Following similar steps convergence has been established for some specific schemes designed for the time–varying case, see for example Johnstone et al. (1982), Canetti and Espana (1989), Bittanti et al. (1990). In view of the close relationship between the forgetting schemes, it is tempting to try to unify the analytical treatment. Such an approach is taken in this chapter, where the general algorithm from chapter 3 is examined in a deterministic environment. This leads to simple and fairly general results that can be applied for examination of any specific forgetting method, see also Parkum, Poulsen and Holst (1991, 1992).

A set of properties that will be considered mandatory for a good forgetting algorithm is outlined in section 4.1. The importance in connection with tracking of time–varying parameters and adaptive control is stressed.

In section 4.2 the general forgetting algorithm is analyzed. This results in sufficient conditions on the choice of time update for P, ensuring that the fundamental asymptotic results will hold. By analyzing the general algorithm once and for all, a significant simplification is obtained: it becomes possible to establish convergence for an arbitrary forgetting method by simply examining the update equation for P.

In section 4.3 the analytical results are applied to the methods discussed in chapter 3. This leads i.a. to a proof of exponential convergence for the matrix forgetting method.

4.1 Some important error and convergence properties

The goal in this section is to define and discuss some basic asymptotic properties which are of importance for the forgetting algorithms. It seems reasonable to demand good theoretical virtues in a deterministic and time–invariant environment as well as robustness in case of minor perturbations from this ideal situation. The importance of the asymptotic properties in connection with tracking of time–varying parameters and adaptive control is emphasized.

Some definitions related to the convergence question are given in section 4.1.1.

In section 4.1.2 certain elementary properties that hold for RLS are discussed. A standard result in the theory of adaptive control is that these properties imply global stability of a quite general class of adaptive controllers. Consequently it is natural to check that they are retained for modified versions of RLS.

With the RLS algorithm the estimates converge to the true parameter values under suitable excitation conditions. However, the convergence rate is not exponential. Exponential convergence in the constant parameter case is extremely important since it implies tracking ability when the algorithm is used for estimation of slowly drifting parameters. Furthermore, it ensures robustness in case of measurement noise, under estimation of the plant order or nonlinearities. These situations lead to error equations that are perturbations of those equations for which exponential convergence has been established, and it is well known that exponentially stable equations are robust in the presence of perturbations. Exponential convergence is discussed in section 4.1.3.

4.1.1 Some definitions

The following definitions are given in terms of the estimation error

$$\tilde{\theta}(t) = \theta^0 - \hat{\theta}(t+1\,|\,t) \tag{4.1.1}$$

$\tilde{\theta}(t) \in R^p$, and the regression vector $\varphi(t) \in R^p$.

Definition 4.1.1

An estimation algorithm is said to be convergent if

$$\lim_{t \to \infty} \tilde{\theta}(t) = 0 \tag{4.1.2}$$

for any choice of $\hat{\theta}(1|0)$.

Definition 4.1.2

An estimation algorithm is said to be exponentially convergent if there exists constants $C > 0$ and $0 \leq \delta < 1$ such that

$$\| \tilde{\theta}(t) \| \leq C \, \delta^t \, \| \tilde{\theta}(0) \| \tag{4.1.3}$$

for all $t \geq 0$ and for any choice of $\hat{\theta}(1|0)$.

It is important to realize that convergence is conditioned on the properties of the data flow. Typically it is necessary to require that the observations contain a sufficient amount of information. The following definitions are needed:

Definition 4.1.3

The vector sequence $\{ \varphi(t) \}$ is said to be bounded if there exists a constant $M > 0$ such that

$$\| \varphi(t) \| < M \tag{4.1.4}$$

for all $t \geq 0$.

Definition 4.1.4

The vector sequence $\{ \varphi(t) \}$ is said to be persistently exciting if there exists constants $\rho > 0$, $N > 0$ such that

$$\sum_{k=t+1}^{t+N} \varphi(k)\ \varphi^T(k) \geq \rho\ I \tag{4.1.5}$$

for all $t \geq 0$.

For the linear plant (2.3.1) the persistent excitation condition can be converted to one containing only input quantities, or alternatively to one containing only output quantities. See Anderson and Johnson (1982).

4.1.2 Basic convergence properties

From (2.2.1) it can be seen that in the deterministic case ($\sigma^2 = 0$) the off line least squares estimate converges in p steps

$$\hat{\theta}_{LS}(p) = \theta^0 \tag{4.1.6}$$

provided that the vectors $\varphi(1)$, ..., $\varphi(p)$ are linearly independent. This is not true for the recursive version of the method. Here erroneous initial guesses will lead to an estimation error

$$\tilde{\theta}(t) \neq 0 \tag{4.1.7}$$

Consequently, the output prediction

$$\hat{y}(t,\hat{\theta}(t|t-1)) = \varphi^T(t)\ \hat{\theta}(t|t-1) \tag{4.1.8}$$

will be associated with an error

$$\begin{aligned}\epsilon(t) &= \epsilon(t,\hat{\theta}(t|t-1)) \\ &= y(t) - \hat{y}(t,\hat{\theta}(t|t-1)) \\ &= \varphi^T(t)\ \tilde{\theta}(t-1) \qquad (4.1.9)\end{aligned}$$

Assume that the RLS method is applied to data generated by (2.1.6) in the case $\sigma^2 = 0$. Let κ be the condition number of $P(1|0)$. Then the algorithm has the following essential properties:

$$\| \tilde{\theta}(t) \|^2 \leq \kappa \| \tilde{\theta}(0) \|^2 \qquad \text{for all } t \qquad (4.1.10)$$

$$\lim_{t \to \infty} \sum_{k=1}^{t} \frac{(y(k) - \varphi^T(k)\ \hat{\theta}(k|k-1))^2}{1 + \varphi^T(k)\ P(k|k-1)\ \varphi(k)} < \infty \qquad (4.1.11)$$

$$\lim_{t \to \infty} \sum_{k=N}^{t} \| \hat{\theta}(k+1|k) - \hat{\theta}(k-N+1|k-N) \|^2 < \infty \quad \text{for any finite } N. \qquad (4.1.12)$$

(Goodwin and Sin, 1984). (4.1.10) – (4.1.12) hold under very weak assumptions. For example, no assumptions have to be made about boundedness or persistent excitation of the signals.

(4.1.10) implies that the estimation error remains bounded.

From (4.1.11) it follows that

$$\lim_{t \to \infty} \frac{\epsilon^2(t)}{1 + \varphi^T(t)\ P(t|t-1)\ \varphi(t)} = 0 \qquad (4.1.13)$$

i.e. the normalized, squared prediction error converges to zero. (4.1.11) shows that the convergence is faster than $1 / t$. Notice that if the error from the filtering version of the algorithm is defined

(4.1.14) $$\eta(t) = y(t) - \varphi^T(t)\ \hat{\theta}(t|t)$$

then

(4.1.15) $$\eta(t) = \frac{\epsilon(t)}{1 + \varphi^T(t)\ P(t|t-1)\ \varphi(t)}$$

i.e. (4.1.11) implies

(4.1.16) $$\lim_{t \to \infty} \eta(t) = 0$$

(4.1.12) indicates that the *squared* estimator step length converges to zero

(4.1.17) $$\lim_{t \to \infty} \| \hat{\theta}(t+1|t) - \hat{\theta}(t+1-N|t-N) \|^2 = 0$$

and the convergence is faster than $1 / t$. Note that this does not guarantee convergence of the sequence $\{ \hat{\theta}(t+1|t) \}$.

A question of primary interest is how the complete system, in which the estimator is applied, performs. A very important target area of application is the field of adaptive control. In the time–varying case, an adaptive controller can be constructed by combining a forgetting algorithm with a control law. An interesting question is: how does the choice of time update for P affect the performance of the total closed loop system? It is reasonable to require that certain properties like stability and asymptotically perfect tracking of the reference signal are guaranteed. Therefore, to illustrate the practical importance of (4.1.10) – (4.1.12), consider the application of the recursive estimation algorithm as part of the adaptive controller described in chapter 2. Let the reference signal be denoted $y^*(t)$. It can be proved (see e.g. Goodwin and Sin, 1984) for the closed loop system that

$\{ u(t) \}$, $\{ y(t) \}$ are bounded for all t

and

$$\| \, y(t) - y^{*}(t) \, \| \quad \text{will converge to zero}$$

provided that the estimates are generated by an algorithm having the properties (4.1.10) – (4.1.12). Hence, (4.1.10) – (4.1.12) ensure global closed loop stability and asymptotically perfect tracking for an adaptive controller consisting of a simple one–step ahead control law combined with a recursive forgetting algorithm. The result can be generalized in several ways and the assumptions can be weakened. For example, it is sufficient to know an upper limit for the polynomial orders n_a, n_b. The time delay may be larger than one. Very similar results apply to algorithms based on the direct and the indirect approach. Extension to multiple input–multiple output models is straight forward. It is also possible to employ a more advanced control principle, for example closed loop pole assignment. See Goodwin and Sin (1984) for a comprehensive discussion of the possible extensions.

The forgetting methods are designed to handle a situation where the parameters may drift in time. However, in the simple time–invariant special case they should retain the elementary properties discussed above. There is no reason to expect that an algorithm with poor performance in the most simple case will behave satisfactory in more difficult situations. On the other hand, (4.1.10) – (4.1.12) are not sufficient requirements in the time–varying case. For example, RLS has these properties, but is unable to cope with drifting parameters.

4.1.3 Exponential convergence

Consider the RLS method applied to the data generated by (2.1.6) with $\sigma^2 = 0$. Let $\alpha_{min}(A)$ denote the smallest eigenvalue for the matrix A. If the excitation condition

$$\alpha_{min} \Big(\sum_{k=1}^{t} \varphi(k) \, \varphi^{T}(k) \Big) \to \infty \qquad \text{for} \quad t \to \infty \tag{4.1.18}$$

is fulfilled, then RLS is convergent

$$\hat{\theta}_{RLS}(t) \rightarrow \theta^0 \qquad \text{for} \quad t \rightarrow \infty \tag{4.1.19}$$

However, the convergence rate is not exponential. Generally it turns out that exponential convergence in the constant parameter case can be regarded as necessary in order to ensure good performance in the time–varying case – see Anderson and Johnson (1982). This will be illustrated by an example.

Example 4.1.1

Consider the scalar example

$$\varphi(t) = 1 \quad \text{for all } t$$

$$P^{-1}(0) = P_0^{-1}$$

$$\tilde{\theta}(0) = \tilde{\theta}_0$$

The goal is to compare the performance of the RLS method with that of EF with a forgetting factor $\lambda < 1$. Notice that $\{ \varphi(t) \}$ is persistently exciting and bounded.

In order to assess the tracking ability of the methods introduce

$$h(t,\ t\text{–}N) = \frac{\| \tilde{\theta}(t\text{–}N) \| - \| \tilde{\theta}(t) \|}{\| \tilde{\theta}(t\text{–}N) \|}$$

$$= 1 - \frac{\| \tilde{\theta}(t) \|}{\| \tilde{\theta}(t\text{–}N) \|} \tag{4.1.20}$$

$h(t,\ t\text{–}N)$ is a measure of the relative improvement of the estimates from $t - N$ to t. If $h(t,\ t\text{–}N)$ converges to 0 for all fixed values of N, the algorithm loses its ability to track parameter variations.

For RLS the information matrix becomes

$$P^{-1}(t) = P^{-1}(t-1) + \varphi(t)\ \varphi^{T}(t)$$

$$= P_0^{-1} + t$$

The estimation error evolves according to

$$\tilde{\theta}(t) = \left(I - \frac{P(t-1)\ \varphi(t)\ \varphi^{T}(t)}{1 + \varphi^{T}(t)\ P(t-1)\ \varphi(t)} \right) \tilde{\theta}(t-1)$$

Simple calculations show that

$$\tilde{\theta}(t) = \frac{P_0^{-1}}{P_0^{-1} + t}\ \tilde{\theta}(0)$$

Hence, $\tilde{\theta}(t)$ converges to zero approximately as $1 / t$. $h(t, t-N)$ becomes

$$h(t, t-N) = \frac{N}{P_0^{-1} + t}$$

and

$$\lim_{t \to \infty} h(t, t-N) = 0$$

for any fixed N. This illustrates the fact that the RLS gain eventually dies out, and the tracking ability is gradually lost.

Now turn to the EF method . Here P(t) is updated in the following way :

$$P^{-1}(t) = \lambda\ P^{-1}(t-1) + \varphi(t)\ \varphi^{T}(t)$$

$$= \lambda^{t}\ P_0^{-1} + \frac{1 - \lambda^{t}}{1 - \lambda}$$

The estimation error becomes

$$\tilde{\theta}(t) = \left(I - \frac{P(t-1)\,\varphi(t)\,\varphi^T(t)}{\lambda + \varphi^T(t)\,P(t-1)\,\varphi(t)} \right) \tilde{\theta}(t-1)$$

$$= \lambda^t \frac{P_0^{-1}\,(1-\lambda)}{\lambda^t\,P_0^{-1}\,(1-\lambda) + (1-\lambda^t)}\,\tilde{\theta}(0)$$

Clearly

$$\tilde{\theta}(t) \le \lambda^t \frac{P_0^{-1}\,(1-\lambda)}{1-\lambda^t}\,\tilde{\theta}(0)$$

$$\le P_0^{-1}\,\lambda^t\,\tilde{\theta}(0)$$

and the algorithm is exponentially convergent. It can readily be seen that this virtue guarantees permanent tracking ability:

$$h(t,t-N) = 1 - \lambda^N \frac{\lambda^{t-N}\,P_0^{-1}\,(1-\lambda) + (1-\lambda^{t-N})}{\lambda^t\,P_0^{-1}\,(1-\lambda) + (1-\lambda^t)}$$

$$\lim_{t \to \infty} h(t,\,t-N) = 1 - \lambda^N > 0$$

∇

The importance of exponential convergence in connection with time–varying plants can be further explained as follows:

Let

(4.1.21) $$y(t) = \varphi^T(t)\,\theta^0(t)$$

(4.1.22) $$\tilde{\theta}(t) = \theta^0(t) - \hat{\theta}(t+1\,|\,t)$$

For a general forgetting algorithm the estimation error obeys the relation

$$\tilde{\theta}(t) = F(t)\ \tilde{\theta}(t-1) + F(t)\ (\ \theta^0(t) - \theta^0(t-1)\ ) \tag{4.1.23}$$

$$F(t) = \left(I - \frac{P(t\,|\,t-1)\ \varphi(t)\ \varphi^T(t)}{1 + \varphi^T(t)\ P(t|t-1)\ \varphi(t)} \right) \tag{4.1.24}$$

If the homogeneous part of (4.1.23) is exponentially stable then

$$\| \theta^0(t) - \theta^0(t-1) \| \leq K \tag{4.1.25}$$

ensures that

$$\lim_{t \to \infty} \sup \| \tilde{\theta}(t) \| \leq C\ K \tag{4.1.26}$$

for some $C > 0$. Hence, exponential stability in the time–invariant case implies that bounded parameter variations lead to bounded estimation errors in the time–varying case. Similar results will hold when the perturbation is caused by for example under estimation of the plant order, nonlinearities or measurement noise.

Now consider again the application of the estimator as part of the adaptive controller described in chapter 2. Robustness considerations analogous to the ones mentioned above for the identification problem apply to the control problem, i. e., a desirable property is that the difference between the output and the reference dies out exponentially fast. Since

$$y(t) - y^*(t) = \varphi^T(t)\ \tilde{\theta}(t) \tag{4.1.27}$$

this property can be established if $\{\ \varphi\ \}$ is bounded and if $\tilde{\theta}$ converges exponentially fast to zero. With proper choice of estimator the crucial question becomes whether or not $\{\ \varphi\ \}$ is persistently exciting when the plant is operating in closed loop. In Johnstone and Anderson (1982) it is shown that this can be obtained by choosing a reference sequence $\{\ y^*\ \}$ with a

sufficiently rich variation.

Motivated by the discussion given in this section exponential convergence in the case of persistently exciting and bounded regressors may be added to the list of desirable properties for forgetting algorithms. The permanent ability to modify the estimates in accordance with the latest information is essential. It guarantees robustness not only in the case of drifting parameters, but also when the error equation is affected by other types of perturbations.

4.2 Analysis of the general forgetting algorithm

In this section the general forgetting algorithm formulated in chapter 3 is considered. Analytical results are presented providing sufficient conditions ensuring that the algorithm has the properties discussed above. This leads to a simplification of the convergence question. For any specific algorithm which is a special case of the general one it is sufficient to verify that these conditions are fulfilled, then the convergence properties have been established. In many cases, as the discussion in section 4.3 and in chapter 5 will show, this can be checked without great difficulties.

Some of the steps in the proofs of the theorems presented in this section can be found elsewhere in the literature, but they are included here for the sake of completeness. Furthermore, the results derived here are much more general due to the weak assumptions imposed on the structure of the estimation scheme.

In section 4.2.1 it is proved that the elementary RLS properties are retained provided that the P–matrix is bounded from above.

If it is required that the P–matrix is bounded from above and from below, and if the data are sufficiently informative, exponential convergence can be established. This result is given in section 4.2.2.

4.2.1 Basic convergence properties

The aim is to show that the elementary properties of the recursive least squares method are retained for a general tracking algorithm provided that the parameter covariance matrix is bounded from above. Hence, the general forgetting algorithm described in chapter 3 is considered. Let the choice of time update for P ensure

$$0 < P(t+1|t) \leq \alpha_{max} I \qquad \text{for all } t \tag{4.2.1}$$

$P(t+1|t)$ is assumed to be a symmetrical $(p \times p)$–matrix for all t and the time update is assumed to guarantee $P(t+1|t) \geq P(t|t)$ for all t.

In (4.2.1) I is the identity matrix and α_{max} is a positive constant, $0 < \alpha_{max} < \infty$. The upper bound is a guarantee against blow up. It will be assumed that measurements are available at $t = 1, 2, \ldots$ and that the algorithm is initialized as follows:

$$\hat{\theta}(0|0) = \hat{\theta}(1|0) = 0 \tag{4.2.2}$$

$$P(0|0) = P(1|0) = a\, I, \qquad 0 < a \leq \alpha_{max} \tag{4.2.3}$$

Note the definition $P(0|0) \equiv P(1|0)$, i.e. no time update is performed between $t = 0$ and $t = 1$. This is a technical detail, which makes the notation a little easier in the following. It is also intuitively very reasonable to assume that the a priori knowledge connected with $\theta(0)$ and $\theta(1)$ is the same, since measurement no. 1 is available at $t = 1$.

To simplify the notation introduce

$$\hat{\theta}(t) = \hat{\theta}(t|t) = \hat{\theta}(t+1|t)) \tag{4.2.4}$$

$$\tilde{\theta}(t) = \theta^0 - \hat{\theta}(t+1|t) \tag{4.2.5}$$

$$S(t) = P(t|t) \tag{4.2.6}$$

$$P(t) = P(t+1|t) \tag{4.2.7}$$

$$K(t) = P(t|t)\ \varphi(t) = \frac{P(t|t-1)\ \varphi(t)}{1 + \varphi^T(t)\ P(t|t-1)\ \varphi(t)} \tag{4.2.8}$$

$$\epsilon(t) = y(t) - \varphi^T(t)\ \hat{\theta}(t|t-1) \tag{4.2.9}$$

Now the algorithm can be rewritten as

$$\hat{\theta}(t) = \hat{\theta}(t-1) + K(t)\ \epsilon(t) \tag{4.2.10}$$

$$S^{-1}(t) = P^{-1}(t-1) + \varphi(t)\ \varphi^T(t) \tag{4.2.11}$$

or

$$S(t) = (I - K(t)\ \varphi^T(t))\ P(t-1) \tag{4.2.12}$$

where

$$P(t) = F\ \{\ S(t),\ \gamma(t)\ \} \geq S(t) \tag{4.2.13}$$

The basic error properties of the algorithm are established in theorem 4.2.1

Theorem 4.2.1

Assume that the algorithm (4.2.10) – (4.2.13) is applied for estimation of the parameters of the system (2.1.6) with $\sigma^2 = 0$ and θ^0 constant. Then provided that (4.2.1) is fulfilled the following is true

$$\| \tilde{\theta}(t) \|^2 \leq \frac{\alpha_{max}}{a} \| \hat{\theta}(0) \|^2 \qquad \text{for all} \quad t \geq 0 \tag{4.2.14}$$

$$\lim_{t \to \infty} \sum_{k=1}^{t} \frac{\epsilon^2(k)}{1 + \varphi^T(k)\ P(k-1)\ \varphi(k)} < \infty \tag{4.2.15}$$

$$\lim_{t \to \infty} \sum_{k=N}^{t} \| \hat{\theta}(k) - \hat{\theta}(k-N) \|^2 < \infty \tag{4.2.16}$$

for any finite N

Proof

First notice that

$$\tilde{\theta}(t) = (I - K(t)\ \varphi^T(t))\ \tilde{\theta}(t-1) \tag{4.2.17}$$

Combination of (4.2.12) and (4.2.17) gives

$$\tilde{\theta}(t) = S(t)\ P^{-1}(t-1)\ \tilde{\theta}(t-1) \tag{4.2.18}$$

or

$$S^{-1}(t)\ \tilde{\theta}(t) = P^{-1}(t-1)\ \tilde{\theta}(t-1) \tag{4.2.19}$$

Now introduce the function:

$$V(t) = \tilde{\theta}^{T}(t)\ S^{-1}(t)\ \tilde{\theta}(t) \tag{4.2.20}$$

Then

$$\begin{aligned} V(t) - V(t-1) &= \tilde{\theta}^{T}(t)\ S^{-1}(t)\ \tilde{\theta}(t) - \tilde{\theta}^{T}(t-1)\ S^{-1}(t-1)\ \tilde{\theta}(t-1) \\ &= \tilde{\theta}^{T}(t)\ P^{-1}(t-1)\ \tilde{\theta}(t-1) - \tilde{\theta}^{T}(t-1)\ S^{-1}(t-1)\ \tilde{\theta}(t-1) \\ &= (\ \tilde{\theta}(t-1) - K(t)\epsilon(t)\ )^{T}\ P^{-1}(t-1)\ \tilde{\theta}(t-1) \\ &\qquad - \tilde{\theta}^{T}(t-1)\ S^{-1}(t-1)\ \tilde{\theta}(t-1) \\ &= -\epsilon(t)\ K^{T}(t)\ P^{-1}(t-1)\ \tilde{\theta}(t-1) \\ &\qquad - \tilde{\theta}^{T}(t-1)\ (\ S^{-1}(t-1) - P^{-1}(t-1))\ \tilde{\theta}(t-1) \\ &= \frac{-\epsilon(t)\ \varphi^{T}(t)\ \tilde{\theta}(t-1)}{1 + \varphi^{T}(t)\ P(t-1)\ \varphi(t)} \\ &\qquad - \tilde{\theta}^{T}(t-1)\ (\ S^{-1}(t-1) - P^{-1}(t-1)\ )\ \tilde{\theta}(t-1) \\ &= \frac{-\epsilon^{2}(t)}{1 + \varphi^{T}(t)\ P(t-1)\ \varphi(t)} \\ &\qquad - \tilde{\theta}^{T}(t-1)\ (\ S^{-1}(t-1) - P^{-1}(t-1)\ )\ \tilde{\theta}(t-1) \end{aligned} \tag{4.2.21}$$

From (4.2.21) it follows that

$$V(t) - V(t-1) \leq 0 \quad \text{for all } t \tag{4.2.22}$$

i.e.

$$\tilde{\theta}^{T}(t)\, S^{-1}(t)\, \tilde{\theta}(t) \leq \tilde{\theta}^{T}(t-1)\, S^{-1}(t-1)\, \tilde{\theta}(t-1)$$

$$\leq \ldots$$

$$\leq \tilde{\theta}^{T}(0)\, S^{-1}(0)\, \tilde{\theta}(0) \qquad (4.2.23)$$

$V(t)$ is thus non–negative and non–increasing.

Now notice that

$$\| \tilde{\theta}(t) \|^2 \leq \alpha_{max}\, \tilde{\theta}^{T}(t)\, S^{-1}(t)\, \tilde{\theta}(t)$$

$$\leq \alpha_{max}\, \tilde{\theta}^{T}(0)\, S^{-1}(0)\, \tilde{\theta}(0)$$

$$= \frac{\alpha_{max}}{a} \| \tilde{\theta}(0) \|^2 \qquad (4.2.24)$$

This proves the first part of the theorem.

From (4.2.21) it can be seen that

$$V(t) = V(0) - \sum_{k=1}^{t} \frac{\epsilon^2(k)}{1 + \varphi^{T}(k)\, P(k-1)\, \varphi(k)}$$

$$- \sum_{k=1}^{t} \tilde{\theta}^{T}(k-1) \left[S^{-1}(k-1) - P^{-1}(k-1) \right] \tilde{\theta}(k-1) \qquad (4.2.25)$$

Clearly, both of the sums involved in this equation are non–negative. Since $V(t)$ is non–negative and $V(0) = \frac{\| \tilde{\theta}(0) \|^2}{a}$ it follows that

$$\lim_{t \to \infty} \sum_{k=1}^{t} \frac{\epsilon^2(k)}{1 + \varphi^T(k)\, P(k-1)\, \varphi(k)} = V(0) - \lim_{t \to \infty} V(t)$$

$$- \lim_{t \to \infty} \sum_{k=1}^{t} \tilde{\theta}^T(k-1) \left[S^{-1}(k-1) - P^{-1}(k-1) \right] \tilde{\theta}(k-1)$$

(4.2.26)
$$\leq \frac{\| \tilde{\theta}(0) \|^2}{a}$$

which proves part two.

Now notice that

(4.2.27)
$$\frac{\epsilon^2(k)}{1 + \varphi^T(k)\, P(k-1)\, \varphi(k)} = \frac{(1 + \varphi^T(k)\, P(k-1)\, \varphi(k))\, \epsilon^2(k)}{(1 + \varphi^T(k)\, P(k-1)\, \varphi(k))^2}$$

From (4.2.26) – (4.2.27) it follows that

(4.2.28)
$$\lim_{t \to \infty} \sum_{k=1}^{t} \frac{(1 + \varphi^T(k)\, P(k-1)\, \varphi(k))\, \epsilon^2(k)}{(1 + \varphi^T(k)\, P(k-1)\, \varphi(k))^2} \leq \frac{\| \tilde{\theta}(0) \|^2}{a}$$

hence

(4.2.29)
$$\lim_{t \to \infty} \sum_{k=1}^{t} \frac{\varphi^T(k)\, P(k-1)\, \varphi(k)\, \epsilon^2(k)}{(1 + \varphi^T(k)\, P(k-1)\, \varphi(k))^2} \leq \frac{\| \tilde{\theta}(0) \|^2}{a}$$

Furthermore

(4.2.30)
$$\hat{\theta}(k) - \hat{\theta}(k-1) = \frac{P(k-1)\, \varphi(k)}{1 + \varphi^T(k)\, P(k-1)\, \varphi(k)}\, \epsilon(k)$$

i.e

$$\| \hat{\theta}(k) - \hat{\theta}(k-1) \|^2 = \frac{\varphi^T(k)\ P^2(k-1)\ \varphi(k)}{(1 + \varphi^T(k)\ P(k-1)\ \varphi(k))^2}\ \epsilon^2(k)$$

$$(4.2.31) \qquad \leq \alpha_{max} \frac{\varphi^T(k)\ P(k-1)\ \varphi(k)}{(1 + \varphi^T(k)\ P(k-1)\ \varphi(k))^2}\ \epsilon^2(k)$$

From (4.2.29) and (4.2.31) it follows that

$$(4.2.32) \qquad \lim_{t \to \infty} \sum_{k=1}^{t} \| \hat{\theta}(k) - \hat{\theta}(k-1) \|^2 \leq \alpha_{max} \frac{\| \tilde{\theta}(0) \|^2}{a}$$

Furthermore

$$\left[\hat{\theta}(k) - \hat{\theta}(k-N) \right] = \left[\hat{\theta}(k) - \hat{\theta}(k-1) + \hat{\theta}(k-1) - \hat{\theta}(k-2) \right.$$

$$(4.2.33) \qquad \left. + \ldots + \hat{\theta}(k-N+1) - \hat{\theta}(k-N) \right]$$

The Schwarz inequality

$$(4.2.34) \qquad \| \sum_{i=1}^{N} a_i \|^2 \leq N \sum_{i=1}^{N} \| a_i \|^2$$

gives

$$\| \hat{\theta}(k) - \hat{\theta}(k-N) \|^2 \leq N \left[\| \hat{\theta}(k) - \hat{\theta}(k-1) \|^2 + \ldots \right.$$

$$(4.2.35) \qquad \left. + \| \hat{\theta}(k-N+1) - \hat{\theta}(k-N) \|^2 \right]$$

Since N is finite, combination of (4.2.32) and (4.2.35) gives

$$(4.2.36) \qquad \lim_{t \to \infty} \sum_{k=N}^{t} \| \hat{\theta}(k) - \hat{\theta}(k-N) \|^2 \leq N^2 \alpha_{max} \frac{\| \tilde{\theta}(0) \|^2}{a}$$

i.e. the third and final part of the theorem has been proved.

□

4.2.2 Exponential convergence

Now assume that P is bounded not only from above but also from below:

$$\alpha_{min}\, I \le P(t+1|t) \le \alpha_{max}\, I \qquad \text{for all } t \tag{4.2.37}$$

In (4.2.37) the constants fulfill $0 < \alpha_{min} < \alpha_{max} < \infty$. The lower bound ensures that the algorithm gain does not converge to zero. All other assumptions are as above and the same notational convention is adopted. Then the following result holds:

Theorem 4.2.2

Assume that the algorithm (4.2.10) – (4.2.13) is applied for estimation of the parameters of the system (2.1.6) with $\sigma^2 = 0$ and θ^0 constant. Let P be bounded as in (4.2.37). If the sequence $\{ \varphi(t) \}$ is persistently exciting and bounded, then the estimation algorithm is exponentially convergent.

Proof

From (4.2.21) it follows that:

$$V(t) - V(t-1) \le - \frac{\epsilon^2(t)}{1 + \varphi^T(t)\, P(t-1)\, \varphi(t)} \tag{4.2.38}$$

where $V(t) = \tilde{\theta}^T(t)\, S^{-1}(t)\, \tilde{\theta}(t)$ and $\epsilon(t) = \varphi^T(t)\, \tilde{\theta}(t-1)$. Now repeat (4.2.38) for $t = j + N, \ldots, t = j + 1$. Addition of the equations gives

$$V(j+N) - V(j) \leq - \sum_{k=j+1}^{j+N} \frac{\epsilon^2(k)}{1 + \varphi^T(k)\ P(k-1)\ \varphi(k)}$$

$$\leq \frac{-1}{1 + M^2\ \alpha_{max}} \sum_{k=j+1}^{j+N} \epsilon^2(k) \tag{4.2.39}$$

Since

$$\tilde{\theta}(t) - \tilde{\theta}(t-1) = - K(t)\ \epsilon(t) \tag{4.2.40}$$

it follows that

$$\tilde{\theta}(k-1) - \tilde{\theta}(j) = - \sum_{i=j+1}^{k-1} K(i)\ \epsilon(i) \tag{4.2.41}$$

This gives

$$\| \varphi^T(k)\ \tilde{\theta}(j) \| = \| \varphi^T(k)\ \tilde{\theta}(k-1) - \varphi^T(k) \left[\tilde{\theta}(k-1) - \tilde{\theta}(j) \right] \|$$

$$= \| \epsilon(k) + \varphi^T(k) \sum_{i=j+1}^{k-1} K(i)\ \epsilon(i) \|$$

$$\leq \sum_{i=j+1}^{k} c(i) \| \epsilon(i) \| \tag{4.2.42}$$

where the coefficients $c(i)$ are restricted by

$$0 \leq c(i) \leq c_{max} = \max (1,\ M^2\ \alpha_{max}) \tag{4.2.43}$$

(Here the relation $\varphi^T(k)\ K(i) = \varphi^T(k)\ S(i)\ \varphi(i) \leq M^2\ \alpha_{max}$ has been used).

Now the inequality (4.2.34) gives

$$\| \varphi^T(k)\ \tilde{\theta}(j) \|^2 \le (k-j)\ c^2_{max} \sum_{i=j+1}^{k} \epsilon^2(i) \tag{4.2.44}$$

Furthermore

$$\sum_{k=j+1}^{j+N} | \varphi^T(k)\ \tilde{\theta}(j) |^2 \le \sum_{k=j+1}^{j+N} \left[(k-j)\ c^2_{max} \sum_{i=j+1}^{k} \epsilon^2(i) \right]$$

$$\le \sum_{k=j+1}^{j+N} \left[(j+N-j)\ c^2_{max} \sum_{i=j+1}^{j+N} \epsilon^2(i) \right]$$

$$= N^2\ c^2_{max} \sum_{i=j+1}^{j+N} \epsilon^2(i) \tag{4.2.45}$$

Alternatively this can be written

$$-\sum_{i=j+1}^{j+N} \epsilon^2(i) \le -\frac{1}{N^2\ c^2_{max}} \sum_{k=j+1}^{j+N} \| \varphi^T(k)\ \tilde{\theta}(j) \|^2 \tag{4.2.46}$$

From (4.2.39) and (4.2.46) it follows that

$$\tilde{\theta}^T(j+N)\ S^{-1}(j+N)\ \tilde{\theta}(j+N) - \tilde{\theta}^T(j)\ S^{-1}(j)\ \tilde{\theta}(j)$$

$$\le \frac{-1}{1 + M^2\ \alpha_{max}} \frac{1}{N^2\ c^2_{max}} \sum_{k=j+1}^{j+N} \| \varphi^T(k)\ \tilde{\theta}(j) \|^2 \tag{4.2.47}$$

The regressors are assumed to be persistently exciting, i.e. they fulfill the inequality

$$\sum_{k=j+1}^{j+N} \varphi(k)\ \varphi^{T}(k) \geq \rho\ I \tag{4.2.48}$$

Hence

$$\sum_{k=j+1}^{j+N} \| \varphi^{T}(k)\ \tilde{\theta}(j) \|^{2} \geq \rho \| \tilde{\theta}(j) \|^{2} \tag{4.2.49}$$

Noting that $S^{-1}(j) = P^{-1}(j-1) + \varphi(j)\ \varphi^{T}(j)$, where $P^{-1}(j-1) \leq \frac{1}{\alpha_{min}} I$ and $\| \varphi(j) \| \leq M$ the following inequality is obtained

$$S^{-1}(j) \leq \kappa\ I \tag{4.2.50}$$

where

$$\kappa = \frac{1}{\alpha_{min}} + M^{2} \tag{4.2.51}$$

i.e.

$$\tilde{\theta}^{T}(j)\ S^{-1}(j)\ \tilde{\theta}(j) \leq \kappa \| \tilde{\theta}(j) \|^{2} \tag{4.2.52}$$

Insertion of (4.2.49) and (4.2.52) into (4.2.47) gives

$$\tilde{\theta}^{T}(j+N)\ S^{-1}(j+N)\ \tilde{\theta}(j+N) - \tilde{\theta}^{T}(j)\ S^{-1}(j)\ \tilde{\theta}(j)$$

$$\leq \frac{-1}{1 + M^{2}\ \alpha_{max}} \frac{1}{N^{2}\ c_{max}^{2}} \frac{\rho}{\kappa} \tilde{\theta}^{T}(j)\ S^{-1}(j)\ \tilde{\theta}(j) \tag{4.2.53}$$

Consequently

$$\tilde{\theta}^{T}(j+N)\ S^{-1}(j+N)\ \tilde{\theta}(j+N) \leq \beta\ \tilde{\theta}^{T}(j)\ S^{-1}(j)\ \tilde{\theta}(j) \tag{4.2.54}$$

where

(4.2.55) $$\beta = 1 - \frac{1}{1 + M^2 \alpha_{max}} \frac{1}{N^2 c^2_{max}} \frac{\rho}{\kappa} < 1$$

Since $S^{-1}(j+N) \geq \frac{1}{\alpha_{max}} I$ it has been proved that

(4.2.56) $$\| \tilde{\theta}(t) \| \to 0$$

at least exponentially fast

□

4.3 Application of the analytical results

The results of section 4.2 provide a simple approach for establishing the basic error and convergence properties for an arbitrary forgetting scheme. The elementary RLS properties (4.2.14) – (4.2.16) are retained if the following is true:

1) the algorithm is a special case of the general forgetting method described in chapter 3

2) $P(t+1|t) > P(t|t)$

3) the P–matrix is bounded from above

Furthermore, if in addition

4) the P–matrix is bounded from below

then the algorithm is also exponentially convergent, provided that $\{ \varphi(t) \}$ is persistently exciting and bounded.

This means that answers to questions concerning the asymptotic behaviour of the algorithms can be found by simply examining the update equation for the P–matrix. Obviously it is much simpler to check 1) – 4) than to go through proofs like the ones presented above for each new algorithm appearing.

In the following discussion the algorithms will be examined in two situations:

a) No assumptions imposed on $\{ \varphi(t) \}$.

b) $\{ \varphi(t) \}$ persistently exciting and bounded.

Results from the existing literature are mentioned. Furthermore, a new result establishing exponential convergence of the matrix forgetting method is presented.

The results from above will also be useful in chapter 5, where a new method with selective forgetting is studied.

Exponential forgetting (EF), Matrix forgetting (MF)

a) As was discussed in chapter 3 P may grow unbounded in the general case. Hence, the elementary properties (4.2.14) – (4.2.16) cannot be established.

b) In this case the convergence properties of EF have been examined for example in Johnstone et al. (1982). It was shown therein that the EF method with a constant forgetting factor λ, $0 < \lambda < 1$ is exponentially convergent.

If a time–varying forgetting factor is applied, the properties of the algorithm may change. For example, if the forgetting factor suggested in Fortescue et al. (1981) is applied, the algorithm will still be convergent, but the convergence rate is no longer exponential. The problem occurs because $\lambda(t)$ can become arbitrarily close to one. It can be resolved by bounding the forgetting factor:

(4.3.1) $$0 < \lambda_{min} \leq \lambda(t) \leq \lambda_{max} < 1$$

(Cannetti and Espana, 1989).

In order to illustrate the power of the approach suggested above consider now the more general matrix forgetting method (section 3.2.3) with

(4.3.2) $$\Phi(t) = \text{diag} \left(1 \; / \; \sqrt{\lambda_k(t)} \right)$$

This algorithm contains EF as a special case. Since the forgetting factors are allowed to be time–varying, the following result will also cover for example Fortescues method and the constant trace algorithm.

Theorem 4.3.1

Consider the MF method with $\Phi(t) = \text{diag}\ (\ 1\ /\ \sqrt{\lambda_k(t)}\ )$ and

$$0 < \lambda_{min} \leq \lambda_k(t) \leq \lambda_{max} < 1 \qquad k = 1, \ldots, p \tag{4.3.3}$$

Let the initial conditions be

$$\hat{\theta}(1|0) = 0$$

$$P(1|0) = a\ I, \quad 0 < a \leq \alpha_{max}$$

Assume that this algorithm is applied for estimation of the parameters in (2.1.6) with $\sigma^2 = 0$ and θ^0 constant. If $\{\ \varphi(t)\ \}$ is persistently exciting and bounded, then the algorithm will have the properties (4.2.14) – (4.2.16) and it is exponentially convergent.

Proof

Since the algorithm is a special case of the general one with $P(t+1|t) \geq P(t|t)$, the result follows from section 4.2 if P is bounded from above and from below. Hence, the task is to prove

$$\alpha_{min}\ I \leq P(t+1|t) \leq \alpha_{max}\ I \qquad \text{for all} \quad t \geq 0 \tag{4.3.4}$$

for some constants

$$0 < \alpha_{min} < \alpha_{max} < \infty \tag{4.3.5}$$

The predictive formulation is used, i.e. $P(t+1|t) = P(t)$. The update is

$$P^{-1}(t) = (\ \Phi^{-1}(t)\ )^T \left[\ P^{-1}(t-1) + \varphi(t)\ \varphi^T(t)\ \right] \Phi^{-1}(t) \tag{4.3.6}$$

Upper bound for P:

$$\left(\frac{1}{\lambda_{min}} \right) P^{-1}(t) \geq P^{-1}(t-1)$$

(4.3.7)

$$\left(\frac{1}{\lambda_{min}} \right)^N P^{-1}(t) \geq P^{-1}(t-N)$$

Addition of the equations gives

$$P^{-1}(t) \sum_{k=1}^{N} \left(\frac{1}{\lambda_{min}} \right)^k \geq \sum_{k=1}^{N} P^{-1}(t-k) \tag{4.3.8}$$

or

$$P^{-1}(t) \geq (1 - \lambda_{min}) \frac{\lambda_{min}^N}{1 - \lambda_{min}^N} \sum_{k=1}^{N} P^{-1}(t-k)$$

$$\geq (1 - \lambda_{min}) \frac{\lambda_{min}^N}{1 - \lambda_{min}^N} \lambda_{min} \sum_{k=1}^{N} \varphi(t-k)\ \varphi^T(t-k)$$

$$\geq (1 - \lambda_{min}) \frac{\lambda_{min}^N}{1 - \lambda_{min}^N} \lambda_{min}\ \rho\ I$$

$$\equiv \frac{1}{\alpha_{max}} I \tag{4.3.9}$$

Lower bound for P:

$$P^{-1}(t) \leq \lambda_{max} \left[P^{-1}(t-1) + \varphi(t)\ \varphi^T(t) \right]$$

$$\leq \lambda_{max} \left[P^{-1}(t-1) + M^2 I \right]$$

$$\leq \lambda_{max} \left[\lambda_{max} P^{-1}(t-2) + (\lambda_{max} + 1) M^2 I \right]$$

$$\leq \lambda_{max} \left[\lambda_{max}^{t-1} P^{-1}(0) + \left(\sum_{k=0}^{t-1} \lambda_{max}^{k} \right) M^2 I \right]$$

$$\leq \lambda_{max} \left[\lambda_{max}^{t-1} \frac{1}{a} + \left(\frac{1 - \lambda_{max}^{t}}{1 - \lambda_{max}} \right) M^2 \right] I$$

$$\leq \lambda_{max} \left[\frac{1}{a} + \frac{1}{1 - \lambda_{max}} M^2 \right] I$$

(4.3.10)
$$\equiv \frac{1}{\alpha_{min}} I$$

□

The result indicates that under the usual conditions connected with data all versions of MF and EF are exponentially convergent provided that the forgetting factors are bounded away from 0 and 1.

Directional forgetting (DF)

The version of the method suggested by Kulhavy and Karny (1984) is considered, i.e. the predictive update for P is

(4.3.11)
$$P^{-1}(t) = P^{-1}(t-1) + \left(\lambda - \frac{1 - \lambda}{\varphi^T(t)\ P(t-1)\ \varphi(t)} \right) \varphi(t)\ \varphi^T(t)$$

for $\varphi^T(t)\ P(t-1)\ \varphi(t) > 0$.

a) The DF method is designed with the blow up situation in mind.

However, consider the scalar case $p = 1$. The update for $P(t+1|t) = P(t)$ is

$$P^{-1}(t) = P^{-1}(t-1) + \left(\lambda - \frac{(1-\lambda)\,P^{-1}(t-1)}{\varphi^2(t)} \right) \varphi^2(t)$$

$$= \lambda\left(P^{-1}(t-1) + \varphi^2(t) \right) \tag{4.3.12}$$

Hence, in this case DF is identical to EF. It is easy to construct an example where $\{ \varphi(t) \}$ is a sequence of positive numbers converging to zero at a rate so fast that

$$P^{-1}(t) \to 0 \tag{4.3.13}$$

This indicates that DF does not offer a complete solution to the problems connected with poorly exciting data.

b) Now consider the case where $\{ \varphi(t) \}$ is persistently exciting and bounded. Here it can be shown that

$$0 < P(t) \leq \alpha_{max}\, I \quad , \quad \alpha_{max} > 0 \tag{4.3.14}$$

No positive lower bound can be guaranteed. Hence, in this case the algorithm has the properties (4.2.14) – (4.2.16). Furthermore, it is convergent but not exponentially convergent. The proof of these statements is given in Bittanti et al. (1990).

Linear forgetting (LF)

Consider the time update

$$P(t+1|t) = P(t|t) + R \tag{4.3.15}$$

a) In the general situation P may grow unboundedly, and the elementary properties cannot be established.

b) Again the properties of the method can be examined by studying the update of P. However, from the Kalman filter theory it is known that the estimation algorithm is exponentially convergent provided that the underlying system is uniformly completely observable and controllable (Jazwinski, 1970). The observability and controllability Gramians are

$$O(t,N) = \sum_{k=t+1}^{t+N} \varphi(k)\ \varphi^{T}(k) \tag{4.3.16}$$

$$C(t,N) = \sum_{k=t+1}^{t+N} R(k) = N\ R \tag{4.3.17}$$

Therefore the condition for exponential convergence is that $O(t,N)$ and $C(t,N)$ are bounded from above and below. This will be fulfilled when $\{\ \varphi(t)\ \}$ is persistently exciting and bounded and the condition number of R is positive and finite.

4.4 Summary

In this chapter some results have been given which can be useful when examining the convergence properties of a forgetting algorithm.

A natural step in the examination of an identification algorithm is to verify that it performs well in an ideal environment without modeling errors, parameter variations and disturbances. A number of properties which will be considered mandatory for any forgetting algorithm have been formulated.

Some basic RLS–properties should be retained: bounded estimation errors, boundedness of the sum of normalized, squared prediction errors and boundedness of the sum of squared estimator step lengths. These properties ensure that if the estimator is applied as the identification part of the simple adaptive controller described in chapter 2 then the resulting input and output signals will remain bounded and the output tracking error will converge to zero.

It is also reasonable to require that the algorithm is exponentially convergent if the experiment provides persistent excitation. This property guarantees that bounded time–variations of the parameter vector result in bounded estimation errors. In connection with adaptive control it ensures that the output tracking error vanishes exponentially fast.

The main results of the chapter are contained in section 4.2. Here the general forgetting algorithm formulated in chapter 3 has been examined and it has been proved that the properties of the algorithm depend critically on the choice of time update for the parameter covariance matrix. If this choice ensures that no eigenvalues for P are decreased in the time update and that P remains bounded from above, then the algorithm has the basic RLS properties. If in addition P is positively bounded from below then the algorithm is also exponentially convergent.

Using the results of this chapter, the convergence question can be dealt with by simply examining the update equation for the P matrix. As an example the matrix forgetting method has been studied. It has been proved that if the

forgetting factors are bounded away from 0 and 1 then persistent excitation of the data will guarantee that the method has the basic RLS properties and that it is exponentially convergent.

5

The selective forgetting method

This chapter contains presentation, discussion and examination of a selective forgetting algorithm. It represents a continuation of the work started in Parkum (1988) and in Parkum, Poulsen and Holst (1990).

The discussion in chapter 3 showed that the discounting principle built into the standard exponential forgetting algorithm has to be modified if situations with spatially non–uniform distribution of the information flow or different rates of parameter variation are likely to occur. In this chapter a forgetting method flexible enough to cope with those types of estimation problems is developed.

Recall from chapter 3 that in the matrix forgetting method a forgetting factor is connected to each parameter of the process model. The forgetting factors are then tuned individually in accordance with the rate of variation and the excitation of the single parameters. However, this idea does not take into account that the parameters may be mutually correlated. In that case it is clear that each forgetting factor will in fact work on all parameters. Therefore, a natural extension would be to change the parameterization and to apply the matrix forgetting principle to a new model with stochastically independent parameters. This is the idea of the selective forgetting method.

The heuristic discussion leading to the formulation of the selective forgetting principle does not take into account the practical difficulties connected with its implementation. In fact, determination of the stochastically independent parameters requires calculation of eigenvectors and eigenvalues for the covariance matrix, and this is a computationally demanding task. In order to transform the ideas into a method suited for practical use it is necessary to find a way to circumvent this difficulty. A solution is suggested in the present chapter. The key observation is that the selective forgetting principle can be obtained by implementing the time update of the covariance matrix as a simple matrix polynomial.

The results in Parkum (1988) and Parkum, Poulsen and Holst (1990) showed very promising performance of algorithms based on the selective forgetting idea. In this chapter new simulation results are presented which back up these results. In addition, a number of important theoretical properties established.

In section 5.1 the selective forgetting method is formulated. The update principle is explained and the choice of forgetting factors is considered. Section 5.2 contains an analytical examination of the method. Finally, in section 5.3 the polynomial version of the method is presented and studied.

5.1 Formulation of the selective forgetting method

In order to obtain a graphical understanding of the forgetting methods it is useful to study the confidence ellipsoids $E(t+i|t)$ ($i = 0, 1$) defined by:

(5.1.1) $$E(t+i|t): \qquad \theta^T P^{-1}(t+i|t)\ \theta = 1$$

Let eigenvector no. k of $P(t+i|t)$ be denoted $v_k(t+i|t)$, $k = 1, ...,p$, and let $\alpha_k(t+i|t)$ be the corresponding eigenvalue. The k'th principal axis of $E(t+i|t)$ is parallel to $v_k(t+i|t)$, and the corresponding diameter is proportional to $\sqrt{\alpha_k(t+i|t)}$. The exponential forgetting (EF) time update $E(t|t) \rightarrow E(t+1|t)$ is illustrated in figure 5.1.1 in the two dimensional case.

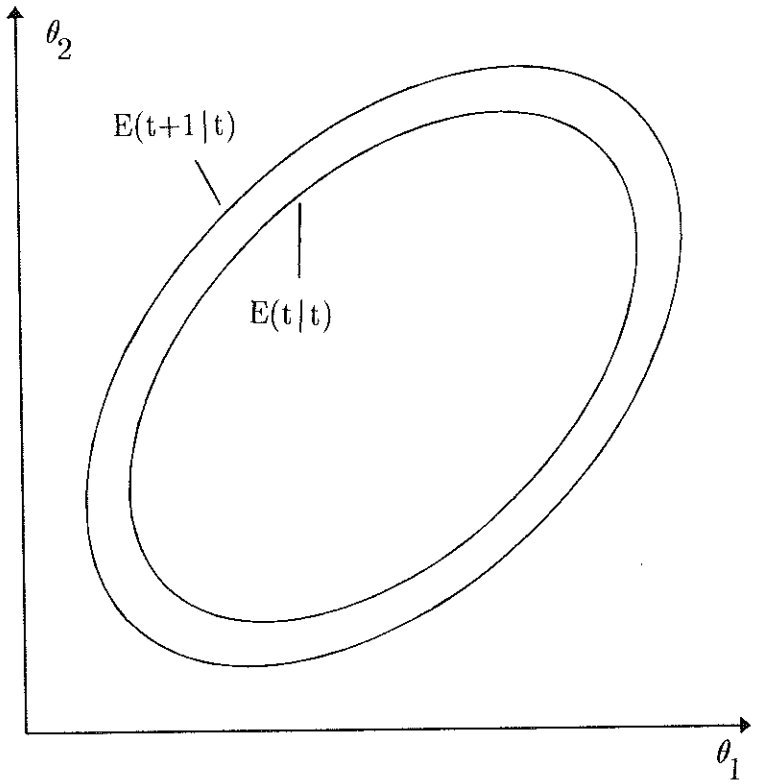

Figure 5.1.1 Illustration of the time update $E(t|t) \rightarrow E(t+1|t)$ for the exponential forgetting method.

The same forgetting factor acts on all directions of the parameter space. If a

certain direction is poorly excited then after some time this direction will become a principal direction of the ellipsoid. The corresponding eigenvalue will continue to grow since no reduction occurs at the measurement updates. Eventually this results in covariance wind up. It is also impossible to utilize a priori knowledge that certain parameters vary more rapidly than others.

The idea in the *selective forgetting* (SF) *algorithm* is to apply an individual forgetting factor for each principal axis of the ellipsoid $E(t|t)$. This is illustrated below. Directions which correspond to nearly constant parameters or poor excitation should be associated with a forgetting factor close to 1. Smaller forgetting factors can be used in directions which correspond to rapid time–variation and sufficient excitation. Intuitively it seems plausible that a method based on this principle offers a complete solution to the covariance wind up problem. A formal proof of this will be given in section 5.2.

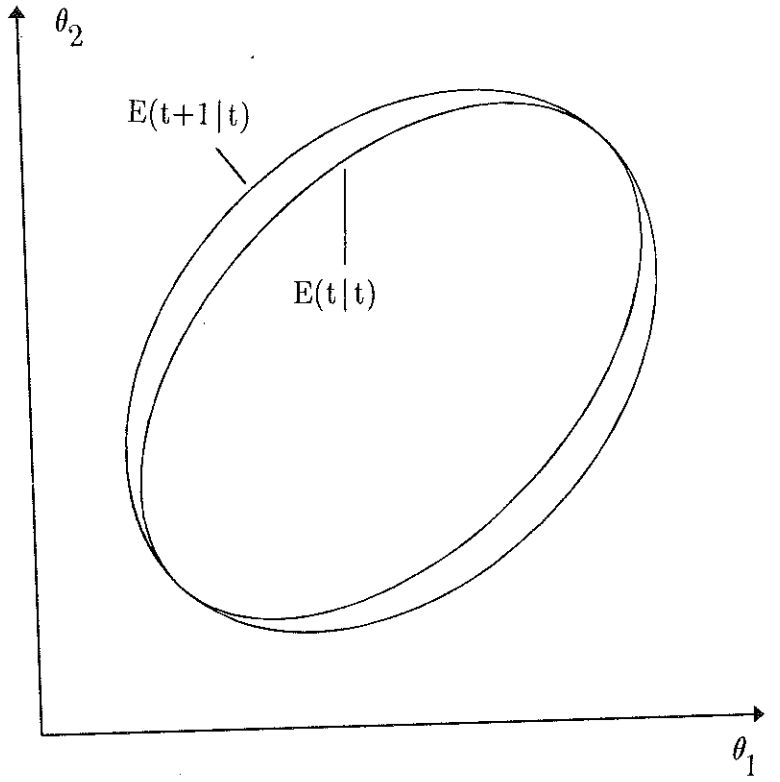

Figure 5.1.2 Illustration of the time update $E(t|t) \rightarrow E(t+1|t)$ for the selective forgetting method.

In the sequel a given method will be regarded as a selective forgetting method if it belongs to the general family formulated in chapter 3 and if the time

update leaves the direction of P's eigenvectors unchanged. With this definition EF and RLS become simple special cases.

To obtain a more precise description of the method notice that $P(t|t)$ can be written

$$P(t|t) = \sum_{k=1}^{p} \alpha_k(t|t)\ v_k(t|t)\ v_k^T(t|t) \tag{5.1.2}$$

where $\alpha_1(t|t), \ldots, \alpha_p(t|t)$ are the eigenvalues of $P(t|t)$ and $v_1(t|t), \ldots, v_p(t|t)$ are the normalized eigenvectors. The EF time update can be written

$$P(t+1|t) = \sum_{k=1}^{p} \frac{\alpha_k(t|t)}{\lambda(t)}\ v_k(t|t)\ v_k^T(t|t) \tag{5.1.3}$$

i.e. all eigenvalues are divided by $\lambda(t)$ and the eigenvectors remain unchanged:

$$\begin{aligned} v_k(t+1|t) &= v_k(t|t) \\ \alpha_k(t+1|t) &= \alpha_k(t|t)\ /\ \lambda(t) \end{aligned} \tag{5.1.4}$$

It can readily be seen that the selective forgetting update discussed above can be obtained by defining

$$P(t+1|t) = \sum_{k=1}^{p} \frac{\alpha_k(t|t)}{\lambda_k(t)}\ v_k(t|t)\ v_k^T(t|t) \tag{5.1.5}$$

This corresponds to the following update equations for the eigenvectors and eigenvalues:

$$v_k(t+1|t) = v_k(t|t)$$

(5.1.6)

$$\alpha_k(t+1|t) = \alpha_k(t|t) \; / \; \lambda_k(t)$$

Now an alternative mathematical formulation will be derived. Assume that the goal is to assign individual forgetting factors $\lambda_1, \ldots, \lambda_p$ to an arbitrary choice of linearly independent parameter linear combinations $u_1^T \theta, \ldots, u_p^T \theta$ ($u_k \in R^p$, $\| u_k \|^2 = 1$). Straightforward generalization of the matrix forgetting method leads to

(5.1.7) $$P(t+1|t) = \Phi(t+1;t) \; P(t|t) \; \Phi^T(t+1;t)$$

(5.1.8) $$\Phi(t+1;t) = U(t) \; G(t) \; U^T(t)$$

(5.1.9) $$G(t) = \mathrm{diag}(\; 1 \; / \; \sqrt{\lambda_k(t)} \;)$$

(5.1.10) $$U(t) = (\; u_1(t) \; \ldots \; u_p(t) \;)$$

Apply the interpretation of $P(t+i|t)$ as the conditional variance of $\theta(t+i)$ given the information $I(t)$ ($i = 0, 1$). The aim is to assign individual forgetting factors to stochastically independent parameter components. This can be obtained by choosing

(5.1.11) $$u_k(t) = v_k(t|t) \qquad , \| v_k(t|t) \|^2 = 1$$

where $v_k(t|t)$ is eigenvector no. k for $P(t|t)$.

The equations given above should be seen as general mathematical descriptions of the selective forgetting method. They do not necessarily represent the form in which the method should be implemented. This will be discussed further in section 5.3.

Applying the Bayesian view on parameter estimation it is possible to give an illustrative interpretation of the selective forgetting method. Let $p(t+i|t)$ denote the conditional probability density function for $\theta(t+i)$ given the information $I(t)$ ($i = 0, 1$). Assume that the distribution is Gaussian:

$$p(t+i|t) = C \exp \{ -\frac{1}{2\sigma^2} (\theta - \hat{\theta}(t+i|t))^T P^{-1}(t+i|t) \times (\theta - \hat{\theta}(t+i|t)) \} \tag{5.1.12}$$

Let $p_k(t+i|t)$ be the corresponding marginal density for $v_k^T(t+i|t)\ \theta$. Due to the independence the following is true

$$p(t|t) = \prod_{k=1}^{p} p_k(t|t) \tag{5.1.13}$$

The EF time update can be written

$$p_k(t+1|t) = (p_k(t|t))^{\lambda(t)} \tag{5.1.14}$$

$$p(t+1|t) = \prod_{k=1}^{p} p_k(t+1|t) \tag{5.1.15}$$

(Peterka, 1980). (5.1.14) – (5.1.15) result in a flattening of the parameter distribution, representing the increased uncertainty during one sampling period.

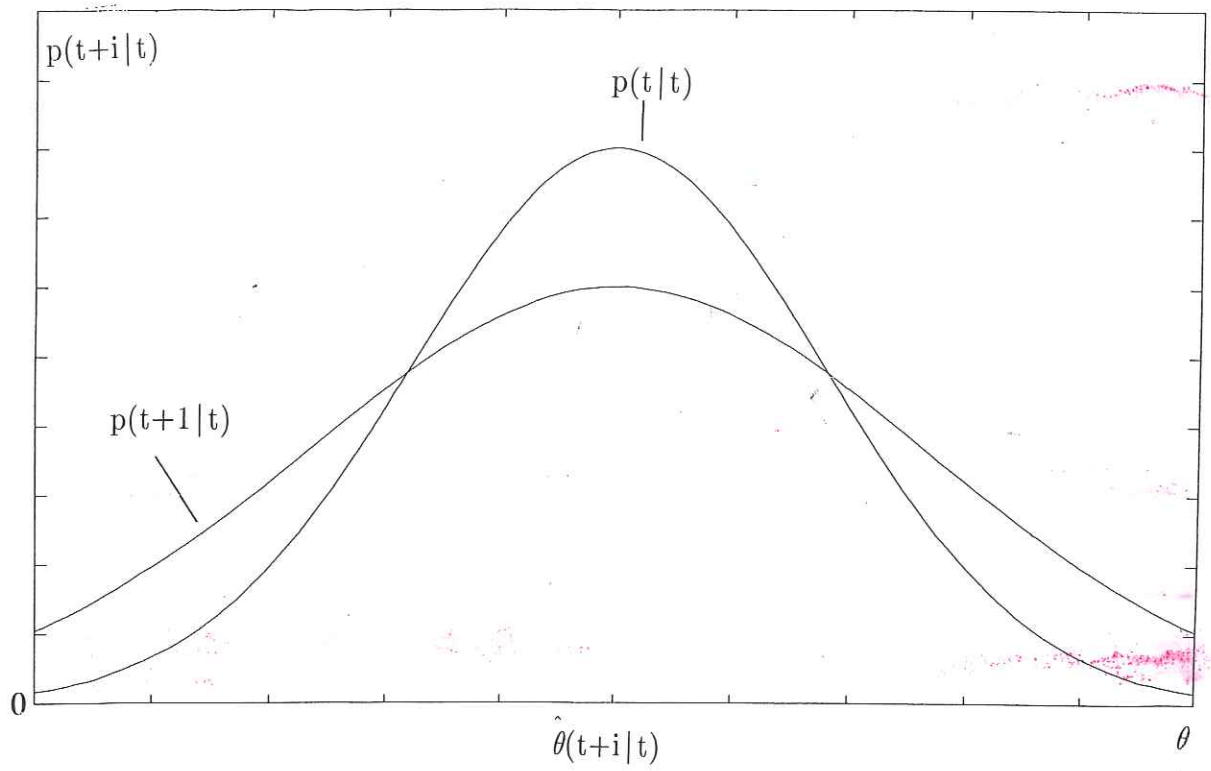

Figure 5.1.3 Flattening of the parameter distribution obtained with the time update of the exponential forgetting method

Analogously, the SF time update can be written

(5.1.16) $$p_k(t+1|t) = (p_k(t|t))^{\lambda_k(t)}$$

(5.1.17) $$p(t+1|t) = \prod_{k=1}^{p} p_k(t+1|t)$$

Hence, p independent flattening operations are performed with different forgetting factors.

Now consider the choice of forgetting factors in the selective forgetting method. At each time update the eigenvectors of P may have new directions, and consequently some kind of forgetting profile connecting a forgetting factor to each direction has to be defined. The value of a forgetting factor associated with a given direction should depend on the amount of information received in

this direction and on the rate of variation of the corresponding parameter linear combination. More formally, if quantities

$INF_k(t)$: a measure of the amount of information received in the direction of the eigenvector $v_k(t|t)$

$VAR_k(t)$: a measure of the variation of $v_k^T(t|t)\ \theta(t)$

can be determined, then choose a suitable function $g(.,.)$ and set

(5.1.18) $$\lambda_k(t) = g(INF_k(t),\ VAR_k(t))$$

In the sequel only the information problem will be considered, i.e. the attention will be directed towards forgetting factors of the type

(5.1.19) $$\lambda_k(t) = g(INF_k(t))$$

However, by formulating a measure $VAR_k(t)$ of the parameter variation, the type of problems connected with different rates of parameter variation can be handled in a quite similar manner.

The following examples illustrate how different choices of forgetting factors can lead to variants of the selective forgetting algorithm which can be regarded as extensions of some well known methods.

Example 5.1.1

In the methods suggested in Kulhavy and Karny (1984) and Hägglund (1983) the idea is to discount information only in the direction where new information is received, i.e. in the direction parallel to $\varphi(t)$. This example shows how a similar principle can be employed in the SF method.

Define

$$\lambda_k(t) = 1 - (1 - \lambda) \frac{\| v_k^T (t|t) \varphi(t) \|}{\| \varphi(t) \|}$$

if $\varphi(t) \neq 0$ and

$$\lambda_k(t) = 1$$

for $\varphi(t) = 0$. Notice that $\| v_k^T(t|t) \varphi(t) \|$ is the length of the projection of $\varphi(t)$ on the normalized eigenvector $v_k(t|t)$. $\lambda_k(t)$ will be identical to 1 if $\varphi(t)$ is perpendicular to $v_k(t|t)$, i.e. if no information is received in the direction $v_k(t|t)$. If $\varphi(t)$ is parallel to $v_k(t|t)$ then $\lambda_k(t)$ becomes identical to the minimum value λ.

In periods of poor excitation, this version of the selective forgetting method behaves like the method suggested by Kulhavy and Karny. To show this, assume that the regression vector φ is identical to a constant, non–zero value for a long time, and that φ eventually becomes the eigenvector for P corresponding to the smallest eigenvalue, which will be denoted α_1.

$P^{-1}(t|t-1)$ is given by

$$P^{-1}(t|t-1) = \frac{1}{\alpha_1(t|t-1)} \frac{\varphi \varphi^T}{\| \varphi \|^2} + \sum_{k=2}^{p} \frac{1}{\alpha_k(t|t-1)} v_k(t|t) v_k^T(t|t)$$

The measurement update gives

$$P^{-1}(t|t) = P^{-1}(t|t-1) + \varphi\ \varphi^T$$

$$= \left(\| \varphi \|^2 + \frac{1}{\alpha_1(t|t-1)} \right) \frac{\varphi\ \varphi^T}{\| \varphi \|^2} + \sum_{k=2}^{p} \frac{1}{\alpha_k(t|t-1)}\ v_k(t|t)\ v_k{}^T(t|t)$$

At the time update the smallest eigenvalue for P will be divided by λ while the remaining $(p - 1)$ eigenvalues will stay unchanged. The eigenvectors will also stay unchanged. This gives:

$$P^{-1}(t+1|t) = \lambda \left(\| \varphi \|^2 + \frac{1}{\alpha_1(t|t-1)} \right) \frac{\varphi\ \varphi^T}{\| \varphi \|^2} + \sum_{k=2}^{p} \frac{1}{\alpha_k(t|t-1)}\ v_k(t|t)\ v_k{}^T(t|t)$$

In total it can be seen that

$$P^{-1}(t+1|t) = P^{-1}(t|t-1) + (1 - \kappa(t))\ \varphi\ \varphi^T$$

where

$$1 - \kappa(t) = \lambda \left(1 + \frac{1}{\alpha_1(t|t-1)\ \| \varphi \|^2} \right) - \frac{1}{\alpha_1(t|t-1)\ \| \varphi \|^2}$$

$$= \lambda - \frac{1 - \lambda}{\alpha_1(t|t-1)\ \| \varphi \|^2}$$

$$= \lambda - \frac{1 - \lambda}{\varphi^T\ P(t|t-1)\ \varphi}$$

i.e.

$$\kappa(t) = (1 - \lambda) (1 + \frac{1}{\varphi^T P(t|t-1) \varphi})$$

It may be concluded that in periods where φ is constant the method becomes identical to the one suggested in Kulhavy and Karny (1984). However, when the excitation is good, the methods will perform differently. This version of the selective forgetting method was suggested and successfully tested in Parkum (1988).

∇

Example 5.1.2

Another possibility is to extend the method suggested in Fortescue et al. (1981). For example, apply the selective forgetting method with

$$\lambda_k(t) = 1 - \frac{\epsilon^2(t)}{J_0 (1 + \varphi^T(t) P(t|t-1) \varphi(t))} h_k(t)$$

The weight function $h_h(t)$ may be chosen as a function of the eigenvalue $\alpha_k(t|t)$, for example

$$h_k(t) = \frac{tr\ P(t|t)}{p\ \alpha_k(t|t)}$$

where $p = \dim \theta$. λ_k may be bounded from below by λ_{min}, $0 < \lambda_{min} < 1$. Small values of α_k result in $\lambda = \lambda_{min}$ while large values will lead to forgetting factors close to 1.

Simulation experiments with this forgetting factor can be found in Parkum, Poulsen and Holst (1990). The results indicate that the modification improves the robustness of the estimation since the risk of covariance wind up is reduced.

∇

The main drawback of the SF versions suggested above is that they require determination of eigenvectors and eigenvalues for the P – matrix. In section 5.3 a version better suited for practical use will be suggested.

5.2 Theoretical study of the selective forgetting method

In this section an analytical study of the selective forgetting method is presented. The goal is to convert the heuristic discussion of the previous section into more formal statements concerning the properties of the algorithm.

The examination will lead to guide lines on how to choose the forgetting factors in order to ensure that P remains bounded. Recall from chapter 4 that a number of convergence results then hold. In addition some other properties are established, for example exponential resetting of P in periods with no excitation.

Focus on the covariance wind up problem and let the forgetting factor $\lambda_k(t)$ be chosen as a function of $\alpha_k(t|t)$ representing the amount of information received in the direction $v_k(t|t)$. The following update is analyzed

$$\alpha_k(t+1|t) = \alpha_k(t|t) \,/\, \lambda_k(t) \equiv f(\alpha_k(t|t)) \tag{5.2.1}$$

Consider the choice of the function $f(\cdot)$. Let α_{min} and α_{max} be constants, $0 < \alpha_{min} < \alpha_{max} < \infty$.

Theorem 5.2.1

Consider the selective forgetting algorithm with the initial values

$$\hat{\theta}(0|0) = 0$$

$$P(0|0) = a\, I, \qquad 0 < a \leq \alpha_{max}.$$

Choose the forgetting factors according to (5.2.1) where the function $f(x)$ is continuous for $0 \leq x \leq \alpha_{max}$ and differentiable for $0 < x < \alpha_{max}$. Then, if the conditions

(5.2.2) $f(0) = \alpha_{min}$

(5.2.3) $f(\alpha_{max}) = \alpha_{max}$

(5.2.4) $f(x) > x \quad \text{if} \quad 0 < x < \alpha_{max}$

(5.2.5) $0 < f'(x) \leq k < 1 \quad \text{if} \quad 0 < x < \alpha_{max}$

are fulfilled, then the following is true

(i) $P(t+1|t) \geq P(t|t) \quad \text{for all} \quad t \geq 0$

(ii) $\alpha_{min} I \leq P(t+1|t) \leq \alpha_{max} I \quad \text{for all} \quad t \geq 0$

(iii) If $\varphi(t) = 0 \quad \text{for all} \quad t \geq 0 \quad \text{then} \quad P(t+1|t) \rightarrow \alpha_{max} I$, and the rate of convergence is at least exponentially fast

Furthermore, if $\alpha_{max} > \alpha_k(t|t) > \alpha_m(t|t) > 0$ then

(iv) $\alpha_k(t+1|t) > \alpha_m(t+1|t)$

and

(v) $$\left[\frac{1}{\alpha_k(t|t)} - \frac{1}{\alpha_k(t+1|t)} \right] < \left[\frac{1}{\alpha_m(t|t)} - \frac{1}{\alpha_m(t+1|t)} \right]$$

Proof

First note that if

(5.2.6) $0 \leq P(t|t) \leq \alpha_{max} I$

then it follows from (5.2.2) – (5.2.5) that

(5.2.7) $$\alpha_{min}\ I \leq P(t+1|t) \leq \alpha_{max}\ I$$

and

(5.2.8) $$P(t+1|t) \geq P(t|t)$$

After the measurement update the eigenvalues of P may be unchanged or reduced in size, i.e.

(5.2.9) $$0 \leq P(t+1|t+1) \leq \alpha_{max} I$$

These results along with the initial conditions can be used to prove (i) and (ii) by induction.

Assume that $\varphi(t) = 0$ for all $t \geq 0$. Then the eigenvalues for P develop according to

(5.2.10) $$\alpha_k(t+1) = f(\alpha_k(t)) \qquad t \geq 0$$

where

(5.2.11) $$\alpha_k(t+1|t) = \alpha_k(t)$$

From the mean value theorem it follows that

(5.2.12) $$f(\alpha_{max}) - f(x) = f'(\eta)\ (\alpha_{max} - x), \qquad 0 \leq x < \alpha_{max}$$

where $x < \eta < \alpha_{max}$. Consequently

(5.2.13) $$\alpha_{max} - f(x) \leq k\ (\alpha_{max} - x)$$

where k is defined in (5.2.5). With x replaced by $\alpha_k(t|t)$ this gives

$$\alpha_{max} - \alpha_k(t) = \alpha_{max} - f(\ \alpha_k(t-1)\ )$$

$$\leq k\ (\ \alpha_{max} - \alpha_k(t-1)\ )$$

$$\leq \ldots$$

(5.2.14) $$\leq k^t(\ \alpha_{max} - \alpha_k(0)\ )$$

i.e.

(5.2.15) $$(\ \alpha_{max} - \alpha_k(t)\ ) \to 0$$

and the convergence is at least exponentially fast. This proves (iii).

(iv) follows immediately from (5.2.5), since this property ensures that $f(x)$ is monotonously increasing for $0 < x < \alpha_{max}$.

Finally (v) can be proved by showing that the function

(5.2.16) $$g(x) = \frac{1}{x} - \frac{1}{f(x)}$$

is monotonously decreasing for $0 < x < \alpha_{max}$. This is true since

$$g'(x) = -\frac{1}{x^2} + \frac{f'(x)}{(f(x))^2}$$

$$\leq -\frac{1}{x^2} + \frac{k}{f(x)^2}$$

$$\leq -\frac{1}{x^2} + \frac{k}{x^2}$$

(5.2.17) $$< 0$$

□

Fig. 5.2.1 illustrates the conditions (5.2.2) – (5.2.5).

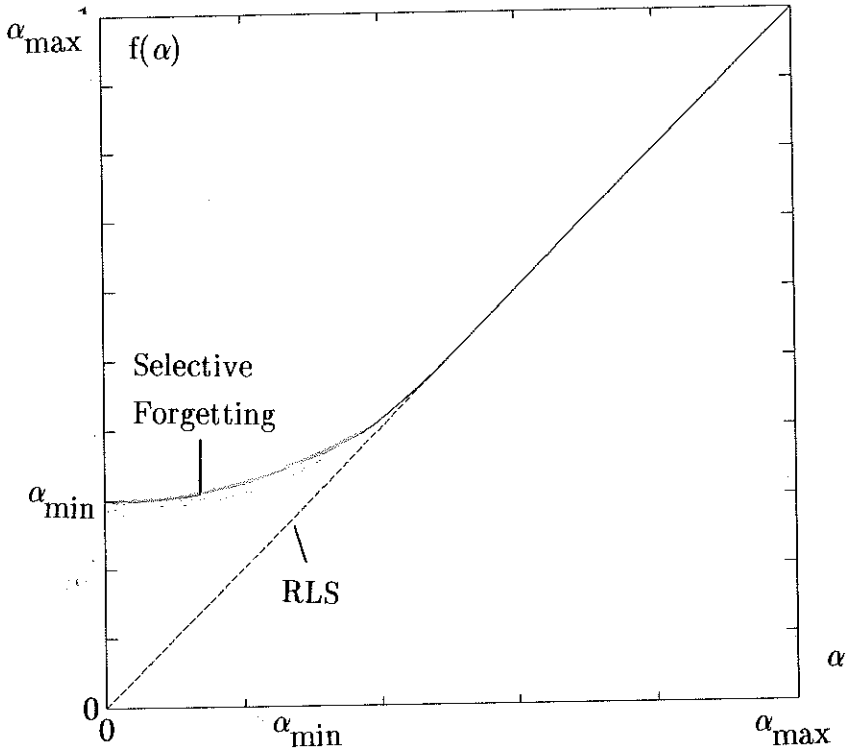

Figure 5.2.1 Graphical comparison of the selective forgetting update subject to the conditions in theorem 5.2.1 and the standard recursive least squares update with no forgetting.

Remarks

Part (i) of the theorem corresponds to the fact that the uncertainty connected to the parameters is increased (in case of time–varying parameters) during a sampling period, where no new information is gathered.

Part (ii) ensures that P has an upper bound (preventing blow up) as well as a lower bound (preventing loss of tracking ability).

Part (iii) shows that the algorithm resets itself exponentially in case of a long period with no new information. This resetting property is important, since the old information gradually loses its value in the time–varying case.

Part (iv) makes sure that the order of the eigenvalues is unchanged, i.e. in this sense the basic structure of P is kept intact.

Finally part (v) can be interpreted in the following manner: if α_k^{-1} is regarded as a measure of the available information (i.e. the part of the received information that has not been discounted) corresponding to the direction determined by v_k, then $\left[\frac{1}{\alpha_k(t|t)} - \frac{1}{\alpha_k(t+1|t)} \right]$ is the amount of information discounted at the time update $t \to t+1$. Part (v) thus ensures that the algorithm forgets more in directions where more information is available.

Theorem 5.2.2

Let the SF method with the forgetting factor (5.2.1) be applied for estimation of the parameters of the system (2.1.6) with $\sigma^2 = 0$ and θ^0 constant. Assume that the conditions in theorem 5.2.1 are fulfilled. Then

$$\| \tilde{\theta}(t) \|^2 \leq \frac{\alpha_{max}}{a} \| \hat{\theta}(0) \|^2 \qquad \text{for all} \quad t \geq 0 \tag{5.2.18}$$

$$\lim_{t \to \infty} \sum_{k=1}^{t} \frac{\epsilon^2(k)}{1 + \varphi^T(k)\, P(k|k-1)\, \varphi(k)} < \infty \tag{5.2.19}$$

$$\lim_{t \to \infty} \sum_{k=N}^{t} \| \hat{\theta}(k+1|k) - \hat{\theta}(k-N+1|k-N) \|^2 < \infty \qquad (5.2.20)$$

for any finite N

Proof

Follows immediately from the theorems 4.2.1 and 5.2.1.

□

Theorem 5.2.3

Assume that the SF method with the forgetting factor (5.2.1) is applied for estimation of the parameters of the system (2.1.6) with $\sigma^2 = 0$ and θ^0 constant. Assume that the conditions in theorem 5.2.1 are fulfilled. If the sequence $\{ \varphi(t) \}$ is persistently exciting and bounded, then the estimation algorithm is exponentially convergent.

Proof

Follows immediately from the theorems 4.2.2 and 5.2.1.

□

Notice that the bounds on P do not depend on the nature of $\{ \varphi(t) \}$. Also, the values of α_{min} and α_{max} can be chosen a priori, i.e. they are tuning parameters of the method.

5.3 The SF1 method

The selective forgetting method has been suggested based on intuitive arguments. The method has been proved to have good theoretical virtues, but there is still a need to convert it into an operational form. This is the aim of the present section.

The key observation in this section is that the polynomial time update equation

$$P(t+1|t) = \sum_{m=-N1}^{N2} a_m(t) \, (\, P(t|t) \,)^m \tag{5.3.1}$$

is a special case of SF. This can be seen by noting that the polynomial update leaves the eigenvectors of P unchanged. It can easily be verified that (5.3.1) corresponds to a selective forgetting method with the forgetting factor

$$\lambda_k(t) = \alpha_k(t|t) \; / \sum_{m=-N1}^{N2} a_m(t) \, (\, \alpha_k(t|t) \,)^m \tag{5.3.2}$$

By implementing the time update as a matrix polynomial, the computational burden connected with determination of eigenvalues and eigenvectors of P is avoided.

In the sequel a selective forgetting method corresponding to the choice of a first order polynomial will be studied in some detail. For the sake of completeness, the full set of equations of the method are given below:

The SF1 method:

Measurement update:

$$\hat{\theta}(t|t) = \hat{\theta}(t|t-1) + P(t|t)\,\varphi(t)\,(\,y(t) - \varphi^{T}(t)\,\hat{\theta}(t|t-1)\,) \tag{5.3.3}$$

$$P^{-1}(t|t) = P^{-1}(t|t-1) + \varphi(t)\,\varphi^{T}(t) \tag{5.3.4}$$

Time update:

$$\hat{\theta}(t+1|t) = \hat{\theta}(t|t) \tag{5.3.5}$$

$$P(t+1|t) = \frac{\alpha_{max} - \alpha_{min}}{\alpha_{max}}\,P(t|t) + \alpha_{min}\,I \tag{5.3.6}$$

Initial values:

$$\hat{\theta}(1|0) = \hat{\theta}(0|0) = 0 \tag{5.3.7}$$

$$P(1|0) = P(0|0) = a\,I, \quad 0 < a \leq \alpha_{max} \tag{5.3.8}$$

Remarks

* The choice of a second order polynomial would result in the so called exponential forgetting and resetting (EFRA) algorithm suggested in Salgado et al. (1988). This method has good theoretical and practical virtues, as was shown by Salgado and his co–authors. However, the following discussion will show that this can actually be achieved without adding the P^2–term.

* Notice that the SF1 update can be regarded as a slightly modified version of the one known from the linear forgetting method (see chapter 3). The multiplication of $P(t|t)$ by the factor $\frac{\alpha_{max} - \alpha_{min}}{\alpha_{max}}$ decreases the covariance wind up tendency.

The SF1 method corresponds to the following choice of forgetting factor

$$\lambda_k(t) = \frac{\alpha_k(t\,|\,t)}{\alpha_{min} + \frac{\alpha_{max} - \alpha_{min}}{\alpha_{max}}\,\alpha_k(t|t)} \tag{5.3.9}$$

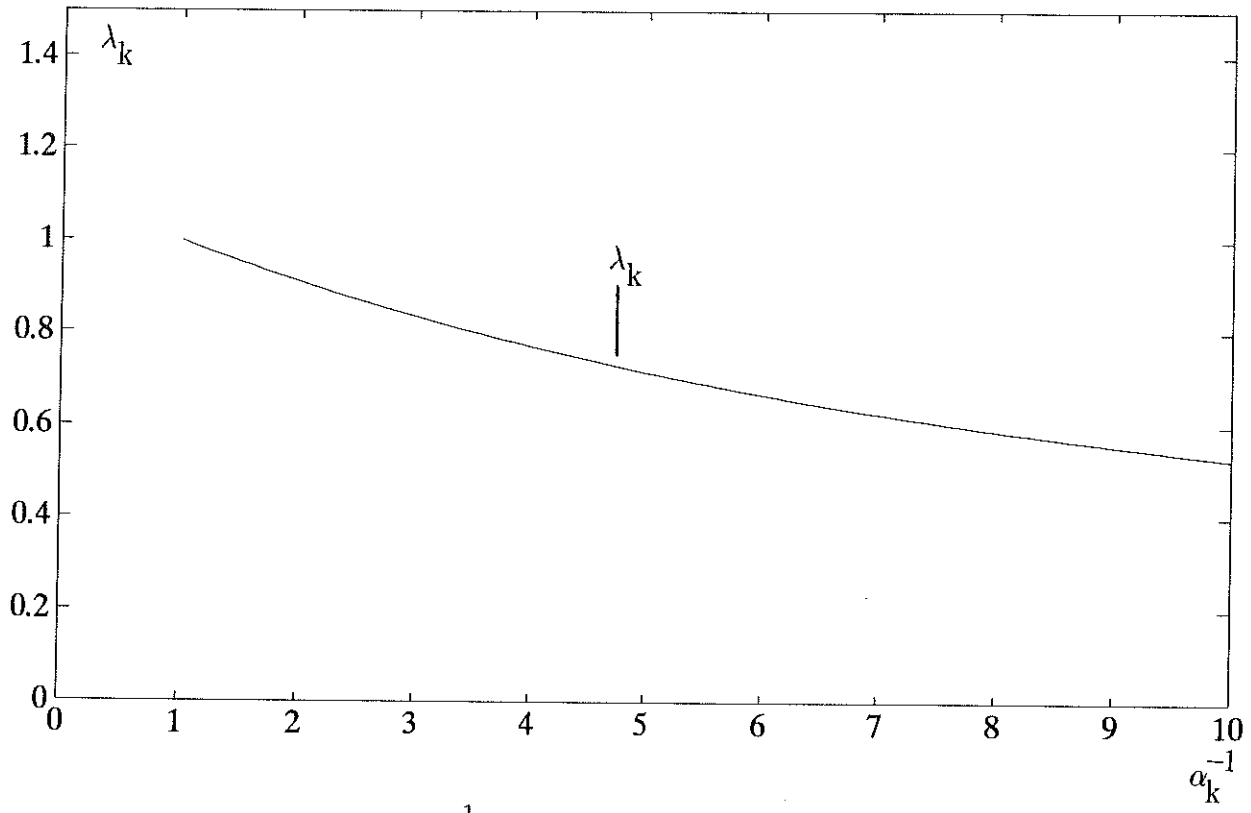

Figure 5.3.1 λ_k versus α_k^{-1} for the SF1 method.

If $\alpha_k(t|t) < \alpha_{max}$, then $\lambda_k(t) < 1$, and information is discounted. If $\alpha_k(t|t) \to \alpha_{max}$, then $\lambda_k(t) \to 1$ and the forgetting is turned off.

In order to check that the SF1 method fulfills all requirements of the theorems 5.2.1 – 5.2.3, notice that the function f(.) introduced in equation (5.2.1) becomes

$$f(x) = \alpha_{min} + \frac{\alpha_{max} - \alpha_{min}}{\alpha_{max}}\,x \tag{5.3.10}$$

f(x) is continuously differentiable for all $x \in R$.

Obviously

(5.3.11) $$f(0) = \alpha_{min}$$

and

(5.3.12) $$f(\alpha_{max}) = \alpha_{max}$$

Furthermore

(5.3.13) $$f(x) = x + \alpha_{min} \left(1 - \frac{x}{\alpha_{max}} \right) > x \quad \text{for } 0 < x < \alpha_{max}$$

and

(5.3.14) $$0 < f'(x) = \frac{\alpha_{max} - \alpha_{min}}{\alpha_{max}} < 1$$

From these expressions it can be seen that all conditions of the theorems of section 5.2 are fulfilled.

With the SF1 time update $P(t+1|t)$ is a weighted sum of $P(t|t)$ and the identity matrix, I. α_{min} and α_{max} become the tuning parameters of the method, and their choice represents the usual trade off between tracking ability and noise sensitivity. Empirically it has been found that a good choice of α_{max} is $\alpha_{max} = K_{prop}\ \alpha_{min}$ where $10 \leq K_{prop} \leq 1000$. Then $P(t+1|t) = (1 - \frac{1}{K_{prop}})\ P(t|t) + \alpha_{min}$ I. The choice of α_{min} is closely related to the choice of the matrix R in the linear forgetting method (see chapter 3). This can be seen by noting that if $K_{prop} >> 1$ then $P(t+1|t) \approx P(t|t) + \alpha_{min}$ I. Therefore, the choice of α_{min} should reflect the expected size of $E \| \theta(t+1)-\theta(t) \|^2 / (\sigma^2 p)$. Experiments have indicated that if the parameter variation is slow, then the method has a relatively low sensitivity to the choice of α_{min} and α_{max}.

Example 5.3.1

Consider the experimental situation described in example 3.1.2. The parameters are constant, and the input signal becomes identical to 1 for t > 100. Recall that this resulted in large variation of the estimates and covariance wind up for the exponential forgetting method. Results obtained with the SF1 method and with

$$\alpha_{\min} = 0.01$$

$$\alpha_{\max} = 0.1$$

are shown in figures 5.3.1 – 5.3.3.

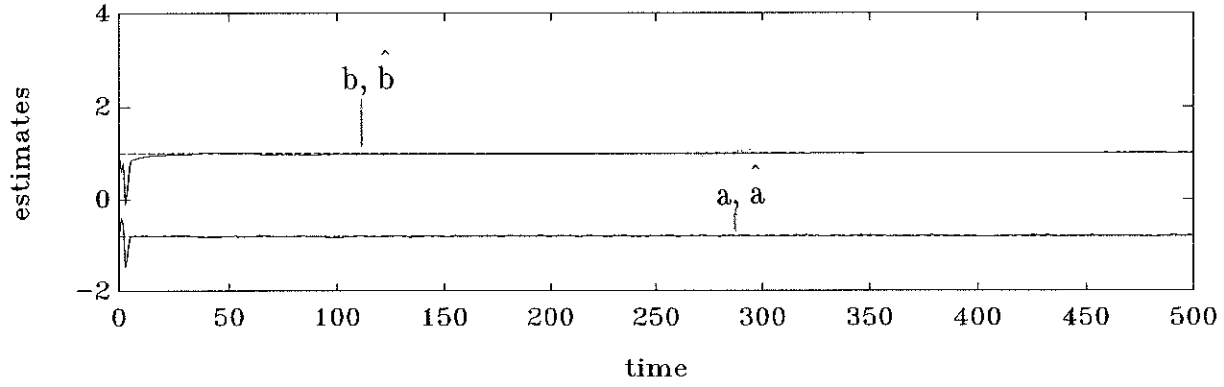

Figure 5.3.1 Parameters (– – –) and estimates (——) for the SF1 method in example 5.3.1. Constant input signal for t > 100.

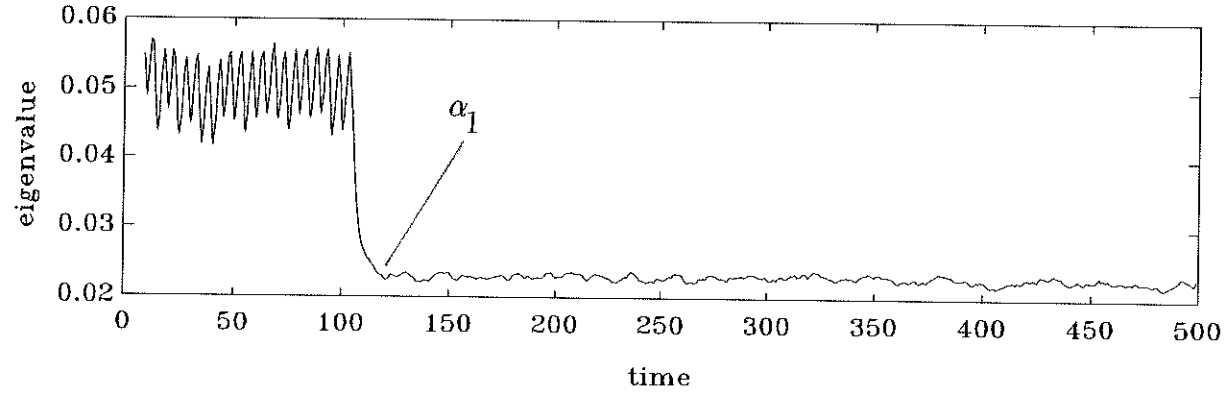

Figure 5.3.2 Eigenvalue no. 1 for the SF1 method in example 5.3.1. Constant input signal for t > 100.

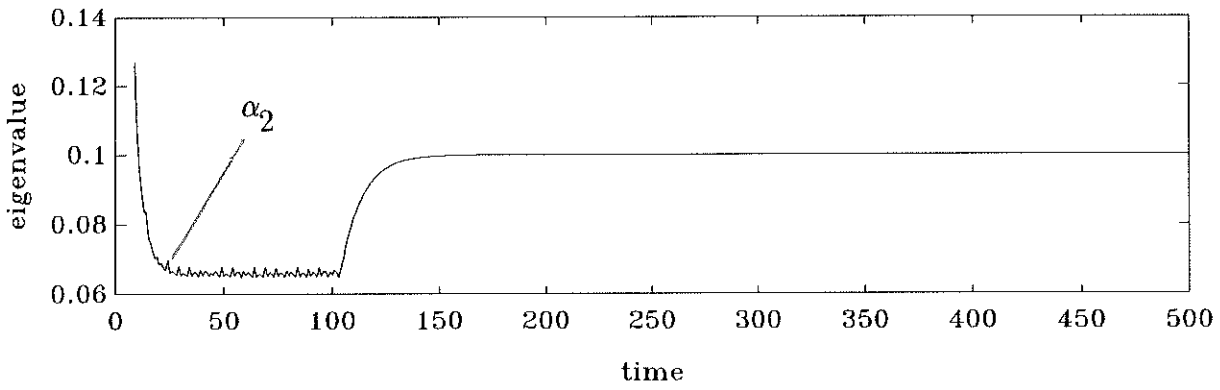

Figure 5.3.3 Eigenvalue no. 2 for the SF1 method in example 5.3.1. Constant input signal for $t > 100$.

Notice that the situation with drifting parameter estimates caused by unbounded growth of P is completely avoided. After $t = 100$ the regression vector is almost constant and eventually it becomes an eigenvector of P. The direction parallel to the regressor is strongly excited in this period. The corresponding eigenvalue for P becomes small, but never smaller than α_{min}. The direction perpendicular to φ is not excited at all, and the second eigenvalue resets itself exponentially fast to the value α_{max}. This is a highly sensible behaviour in the case of time–varying parameters, since the information related to the non–excited direction eventually loses its value.

∇

Example 5.3.2

In this example the performance of the SF1 algorithm is compared to that of the EF and RLS algorithms. The sensitivity of the results to the choice of tuning parameters and to the noise level is studied.

The example will focus on the adaptive control application. The following process is considered:

$$y(t) = \varphi^T(t)\ \theta + e(t)$$

$$\varphi(t) = (\ u(t-1),\ u(t-2),\ u(t-3)\ )^T$$

$$\theta = (\ b_1,\ b_2,\ b_3\ )^T$$

$$\{e(t)\}:\ N(0,\ \sigma^2)\ \text{white noise}$$

The control objective is to track the reference signal $y^*(t)$. The adaptive controller outlined in chapter 2 is applied with different forgetting algorithms handling the identification problem.

First study the case where the parameters are constant

$$b_1 = 1.0$$

$$b_2 = -\ 0.62$$

$$b_3 = 0.5$$

and the reference signal is given by

$$y^*(t) = 2.0 + \text{sign}\ (\ \sin\ (\ \frac{2\ \pi\ t}{50}\ )\ )$$

The initial values for all estimators are

$$\hat{\theta}(0|0) = (0.9,\ -0.5,\ 0.7)^{T}$$

$$P(0|0) = 1000\ I$$

Results obtained with RLS identification for $\sigma = 0.1$ are shown below.

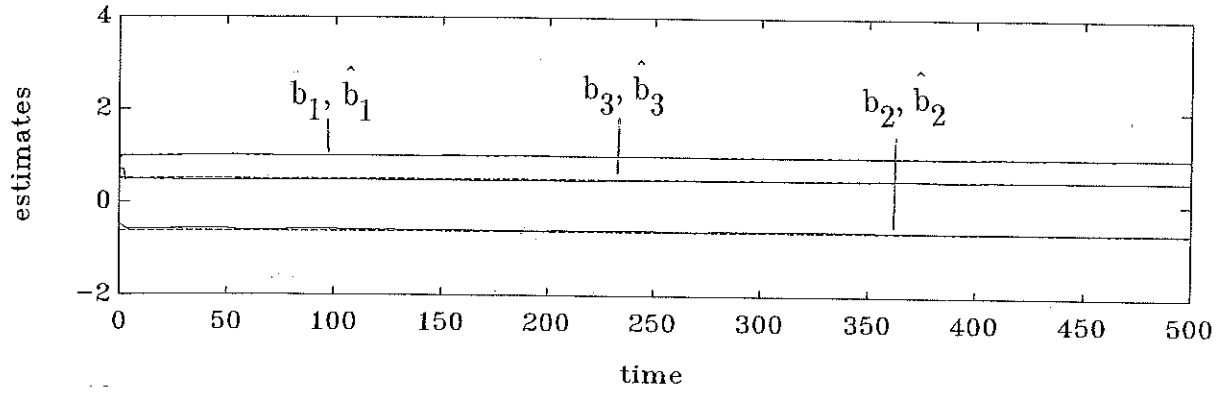

Figure 5.3.4 RLS–estimates. $\sigma = 0.1$. Constant parameters and sufficiently exciting reference signal.

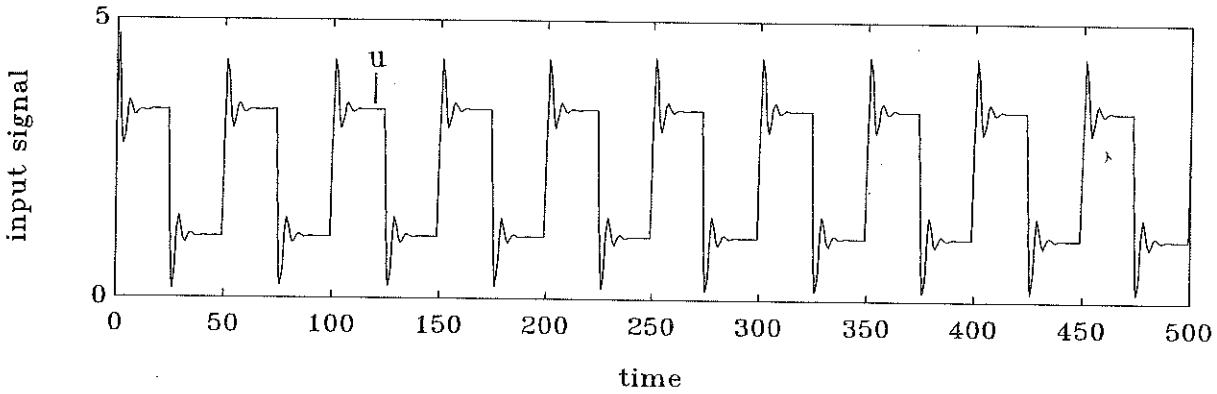

Figure 5.3.5 Input signal obtained with the adaptive controller based on RLS–identification. $\sigma = 0.1$. Constant parameters and sufficiently exciting reference signal.

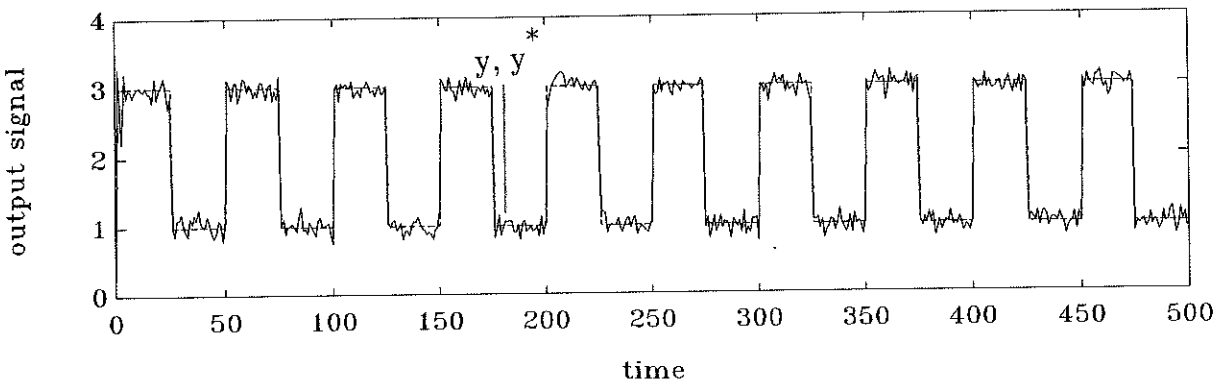

Figure 5.3.6 Output signal obtained with the adaptive controller based on RLS–identification. $\sigma = 0.1$. Constant parameters and sufficiently exciting reference signal.

An interesting question is how the modified versions of RLS handle this situation. To measure the quality of the estimation and control, respectively, define

$$L = \sum_{t=1}^{500} \| \theta(t) - \hat{\theta}(t|t) \|^2$$

$$J = \sum_{t=1}^{500} \| y^*(t) - y(t) \|^2$$

Now replace the estimation part first with EF (tuning parameter: λ), then with SF1 ($\alpha_{max} = 1.0$, tuning parameter: α_{min}). Results are shown below.

Not surprisingly the experiment confirms that RLS is the best choice when the plant is known to be time–invariant. The quality of both estimation and control decreases as the tracking ability and the noise level increases. Roughly speaking, the EF and the SF1 method seem to perform equally well in this experimental situation.

L	$\sigma = 0.00$	$\sigma = 0.05$	$\sigma = 0.1$
$\lambda = 1.00$	0.0944	0.3103	0.9054
$\lambda = 0.99$	0.0944	0.3292	2.1115
$\lambda = 0.96$	0.0944	0.8142	1.6937
$\lambda = 0.90$	0.0944	1.3353	2.8281

Table 5.3.1 Values of the loss function L. Estimation method: EF. Constant parameters and sufficiently exciting reference signal.

J	$\sigma = 0.00$	$\sigma = 0.05$	$\sigma = 0.1$
$\lambda = 1.00$	0.7157	2.0810	5.4980
$\lambda = 0.99$	0.7157	2.1170	5.5512
$\lambda = 0.96$	0.7157	2.1062	6.7772
$\lambda = 0.90$	0.7157	2.1171	6.4218

Table 5.3.2 Values of the loss function J. Estimation method: EF. Constant parameters and sufficiently exciting reference signal.

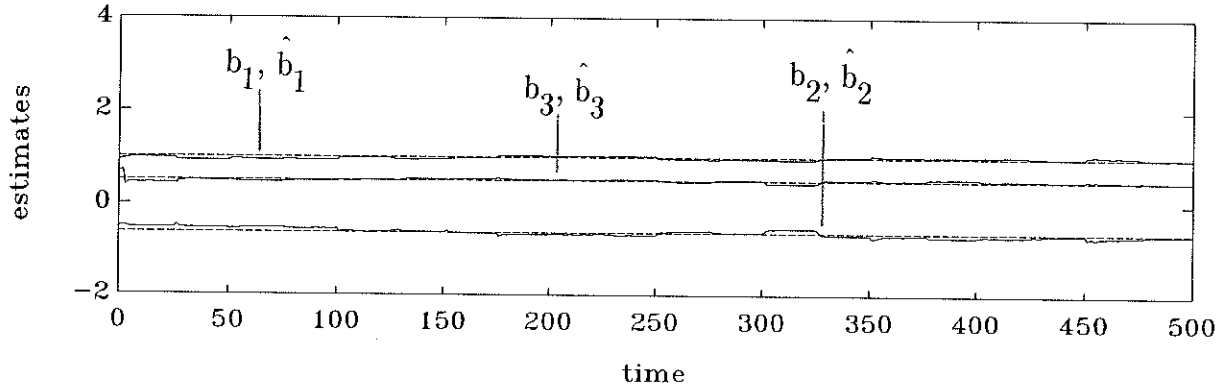

Figure 5.3.7 EF–estimates. $\sigma = 0.1$, $\lambda = 0.96$. Constant parameters and sufficiently exciting reference signal.

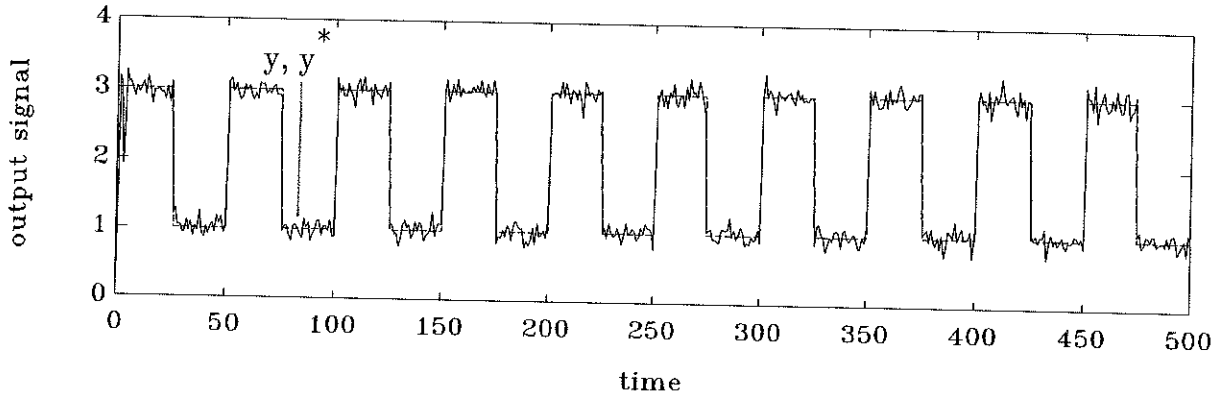

Figure 5.3.8 Output signal obtained with the EF method. $\sigma = 0.1$, $\lambda = 0.96$. Constant parameters and sufficiently exciting reference signal.

L	$\sigma = 0.00$	$\sigma = 0.05$	$\sigma = 0.1$
$\alpha_{\min} = 0.001$	0.0944	0.1867	0.9115
$\alpha_{\min} = 0.005$	0.0944	0.3190	0.9475
$\alpha_{\min} = 0.01$	0.0944	0.5991	1.4243
$\alpha_{\min} = 0.05$	0.0944	0.7005	2.0194
$\alpha_{\min} = 0.10$	0.0944	0.5512	1.5937

Table 5.3.3 Values of the loss function L. Estimation method: SF1. Constant parameters and sufficiently exciting reference signal.

J	$\sigma = 0.00$	$\sigma = 0.05$	$\sigma = 0.1$
$\alpha_{\min} = 0.001$	0.7157	1.9466	6.1539
$\alpha_{\min} = 0.005$	0.7157	2.3671	6.0934
$\alpha_{\min} = 0.01$	0.7157	2.3587	5.8546
$\alpha_{\min} = 0.05$	0.7157	2.0482	7.9919
$\alpha_{\min} = 0.10$	0.7157	2.5470	8.5110

Table 5.3.4 Values of the loss function J. Estimation method: SF1. Constant parameters and sufficiently exciting reference signal.

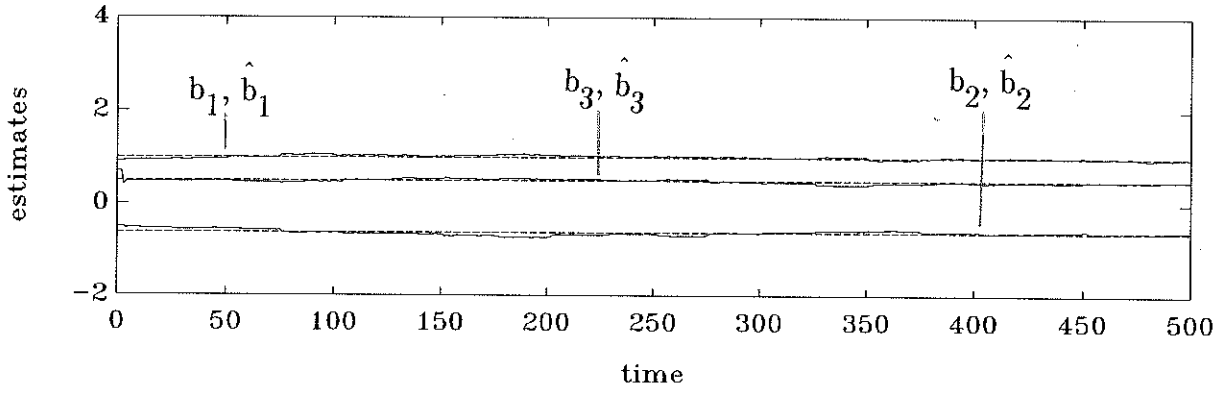

Figure 5.3.9 SF1–estimates. $\alpha_{\min} = 0.01$, $\alpha_{\max} = 1.0$, $\sigma = 0.1$. Constant parameters and sufficiently exciting reference signal.

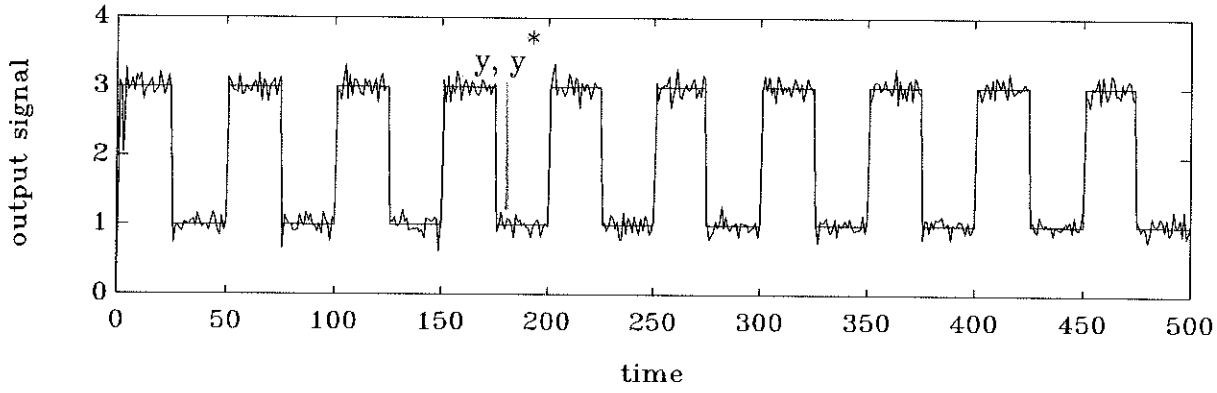

Figure 5.3.10 Output signal obtained with the SF1 method. $\alpha_{min} = 0.01$, $\alpha_{max} = 1.0$, $\sigma = 0.1$. Constant parameters and sufficiently exciting reference signal.

However, assume that the time–invariant description is known to be only approximately valid and that a study of the underlying physical process reveals that b_1 can be expected to exhibit a slow time–variation. To examine the robustness of the forgetting methods the simulation experiment is repeated with

$$b_1(t) = 1.0 + 0.5 \sin \left(\frac{2 \pi t}{500} \right)$$

From the results shown below it can clearly be seen that the RLS method (corresponding to EF with $\lambda = 1$) is unable to cope with the variation of $b_1(t)$.

The EF method is able to track the variation of $b_1(t)$, but only if a very small value of λ is applied.

The SF1 method handles the situation better, and good results are obtained with all tested values of the tuning parameter.

L	$\sigma = 0.00$	$\sigma = 0.05$	$\sigma = 0.1$
$\lambda = 1.00$	59.1477	57.3793	59.2933
$\lambda = 0.99$	23.4160	23.1354	22.6787
$\lambda = 0.96$	8.3721	10.1473	11.6861
$\lambda = 0.90$	3.4997	4.8168	6.9037

Table 5.3.5 Values of the loss function L. Estimation method: EF. Time–varying $b_1(t)$ and sufficiently exciting reference signal.

J	$\sigma = 0.00$	$\sigma = 0.05$	$\sigma = 0.1$
$\lambda = 1.00$	297.1138	299.7316	301.7858
$\lambda = 0.99$	147.5016	148.9426	149.4387
$\lambda = 0.96$	43.6723	45.8328	50.4577
$\lambda = 0.90$	10.5945	12.2882	16.1140

Table 5.3.6 Values of the loss function J. Estimation method: EF. Time–varying $b_1(t)$ and sufficiently exciting reference signal.

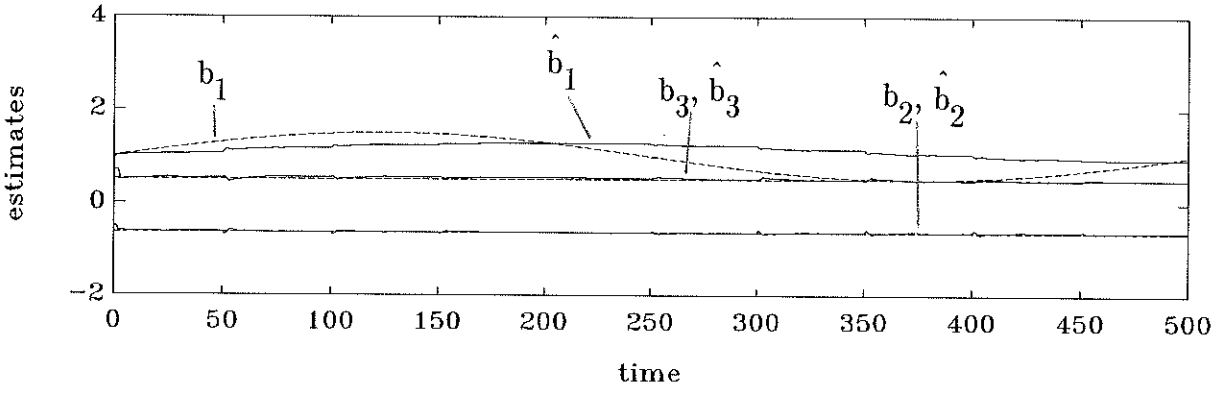

Figure 5.3.11 RLS–estimates. $\sigma = 0.05$. Time–varying $b_1(t)$ and sufficiently exciting reference signal.

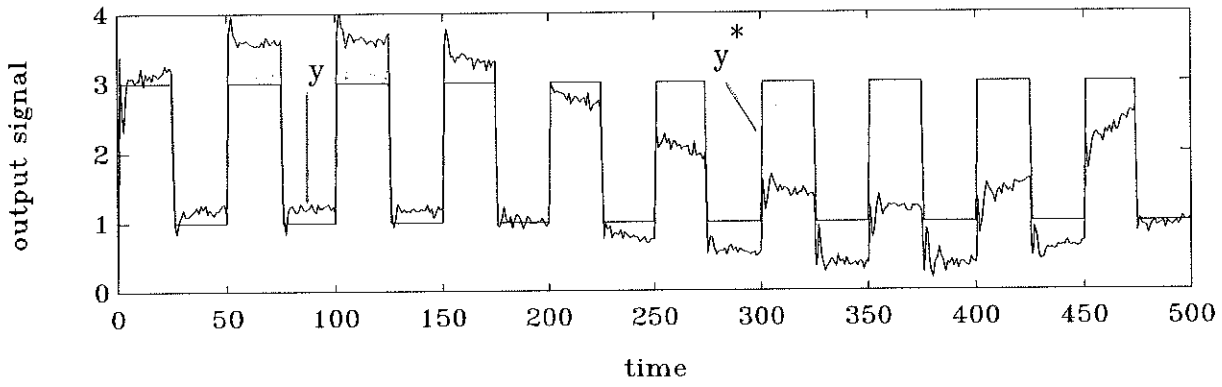

Figure 5.3.12 Output signal obtained with the RLS method. $\sigma = 0.05$. Time–varying $b_1(t)$ and sufficiently exciting reference signal.

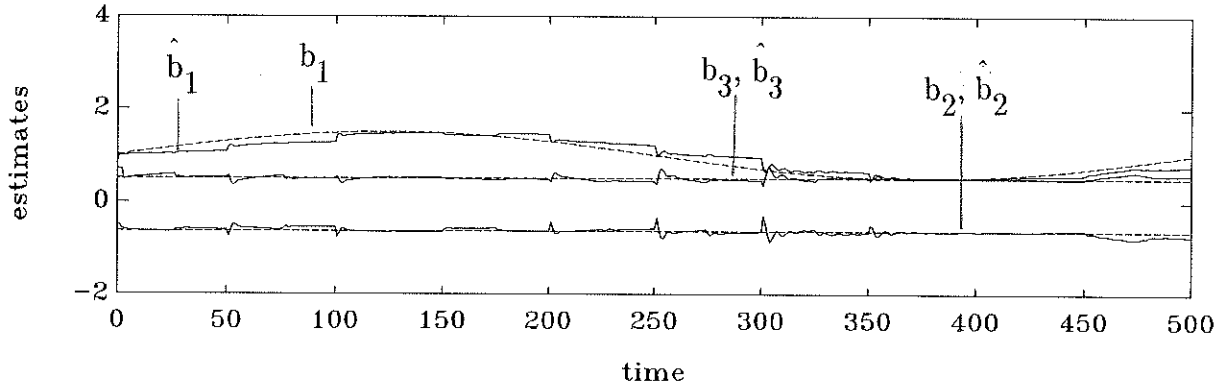

Figure 5.3.13 EF–estimates. $\lambda = 0.96$, $\sigma = 0.05$. Time–varying $b_1(t)$ and sufficiently exciting reference signal.

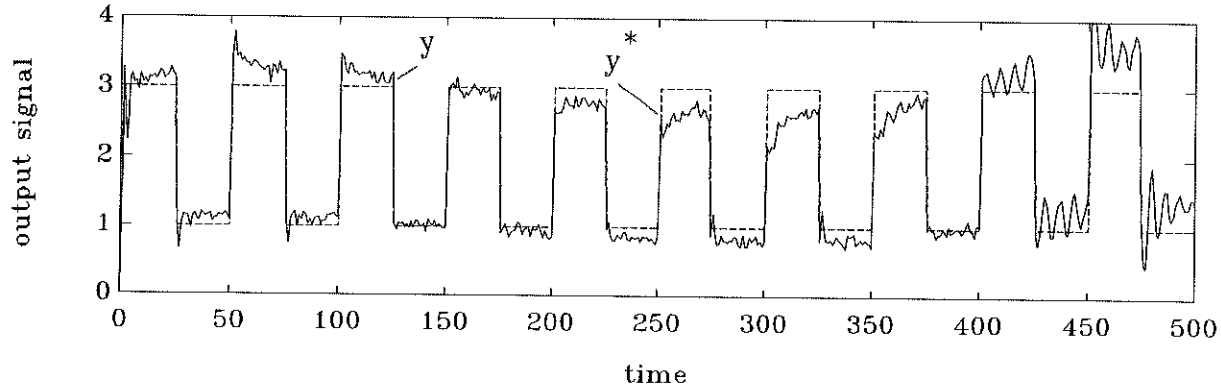

Figure 5.3.14 Output signal obtained with the EF method. $\lambda = 0.96$, $\sigma = 0.05$. Time–varying $b_1(t)$ and sufficiently exciting reference signal.

L	$\sigma = 0.00$	$\sigma = 0.05$	$\sigma = 0.1$
$\alpha_{\min} = 0.001$	14.3295	15.3916	16.3202
$\alpha_{\min} = 0.005$	7.8964	8.0388	10.5326
$\alpha_{\min} = 0.01$	6.1958	6.9791	6.9030
$\alpha_{\min} = 0.05$	4.4156	5.7557	6.9227
$\alpha_{\min} = 0.10$	4.5784	4.2895	5.6734

Table 5.3.7 Values of the loss function L. Estimation method: SF1. Time–varying $b_1(t)$ and sufficiently exciting reference signal.

J	$\sigma = 0.00$	$\sigma = 0.05$	$\sigma = 0.1$
$\alpha_{\min} = 0.001$	10.0012	11.4366	14.8886
$\alpha_{\min} = 0.005$	4.4027	6.1066	10.5987
$\alpha_{\min} = 0.01$	3.3935	4.9473	10.5930
$\alpha_{\min} = 0.05$	2.4735	4.3858	11.6555
$\alpha_{\min} = 0.10$	2.4923	5.3436	11.0261

Table 5.3.8 Values of the loss function J. Estimation method: SF1. Time–varying $b_1(t)$ and sufficiently exciting reference signal.

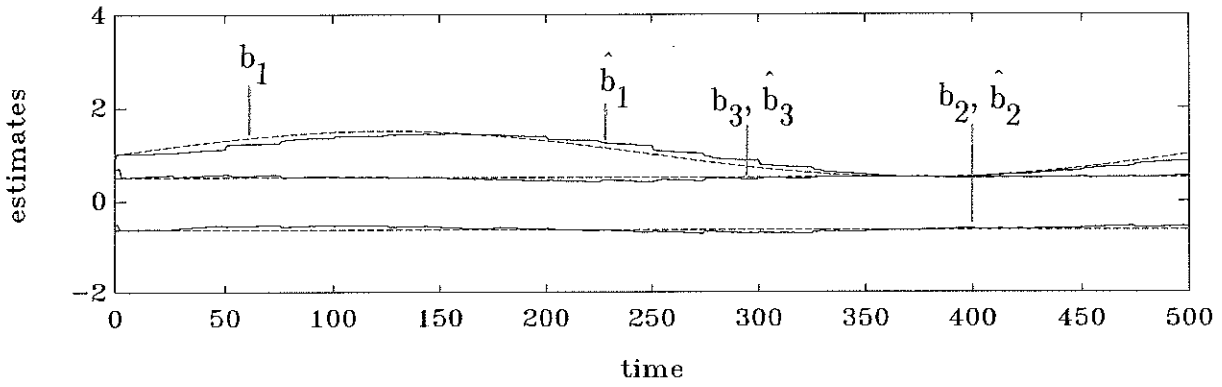

Figure 5.3.15 SF1–estimates. $\alpha_{min} = 0.01$, $\alpha_{max} = 1.0$, $\sigma = 0.05$. Time–varying $b_1(t)$ and sufficiently exciting reference signal.

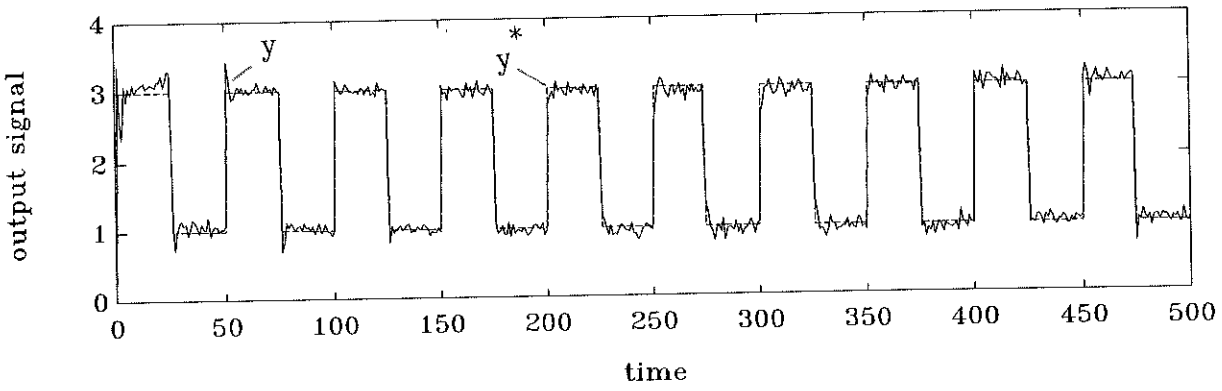

Figure 5.3.16 Output signal obtained with the SF1 method. $\alpha_{min} = 0.01$, $\alpha_{max} = 1.0$, $\sigma = 0.05$. Time–varying $b_1(t)$ and sufficiently exciting reference signal.

The reference signal considered so far provides good excitation. However, it may be of interest to use the adaptive controller in situations where y^* has a lower level of variation. The experiment is therefore repeated with constant parameters and with $y^*(t)$ constant ($= 3$) for $t \geq 100$.

For EF the covariance matrix starts growing unboundedly, and when measurement noise is present, the estimates start to drift erroneously. The tendency reduces when λ approaches 1.

L	$\sigma = 0.00$	$\sigma = 0.05$	$\sigma = 0.1$
$\lambda = 1.00$	0.0944	0.1959	0.4331
$\lambda = 0.99$	0.0944	0.5213	1.3020
$\lambda = 0.96$	0.0944	308.6911	3.64×10^4
$\lambda = 0.90$	0.0944	8.65×10^5	226.6053

Table 5.3.9 Values of the loss function L. Estimation method: EF. Constant parameters and poor excitation for $t \geq 100$.

J	$\sigma = 0.00$	$\sigma = 0.05$	$\sigma = 0.1$
$\lambda = 1.00$	0.7157	1.8920	6.5327
$\lambda = 0.99$	0.7157	2.0222	6.3096
$\lambda = 0.96$	0.7157	2.4345	6.4014
$\lambda = 0.90$	0.7157	3.4843	157.7480

Table 5.3.10 Values of the loss function J. Estimation method: EF. Constant parameters and poor excitation for $t \geq 100$.

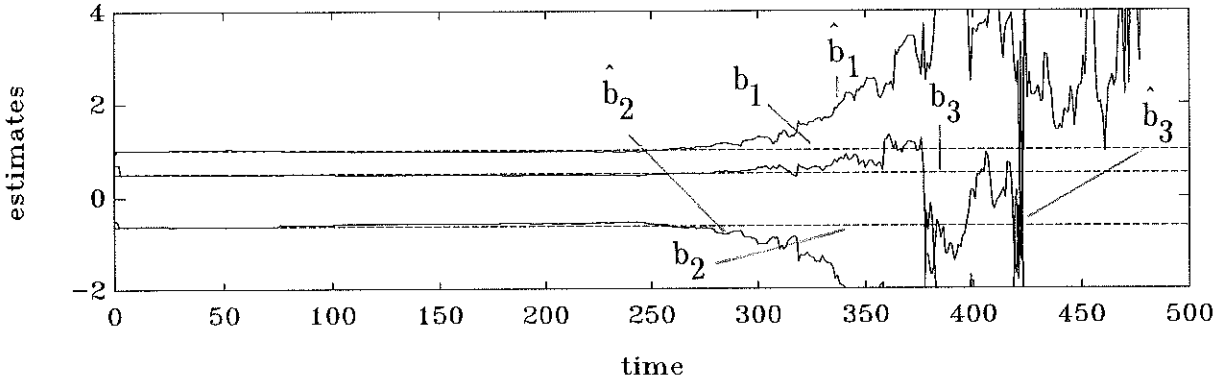

Figure 5.3.17 EF–estimates. $\lambda = 0.96$, $\sigma = 0.05$. Constant parameters and poor excitation for $t \geq 100$.

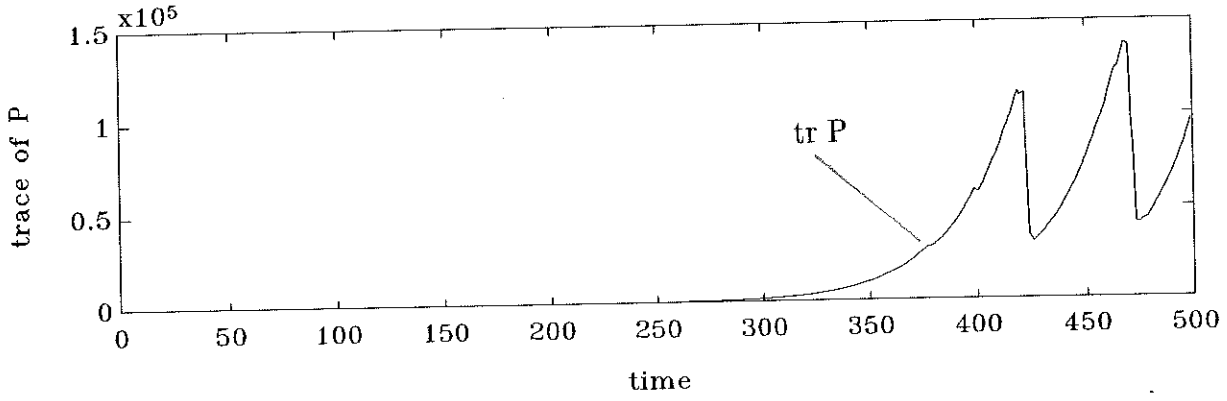

Figure 5.3.18 Trace of the P matrix for the EF method. $\lambda = 0.96$, $\sigma = 0.05$. Constant parameters and poor excitation for $t \geq 100$.

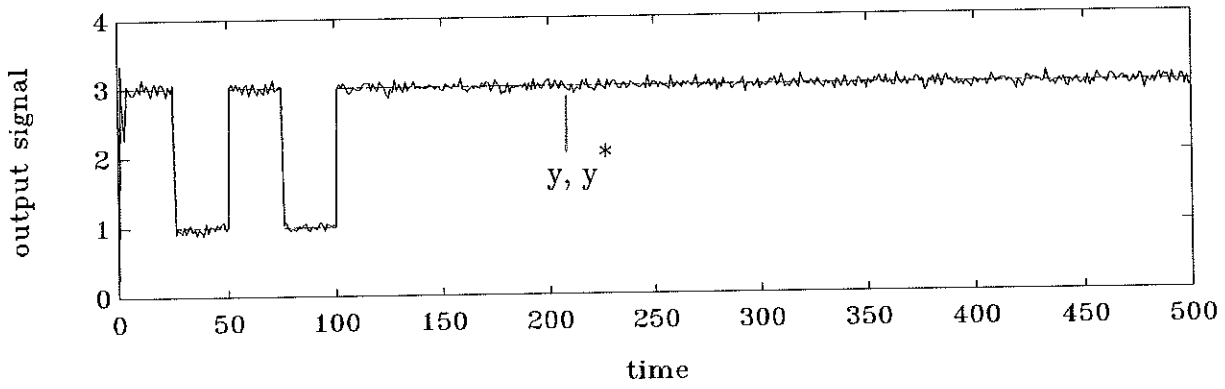

Figure 5.3.19 Output signal obtained with the EF method. $\lambda = 0.96$, $\sigma = 0.05$. Constant parameters and poor excitation for $t \geq 100$.

L	$\sigma = 0.00$	$\sigma = 0.05$	$\sigma = 0.1$
$\alpha_{\min} = 0.001$	0.0944	0.1518	0.2965
$\alpha_{\min} = 0.005$	0.0944	0.3794	1.4040
$\alpha_{\min} = 0.01$	0.0944	0.3447	2.2846
$\alpha_{\min} = 0.05$	0.0944	0.9125	10.9800
$\alpha_{\min} = 0.10$	0.0944	1.5434	8.8343

Table 5.3.11 Values of the loss function L. Estimation method: SF1. Constant parameters and poor excitation for $t \geq 100$.

J	$\sigma = 0.00$	$\sigma = 0.05$	$\sigma = 0.1$
$\alpha_{\min} = 0.001$	0.7157	2.0175	6.7913
$\alpha_{\min} = 0.005$	0.7157	2.1976	7.1263
$\alpha_{\min} = 0.01$	0.7157	2.6009	7.5121
$\alpha_{\min} = 0.05$	0.7157	2.4386	9.1535
$\alpha_{\min} = 0.10$	0.7157	2.8206	8.4409

Table 5.3.12 Values of the loss function J. Estimation method: SF1. Constant parameters and poor excitation for $t \geq 100$.

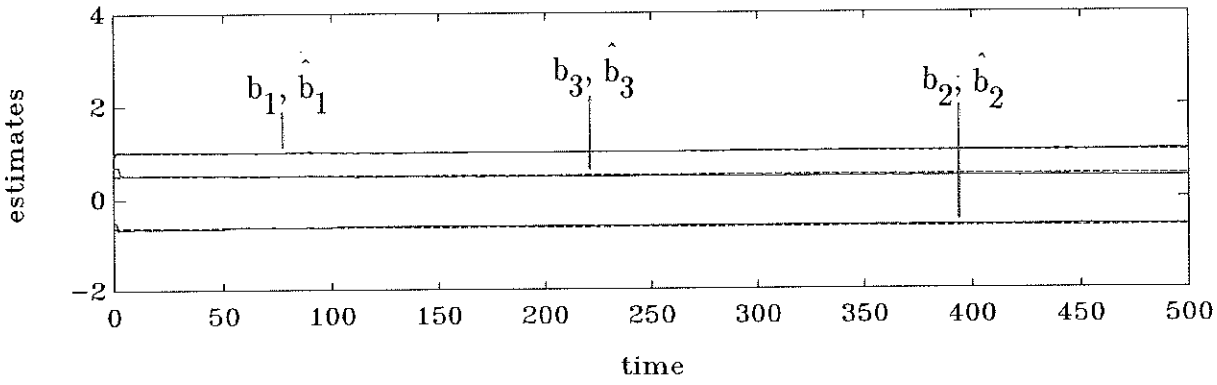

Figure 5.3.20 SF1–estimates. $\alpha_{\min} = 0.01$, $\alpha_{\max} = 1.0$, $\sigma = 0.05$. Constant parameters and poor excitation for $t \geq 100$.

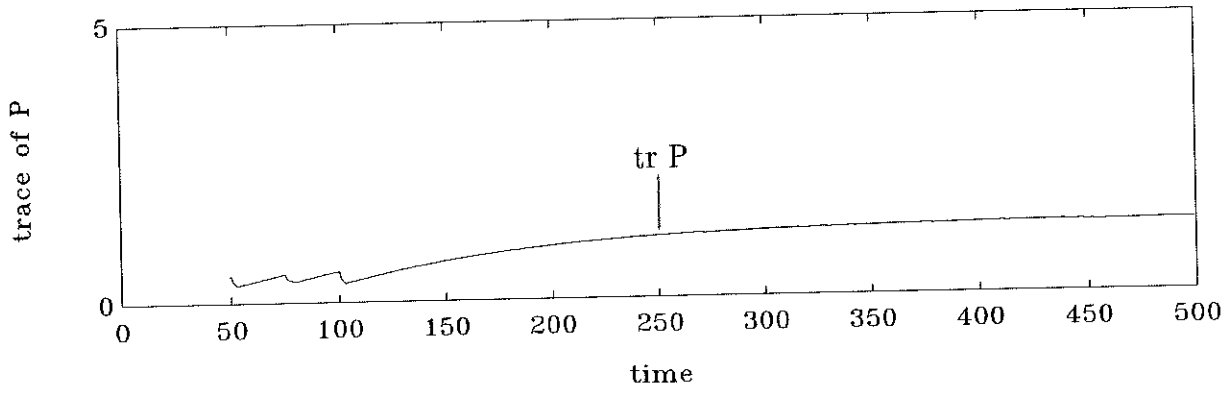

Figure 5.3.21 Trace of P for the SF method. $\alpha_{\min} = 0.01$, $\alpha_{\max} = 1.0$, $\sigma = 0.05$. Constant parameters and poor excitation for $t \geq 100$.

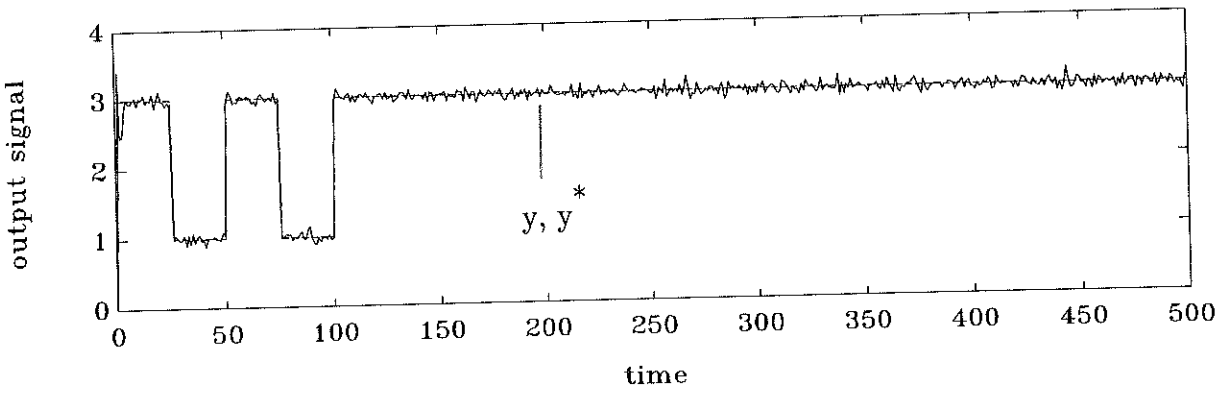

Figure 5.3.22 Output signal obtained with the SF1 method. $\alpha_{min} = 0.01$, $\alpha_{max} = 1.0$, $\sigma = 0.05$. Constant parameters and poor excitation for $t \geq 100$.

Again the SF method proves to have a much more robust behaviour. The P matrix remains bounded, and therefore undesired drift of the estimates in the period with poor excitation is avoided. Good results are obtained for all tested values of the tuning parameter.

In total, the results of the example illustrate that the sensitivity of the SF1 method to the choice of tuning parameters is small. The value of α_{min} has been changed with a factor 100 and the quality of the identification and control has remained good. The algorithm is flexible enough to handle all three test situations well.

∇

5.4 Summary

In this chapter a method based on selective forgetting has been formulated and studied.

An important restriction in the conventional exponential forgetting technique is that the same forgetting factor is used in all directions of the parameter space. As was discussed in chapter 3 this leads to problems both when the information flow does not excite all directions equally well and also when some parameters are known to have a faster variation than others.

In the selective forgetting method the parameters are aggregated into stochastically independent linear combinations, and individual forgetting factors are assigned to each linear combination. This new update principle makes it possible to discount information non–uniformly in the parameter space.

The selective forgetting method was first presented in Parkum (1988) and later published in Parkum, Poulsen and Holst (1990). In this chapter several improvements have been suggested. The theoretical virtues of the method have been examined, and it has been shown that if the choice of forgetting factors meets certain requirements, then the parameter covariance matrix will remain bounded from above and from below. An important detail is that this result will hold regardless of the experimental conditions. Furthermore, the diameters of the spherical bounds become tuning parameters for the method, i.e. they can be chosen prior to the experiment. Applying the results from chapter 4 it follows that the fundamental deterministic RLS properties are retained and that a persistent flow of information leads to exponential convergence of the estimates to their true values. In addition it has been verified that in periods without excitation the algorithm smoothly resets itself and eventually becomes a gradient method. This is a very desirable property since the information stored in the covariance matrix gradually loses its value if the parameters are time–varying. Finally it has been formally proved that the selective forgetting update always discounts more information in directions where more information has been accumulated.

It has been shown that it is possible to implement the selective forgetting time update for the covariance matrix as a polynomial. The choice of a first order polynomial has been recommended. The resulting estimation algorithm has all the theoretical properties mentioned above, and it has low computational requirements.

Finally, some simulation experiments have been presented. They confirm that the method performs well in connection with estimation of constant parameters, that it handles slow parameter variations very well and that it is robust in potential blow up situations. It has also been illustrated that the sensitivity to the choice of tuning parameters is relatively small.

6

Application example: adaptive control of the nitrification process in wastewater treatment

In this chapter an application example is studied. A non–linear adaptive controller for the nitrification process in wastewater treatment is developed. The estimation of a key parameter of the process model is important since this parameter cannot be measured directly. Hence, a recursive estimation algorithm is combined with a modified minimum variance control law to obtain an algorithm which is able both to produce reliable parameter estimates and to control the process.

Biological wastewater treatment based on the activated sludge processes has been in existence for many years. Traditionally, attention has been focused mainly on plant design rather than on operation. Control strategies have often been based on the experience of the operator. Sometimes one or two automatic, single loop controllers, which regulate for example pH or the dissolved oxygen concentration, have been applied. However, the increasing energy costs and the growing appreciation of the need for improved sewage treatment has lead to a greater interest in data acquisition and control of the processes – see for example Andrews (1975) and Olsson (1976). Considering the dynamic behaviour of wastewater treatment processes and the time–varying nature of the flow rate and composition of influent wastewater, it seems likely that application of advanced adaptive control techniques can lead to improved plant efficiency.

Traditionally, the major part of the research effort in the area of control has been centered around linear systems. However, biological systems are inherently non–linear, and a linear description is typically quite inadequate (Williamson, 1977). It is reasonable to believe that the best control results can be obtained if the control law is based on a simplified, but physically meaningful and realistic model capturing the fundamental non–linear interactions between the variables. This philosophy has been adopted by several authors, see for example Goodwin et al. (1982), Dochain and Bastin (1984) and Marsili–Libelli (1989).

The adaptive controllers presented by Goodwin et al. (1982) (dissolved oxygen control and pH neutralization) and Dochain and Bastin (1984) (bacterial growth systems) are based on the minimum variance principle described in chapter 2. The one–step ahead prediction of the output is set

equal to the desired reference value. The unknown parameters are regarded as constants to be estimated on–line using recursive least squares. A similar approach is applied in the present chapter, where control of the nitrification process is considered. However, certain modifications are necessary. The problem setting as well as the nonlinear model employed to describe the process naturally has its own distinctive characteristics, making it necessary to tailor the algorithm individually for this application. The minimum variance control law has to be modified since values of the output variable (the ammonia concentration) below the reference value are completely acceptable. Furthermore, the control signal is bounded. Hence, it is necessary to reconsider all details when studying the properties of the algorithm. The theoretical discussion will assume that the plant is time–invariant. Time–variations and modeling errors due to the simplification are accommodated by considering the parameter as a slowly drifting quantity which can be tracked by a recursive forgetting method.

Section 6.1 contains a brief discussion of the nitrification process. In section 6.2 the mathematical model is formulated and its predictions are compared to real life data. The adaptive controller is developed in section 6.3. An analytical result based on idealized assumptions is given in section 6.4. Finally, in section 6.5 the practical performance is examined via computer simulation.

6.1 Background

Nitrogen is an important nutrient for the growth of microorganisms and plants. However, several problems can be caused by the discharge of wastewater containing excess nitrogen:

* Excess nitrogen leads to undesirable algae and plant growth (eutrophication) in receiving waters.

* The nitrification process

$$2\ NH_4^+ + 3\ O_2 \rightarrow 2\ NO_2^- + 2\ H_2O + 4\ H^+$$

$$2\ NO_2^- + O_2 \rightarrow 2\ NO_3^-$$

  requires oxygen. If a significant concentration of nitrifying bacteria (nitrosomas and nitrobacter) is present in the recipient, excess ammonia nitrogen may lead to an oxygen sag.

* Ammonia nitrogen is toxic to fish.

Consequently an important task is to develop efficient methods for the removal of nitrogen from wastewaters. This is of particular importance when treating wastewaters that discharge to lakes or reservoirs. It is also necessary when treating wastewaters used for recharge of groundwaters for public supplies. Additional information on the various aspects of nitrogen pollution can be found for example in Albertson (1983).

In order to achieve efficient removal of the total amount of nitrogen (ammonia + nitrite + nitrate) the wastewater treatment plant has to perform not only nitrification, but also denitrification (conversion of nitrate nitrogen to nitrogen gas). Denitrification requires anoxic conditions, and consequently the processes cannot occur simultaneously. However, nitrification is normally the bottleneck in the nitrogen removing process, and therefore, in the sequel, the attention will be directed to this subproblem.

A single, completely mixed wastewater treatment reactor will be considered. The reactor may be part of a complex plant configuration. Oxygen is supplied to the reactor via an aeration equipment. Hereby the aerobic environment necessary for nitrification can be provided. It will be assumed that the dissolved oxygen process can be perfectly manipulated with a negligible time delay. This is a reasonable assumption due to the comparatively fast variation of the dissolved oxygen concentration and the availability of reliable sensors. (Dissolved oxygen controllers are described for example in Flanagan et al., 1977, Goodwin et al., 1982 and Holmberg, 1986).

The aim is to construct an algorithm determining an oxygen set point sequence leading to a satisfactory degree of nitrification with low operational costs. A common strategy is to keep the dissolved oxygen concentration constant at some high level. However, the oxygen supply is quite power demanding. Also, the load conditions may vary significantly. In some periods the concentration of ammonia in the inlet may be low and oxygen supply is perhaps not required. Consequently this strategy will lead to unnecessary economical expenses. In the sequel an alternative approach will be suggested.

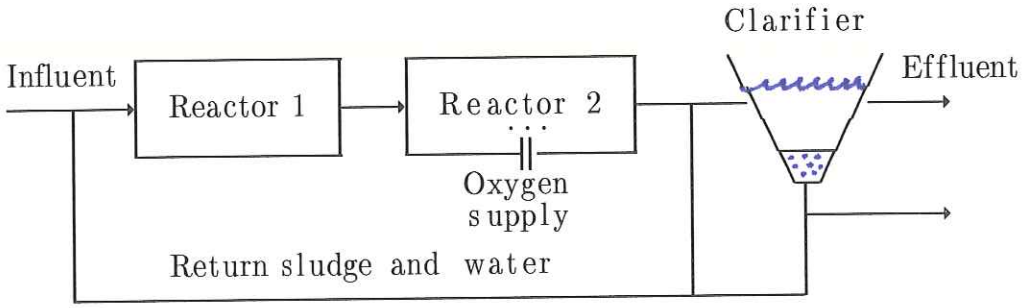

Figure 6.1.1 A typical wastewater treatment plant.

A typical example of a wastewater treatment plant where the considered control algorithm might be employed, is shown above. The system consists of two completely mixed reactors and a clarifier. Denitrification takes place in reactor 1, nitrification in reactor 2. In reactor 2 aerobic conditions are obtained by the use of an aerator. A fraction of the wastewater is recycled after passing reactor 2. In the clarifier the sludge is separated from the wastewater. In order to maintain a high concentration of microorganisms in the system, a part of the sludge is recycled.

6.2 A simplified model for the nitrification process

In this section a simplified model of the nitrification process in a completely mixed wastewater treatment reactor is formulated. The validity of the model is checked by comparing with real life data.

The nitrification process involves two successive steps: oxidation of ammonia (NH_4^+) to nitrite (NO_2^-) and of nitrite to nitrate (NO_3^-). The first step is rate limiting. Once ammonia has been converted into nitrite, the process will readily continue. Consequently nitrification can be modeled as a single step process.

A number of papers dealing with modeling of the nitrification process have been published, see for example Poduska and Andrews (1975), Stenström and Poduska (1980), Ohgaki and Wantawin (1989). Based on these references the following model can be applied to describe the development of the ammonia concentration in the reactor

$$\dot{S}(\tau) = -\frac{\mu_m}{Y_a} X(\tau) \frac{S(\tau)}{K_1 + S(\tau)} \frac{c(\tau)}{K_2 + c(\tau)} + \frac{q(\tau)}{V} (S_i(\tau) - S(\tau)) \tag{6.2.1}$$

where

S = reactor ammonia concentration
c = reactor dissolved oxygen concentration
X = reactor nitrifying biomass concentration
μ_m = maximum specific growth rate of nitrifiers
Y_a = yield coefficient
K_1, K_2 = half saturation constants
q = influent flow rate
S_i = influent ammonia concentration
V = reactor volume

It will be assumed that S, c, q and S_i can be measured on–line, and that the constants V, K_1 and K_2 are known. Typical values of K_1 and K_2 can be found in the literature – see for example Henze et al. (1986). However, it can be recommended that their values are determined and tabulated for the specific wastewater treatment plant of interest for different values of pH and temperature.

The first term in (6.2.1) is the conversion process rate, while the second term is due to transport. The process rate term is proportional to the concentration of nitrifying bacteria, X, and to the double Monod expression

$$\frac{S(\tau)}{K_1 + S(\tau)} \frac{c(\tau)}{K_2 + c(\tau)} \tag{6.2.2}$$

Normally the model is extended by including a differential equation for X. However, it is well known that the growth rate of nitrifiers is slow compared to the variation of ammonia. Furthermore, the concentration X cannot be measured directly. To avoid unnecessary complication of the model this quantity is therefore regarded as approximately constant.

Introducing

$$\theta = \frac{\mu_m}{Y_a} X \tag{6.2.3}$$

and

$$u(\tau) = \frac{c(\tau)}{K_2 + c(\tau)} \tag{6.2.4}$$

the following differential equation is obtained

$$\begin{aligned} \dot{S}(\tau) &= f(S(\tau),u(\tau),q(\tau),S_i(\tau)) \\ &= -\theta \frac{S(\tau)}{K_1 + S(\tau)} u(\tau) + \frac{q(\tau)}{V} (S_i(\tau) - S(\tau)) \end{aligned} \tag{6.2.5}$$

Assume that

(6.2.6) $$0 \leq q(\tau) \leq q_{max}$$

(6.2.7) $$0 \leq u(\tau) \leq u_{max} < 1$$

(6.2.8) $$0 \leq S_i(\tau) \leq S_{max}$$

and

(6.2.9) $$0 \leq S(0) \leq S_{max}$$

Then the ammonia concentration remains bounded:

(6.2.10) $$0 \leq S(\tau) \leq S_{max}$$

for all $\tau \geq 0$. This can be seen by noting that if

(6.2.11) $$S(\tau) \geq S_{max}$$

then

$$\dot{S}(\tau) \leq f(S_{max},0,q(\tau),S_i(\tau))$$

$$= \frac{q(t)}{V} (S_i(\tau) - S(\tau))$$

(6.2.12) $$\leq 0$$

On the other hand, if

(6.2.13) $$S(\tau) \leq 0$$

then

(6.2.14) $\dot{S}(\tau) \geq 0$

Hence, (6.2.10) will hold.

The purpose is to derive a discrete time control law, i.e. an algorithm for determination of a piecewise constant oxygen set point, the value of which is changed only at the sampling instants. For this purpose, a discrete time model is more suited. Using the sampling period T, the Euler approximation

(6.2.15) $$\dot{S}(\tau) \approx \frac{S(\tau+T) - S(\tau)}{T}$$

and the normalized time scale

(6.2.16) $t = \tau / T$

the following model is obtained

(6.2.17) $S(t+1) = S(t) + T\ f(S(t),u(t),q(t),S_i(t))$

The choice of sampling period has to be considered carefully. Too slow sampling leads to loss of information, while too fast sampling will increase the load on the computer handling the communication with the process. A reasonable compromise for the nitrification process seems to be $T \approx 1 - 10$ min, see for example Henze et al. (1986).

In order to check the validity of the resulting discrete time model, its predictions were compared with measurements from the sewage treatment plant at Frederikssund, Denmark. The measurements are from an experiment performed in 1978, see Behrens and Jansen (1979) for a detailed description. The variables of the model were measured every 10 minutes during 24 hours.

The model parameters were determined off–line with the least squares method. First the process $\{ S(t) \}$ was simulated from $t = 0$ feeding only the measured initial value $S(0)$ and the values of $q(t)$, $c(t)$ and $S_i(t)$ to the prediction model. This corresponds to the simulation model

$$\hat{S}_{sim}(t+1) = \hat{S}_{sim}(t) + T\ f(\hat{S}_{sim}(t),u(t),q(t),S_i(t)) \tag{6.2.18}$$

The result in figure 6.2.1 shows that the agreement is quite good. Clearly the model captures the basic features of the variation and gives a reliable description of the process. Next, the one step ahead prediction function

$$\hat{S}(t+1) = S(t) + T\ f(S(t),u(t),q(t),S_i(t)) \tag{6.2.19}$$

was compared with the data. The resulting residual sequence $\{ \epsilon(t) \}$ is plotted in fig. 6.2.2.

The auto correlation function $\rho_{\epsilon\epsilon}(k)$ and the cross correlation functions $\rho_{\epsilon c}(k)$, $\rho_{\epsilon S_i}(k)$ etc. were computed and compared with 95 % confidence intervals based on a normality assumption. The results indicated that the prediction errors can be regarded as white noise, i.e. the major part of the available information contained in data has been utilized.

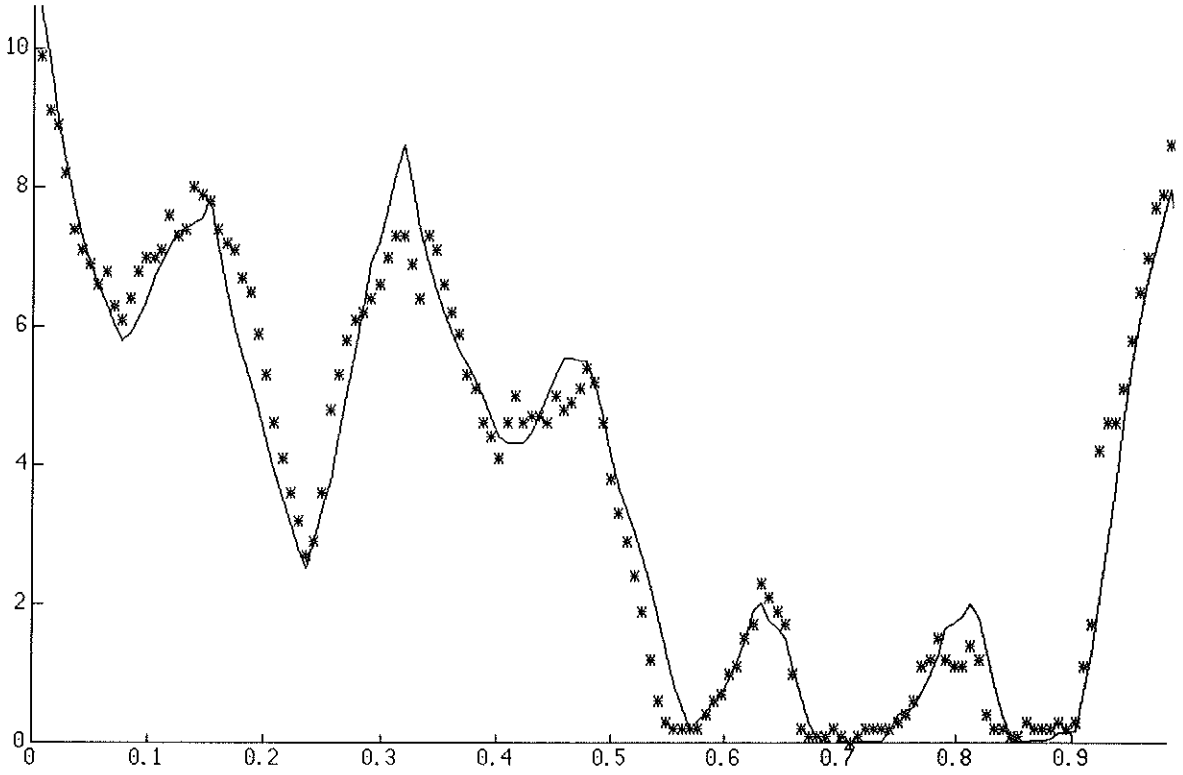

Figure 6.2.1 Simulated (——) and measured (*) values of the ammonia concentration.

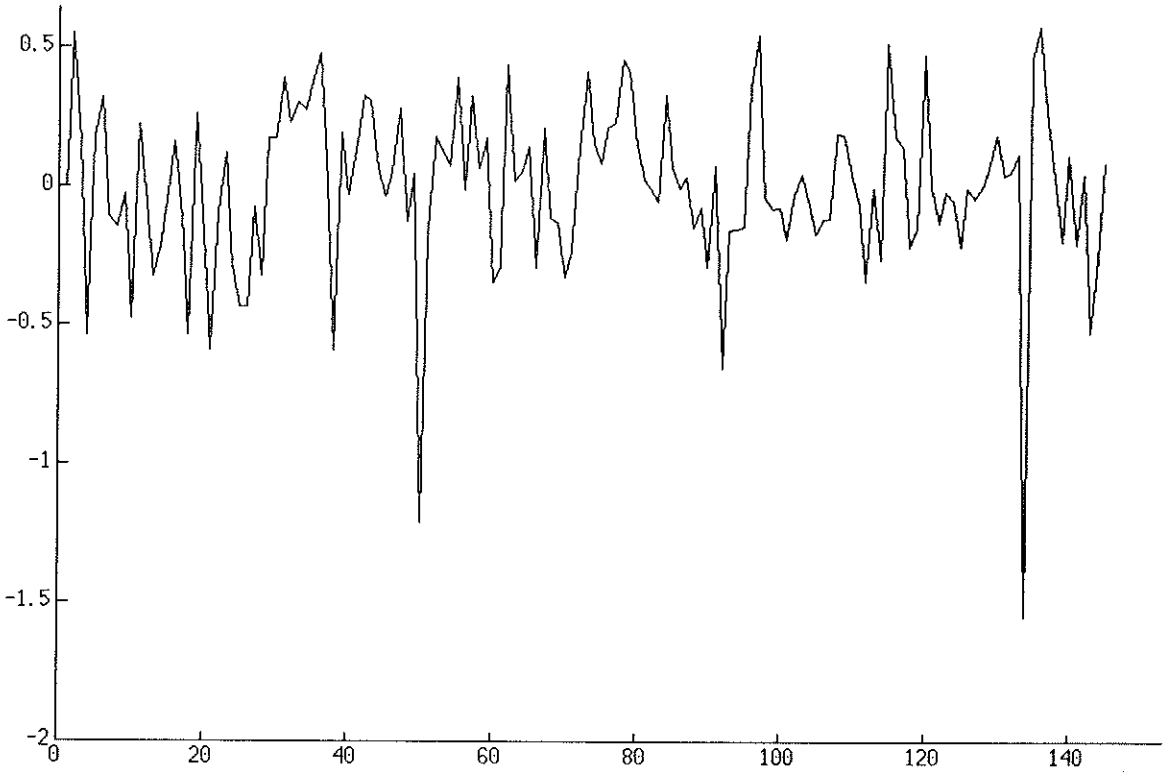

Figure 6.2.2 One step ahead prediction errors.

6.3 Adaptive control algorithm

Assume that an acceptable value S^* for the ammonia concentration $S(t)$ in the effluent can be specified. Let the control objective be to supply just the amount of oxygen needed for bringing $S(t)$ down to the level S^*. In order to save energy, no attempt is made to reduce $S(t)$ further. However, if the conditions are such that $S(t)$ becomes smaller than S^* when no oxygen is supplied, then this of course acceptable.

The adaptive controller consists of two parts: An identification algorithm producing on–line estimates of the unknown parameter θ and a control law.

Consider first the identification part. Introduce

$$\varphi(t+1) = - T \frac{S(t)}{K_1 + S(t)} u(t) \tag{6.3.1}$$

and the signal

$$y(t+1) = S(t+1) - S(t) - T \frac{q(t)}{V} (S_i(t) - S(t))$$

$$= \varphi(t+1)\ \theta \tag{6.3.2}$$

This is the linear regression model structure. In view of the relationship $\theta = \frac{\mu_m}{Y_a} X$, where X is the concentration of nitrifying biomass, it is reasonable to regard the parameter as slowly time–varying. Therefore a forgetting algorithm belonging to the general family described in chapter 3 will be applied for producing an estimate $\hat{\theta}(t+1|t)$ at time t. Since θ is known to be positive and in order to avoid division by zero in the control law described below, it is assumed that the estimate is bounded away from zero:

$$\hat{\theta}(t+1|t) \geq \theta_{min} > 0 \tag{6.3.3}$$

for some positive constant θ_{min}. This can be ensured by use of the technique

described in appendix 1. The one step ahead predictions of y and S now become

$$\hat{y}(t+1) = \varphi(t+1)\ \hat{\theta}(t+1|t) \tag{6.3.4}$$

and

$$\hat{S}(t+1) = \hat{y}(t+1) + S(t) + T\,\frac{q(t)}{V}\,(\,S_i(t) - S(t)\,) \tag{6.3.5}$$

The following control law is applied:

$$\begin{aligned} u(t) &= u^0(t) && \text{if } 0 \le u^0(t) \le u_{max} \\ &= u_{max} && \text{if } u^0(t) > u_{max} \\ &= 0 && \text{if } u^0(t) < 0 \end{aligned} \tag{6.3.6}$$

where

$$u^0(t) = \frac{S(t) + T\,\frac{q(t)}{V}\,(\,S_i(t) - S(t)\,) - S^*}{T\ \hat{\theta}(t+1|t)\,\frac{S(t)}{K_1 + S(t)}} \qquad \text{if } S(t) > 0$$

$$= 0 \qquad \text{else}$$

The idea of the control law can be explained in the following way for $S(t) > 0$: Let the prediction corresponding to $u(t) = 0$ be denoted $\hat{S}(t+1)|_{u=0}$, i.e.

$$\hat{S}(t+1)|_{u=0} = S(t) + T\,\frac{q(t)}{V}\,(\,S_i(t) - S(t)\,) \tag{6.3.7}$$

Whenever $\hat{S}(t+1)|_{u=0} \ge S^*$, $u^0(t)$ will be non–negative, and the value of $u(t)$ which will force $\hat{S}(t+1) = S^*$ is applied. However, $u(t)$ is restricted by a maximum value u_{max}, and this naturally has to be respected. If, on the other hand, $\hat{S}(t+1)|_{u=0} < S^*$, then no oxygen supply is required. In this

situation $u^0(t)$ will become negative, and the input signal $u(t)$ is set equal to zero.

6.4 Convergence analysis

In this section the purpose is to establish some basic convergence properties of the adaptive control algorithm outlined in section 6.3.

First a result concerning the accuracy of the ammonia predictions is given:

Theorem 6.4.1

Let the adaptive controller defined in section 6.3 be applied to the nitrification process (6.2.17). Assume that the parameter θ is constant and that the variables $q(t)$, $S_i(t)$ and $u(t)$ are bounded. Let $\hat{S}(t)$ be the prediction defined in (6.3.5). If the identification part of the adaptive controller consists of a forgetting algorithm fulfilling the assumptions mentioned in theorem 4.2.1, chapter 4, then

$$\lim_{t \to \infty} (S(t) - \hat{S}(t))^2 = 0 \qquad (6.4.1)$$

Proof

It was assumed in theorem 4.2.1 that the parameter covariance matrix $P(t|t-1)$ of the estimator is bounded from above, and it was shown that then the recursive estimator will have the property

$$\lim_{t \to \infty} \frac{(y(t) - \hat{y}(t))^2}{1 + \varphi^T(t)\, P(t|t-1)\, \varphi(t)} = 0$$

When the inlet variables are bounded then so is $S(t)$, see section 6.2. Hence, $\{ \varphi(t) \}$ and $P(t|t-1)$ are bounded, and

$$\lim_{t \to \infty} (y(t) - \hat{y}(t))^2 = 0$$

Since $y(t) - \hat{y}(t) = S(t) - \hat{S}(t)$, the result of the theorem follows directly.

□

In order to show that the adaptive controller asymptotically will have the desired performance, consider the function $\Omega(.,.)$ defined by

$$\Omega(z_1, z_2) = (z_1 - z_2)^2 \qquad \text{if } z_1 > z_2$$

(6.4.2)

$$= 0 \qquad \text{if } z_1 \leq z_2$$

From the discussion in section 6.3 it follows that ideally the adaptive controller should keep $\Omega(S(t), S^*)$ close to zero. This is equivalent with saying that $S(t)$ should be kept smaller than or equal to the acceptable level S^*. The following theorem formally establishes this property:

Theorem 6.4.2

Let the assumptions of theorem 6.4.1 be fulfilled. Define

$$M = S^* - T \frac{q_{max}}{V} S_{max} \tag{6.4.3}$$

and

$$L = T \left(- \theta \frac{M}{K_1 + M} u_{max} + \frac{q_{max}}{V} (S_{max} - M) \right) \tag{6.4.4}$$

If $M > 0$ and $L < 0$ then the closed loop system will have the property

$$\lim_{t \to \infty} \Omega(S(t), S^*) = 0 \tag{6.4.5}$$

Proof

The proof of theorem 6.4.2 requires a number of preliminary results. They will be given as four separate lemmas. Throughout the proof all assumptions behind theorem 6.4.1 are assumed to be fulfilled.

Recall that data are generated by

$$S(t+1) = S(t) + T\ f(S(t),u(t),q(t),S_i(t))$$

where

$$f(S,u,q,S_i) = -\theta \frac{S}{K_1 + S} u + \frac{q}{V} (S_i - S)$$

Now let ϵ be an arbitrary positive constant. The result of theorem 6.4.1 indicates that for some $t_1 > t_0$:

$$\| S(t) - \hat{S}(t) \| < \epsilon \qquad \text{for all } t \geq t_1 \tag{6.4.6}$$

In the sequel it will be assumed that $t \geq t_1$. Define the interval

$$\Upsilon = [\ 0,\ S^* + \epsilon\ ]$$

and the constants

$$M = S^* - T \frac{q_{max}}{V} S_{max}$$

$$L = T\ (\ -\theta \frac{M}{K_1 + M} u_{max} + \frac{q_{max}}{V} (\ S_{max} - M\ )\ )$$

Lemma 1

If $S(t+1) > S^* + \epsilon$ and $u(t) = u_{max}$ then

$$S(t+1) < S(t) + L$$

Proof

Since $u(t) = u_{max}$ the development of S is restricted by

$$S(t+1) - S(t) \leq T\ f(0, u_{max}, q_{max}, S_{max})$$

$$= T \frac{q_{max}}{V} S_{max}$$

Applying $S(t+1) > S^* + \epsilon$ this gives

$$S(t) > S^* + \epsilon - T \frac{q_{max}}{V} S_{max}$$

$$> M$$

Now it follows that

$$S(t+1) - S(t) < T\ f(M, u_{max}, q_{max}, S_{max})$$

or

$$S(t+1) < S(t) + L$$

□

Lemma 2

Assume that $M > 0$ and $L < 0$. If $u(t) = 0$ then

$$S(t+1) \in \Upsilon$$

Proof

First assume $S(t) = 0$. In this case

$$S(t+1) = T \frac{q(t)}{V} S_i(t)$$

and the result follows immediately using $M > 0$.

Now consider the case $S(t) > 0$. From the definition of the control law it can be concluded that $u^0(t) \leq 0$. Hence $\hat{S}(t+1) \leq S^*$. From (6.4.6) it follows that

$$S(t+1) < \hat{S}(t+1) + \epsilon$$

$$\leq S^* + \epsilon$$

□

Lemma 3

If $M > 0$ and $L < 0$ then $S(t)$ eventually will belong to Υ, i.e.

$$S(t) \in \Upsilon$$

for some $t = t_2 \geq t_1$.

Proof

Assume $S(t) \notin \Upsilon$ i.e. $S(t) > S^* + \epsilon$.

If $u(t) = 0$ then lemma 2 shows $S(t+1) \in \Upsilon$

Now consider the case $u(t) > 0$. Obviously $S(t) > 0$ and there are two possibilities:

1) If $u(t) = u^0(t)$ then $\hat{S}(t+1) = S^*$ and

$$S(t+1) < \hat{S}(t+1) + \epsilon$$

$$= S^* + \epsilon$$

i.e. $S(t+1) \in \Upsilon$

2) If $u(t) = u_{max}$ then lemma 1 and $L < 0$ ensures that S decreases at each step by an amount bounded away from zero, until eventually for some $t_2 > t_1$ S will belong to Υ, i.e. $S(t_2) \in \Upsilon$. If the sequence terminates before $S < S^* + \epsilon$ so that $S > S^* + \epsilon$ and $u \neq u_{max}$ then again $u = u^0$ and $S \in \Upsilon$.

□

Lemma 4

Let $M > 0$ and $L < 0$. If

$$S(t) \in \Upsilon$$

then

$$S(t+1) \in \Upsilon$$

Proof

It is given that $S(t) \in \Upsilon$. There are then three possibilities:

1) If $u(t) = u^0 \neq 0$ then $\hat{S}(t+1) = S^*$ and $S(t+1) \in \Upsilon$

2) If $u(t) = 0$ then lemma 2 shows $S(t+1) \in \Upsilon$

3) If $u(t) = u_{max}$ then assume $S(t+1) > S^* + \epsilon$. Lemma 1 shows

$$S(t+1) < S(t) + L$$

$$\leq S^* + \epsilon + L$$

$$< S^* + \epsilon$$

in contradiction with the assumption. Hence $S(t+1) \in \Upsilon$

□

Now theorem 6.4.2 can be proved: By induction using the lemmas 3 and 4 it follows that there exists a t_2 such that $S(t) \in \Upsilon$ for all $t \geq t_2$. Since ϵ can be chosen arbitrarily small the result of the theorem has been established.

□

Recall from chapter 4 that for the minimum variance adaptive controller designed to handle linear plants convergence of the output signal to the reference value can be established. Theorem 6.4.2 is analogous to this result. However, for the nitrification process values of the output signal smaller than the reference value are also acceptable. Therefore $\lim_{t \to \infty} \Omega(S(t), S^*)$ is considered instead of $\lim_{t \to \infty} (S(t) - S^*)$. As in chapter 4, a necessary requirement is that the parameter estimation algorithms has certain basic convergence properties. This is ensured by demanding that the parameter covariance matrix of the estimator is bounded.

Other key assumptions in theorem 6.4.2 are $M > 0$ and $L < 0$. Those demands are related to the fact that the input signal is restricted by the maximum value u_{max}. If, for example, the inlet ammonia concentration (maximum value S_{max}) becomes too high, or if the concentration of biomass in the reactor (proportional to θ) becomes too small, then those

assumptions will not longer hold, and it will not be possible to keep the ammonia concentration at the level S^*.

6.5 Simulation experiments

The results derived in the previous section show that in an idealized environment the adaptive ammonia controller has good theoretical properties. In this section the practical performance is studied via simulation experiments.

In the following examples the SF1 algorithm described in chapter 5 is employed as the identification part of the adaptive controller. The algorithm is implemented in such a way that the normalized parameter

$$\theta_1 = \frac{\theta}{500} \frac{\text{day m}^3}{\text{g}}$$

is estimated. The initial estimates are

$$\hat{\theta}_1(1|0) = 0.1$$

and

$$P_1(1|0) = 1000$$

The tuning parameters are

$$\alpha_{\min} = 0.01$$

$$\alpha_{\max} = 1.0$$

Example 6.5.1

The purpose in this example is to highlight some typical features of the adaptive controller. The data are generated by

$$S(t+1) = S(t) + T \left(- \theta \frac{S(t)}{K_1 + S(t)} u(t) + \frac{q(t)}{V} (S_i(t) - S(t)) \right)$$

where

$$T = 1 \ / \ 144 \text{ day} = 10 \text{ min}$$
$$K_1 = 0.8 \ \frac{g}{m^3}$$
$$V = 20.0 \ m^3$$

Hence, the model structure on which the control law is based is correct.

A constant reference value is applied

$$S^* = 1.5 \ \frac{g}{m^3}$$

and the inlet flow is also constant

$$q(t) = 100 \ \frac{m^3}{day}$$

The maximum value of the control signal is

$$u_{max} = 0.9$$

The first experiment focuses on the identification part of the algorithm.

Experiment no. 1

θ is proportional to the concentration of nitrifying bacteria, and can be expected to be time–varying. In this experiment a ramp variation is assumed for θ. The inlet ammonia concentration is a square wave with a moderately high level. The SF1 method overcomes the initialization error rapidly and tracks the parameter variation without great difficulties.

The following experiments illustrate the performance of the control law. Three different modes of operation will be shown. θ is constant:

$$\theta = 500 \ \frac{g}{m^3 \ day}$$

Experiment no. 2

The inlet ammonia concentration is as in experiment 1 and the value of $u^0(t)$ remains in the interval $0 \leq u^0(t) \leq u_{max}$ for all t. The algorithm acts like a one step ahead controller and $S(t)$ is kept at the level S^*. The variation of $S_i(t)$ is efficiently out–balanced.

Experiment no. 3

Now the experiment is repeated, but this time the value of the inlet ammonia concentration becomes very low, so that $u^0(t) < 0$ in a certain period. When the inlet concentration $S_i(t)$ is moderately high, the controller uses the one step ahead principle. However, in the period where $S_i(t)$ is close to zero, no nitrification is required and the controller sets $u(t) = 0$. The observed behaviour is in accordance with the control objective where values of S below S^* are acceptable.

Experiment no. 4

The simulation is repeated but with a very high ammonia concentration for $300 \leq t \leq 400$. In this period $u^0(t)$ becomes larger than u_{max}, and the controller is unable to keep $S(t)$ at the level S^*. However, $u(t)$ is set to the maximum value, i.e. oxygen is supplied at the highest possible rate. This is the best that can be done with the given control possibilities.

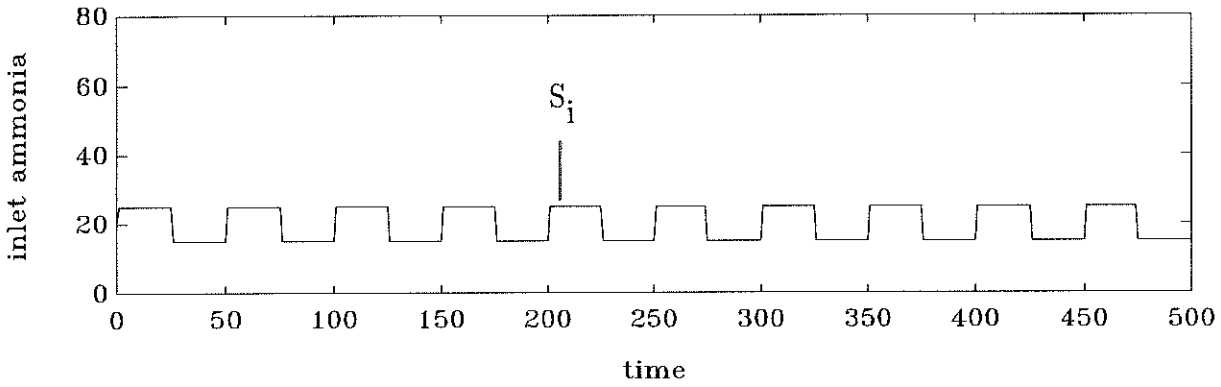

Figure 6.5.1 Inlet ammonia concentration in experiments no.1 and 2.

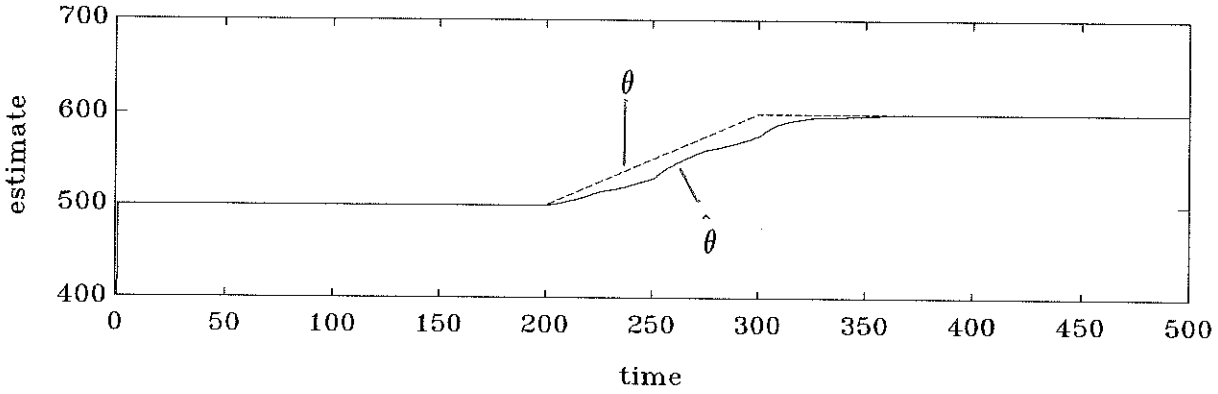

Figure 6.5.2 Tracking of the time–varying parameter $\theta(t) = 500\ \theta_1(t)$ in experiment no. 1. Estimation method: SF1.

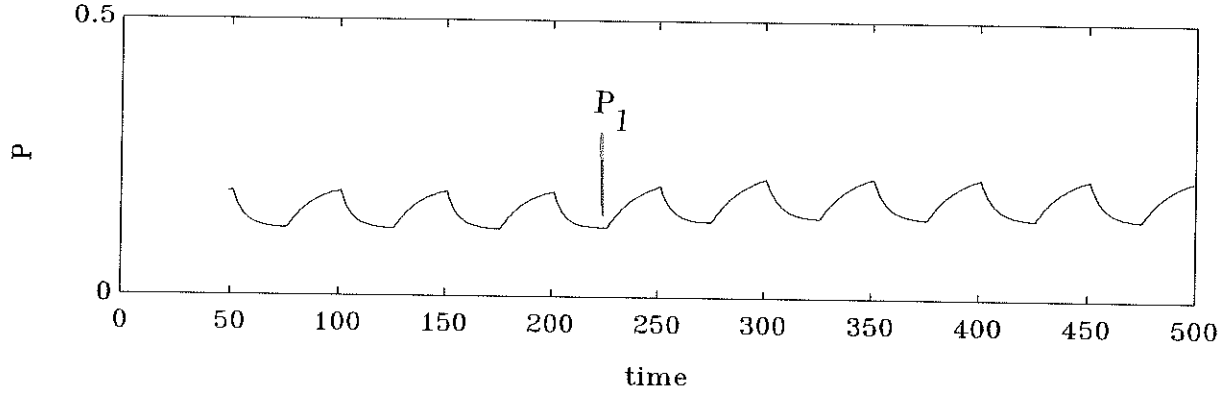

Figure 6.5.3 The parameter covariance $P_1(t)$ obtained with the SF1 method in experiment no. 1.

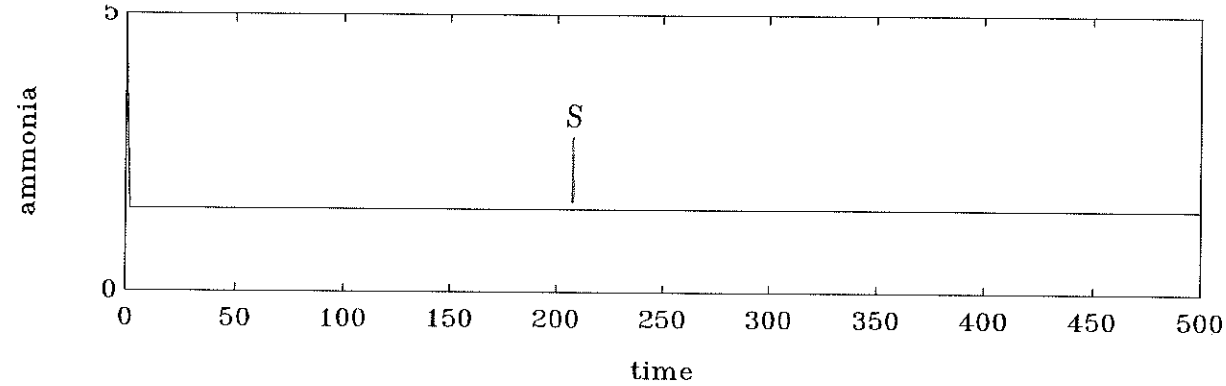

Figure 6.5.4 Reactor ammonia concentration in experiment no. 2. The value of $S_i(t)$ is moderately high for all t.

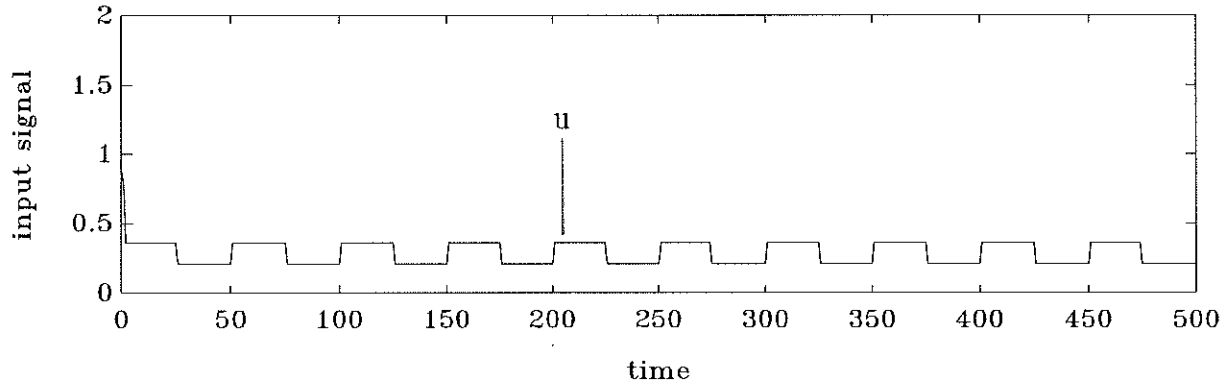

Figure 6.5.5 Control signal in experiment no. 2. The value of $S_i(t)$ is moderately high for all t.

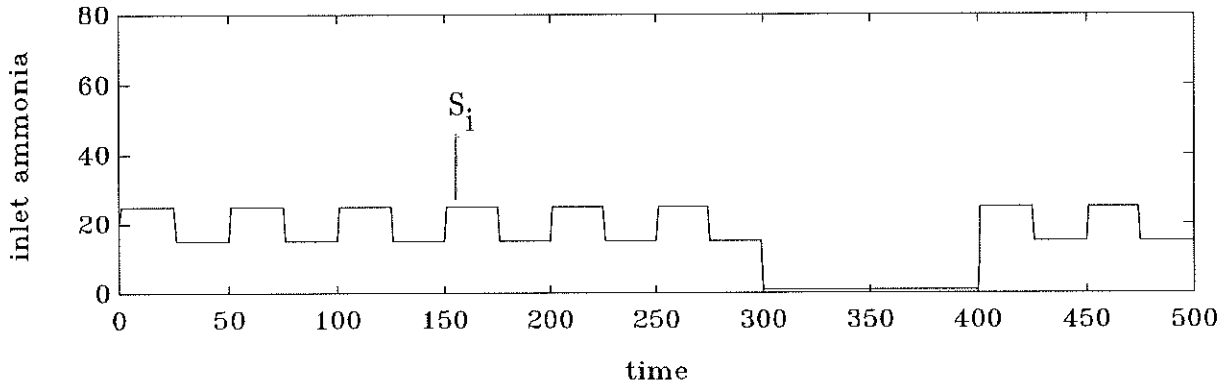

Figure 6.5.6 Inlet ammonia concentration in experiment no.3.

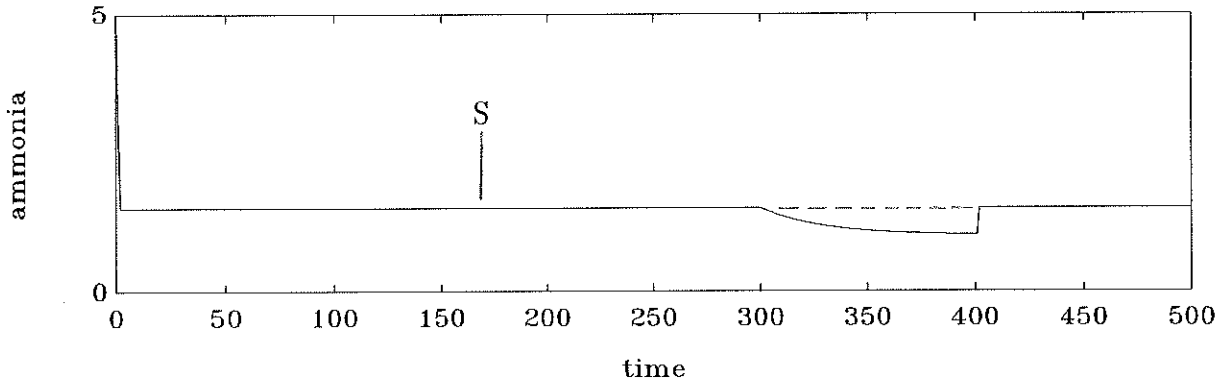

Figure 6.5.7 Reactor ammonia concentration in experiment no. 3. The value of $S_i(t)$ is low for $300 \leq t \leq 400$.

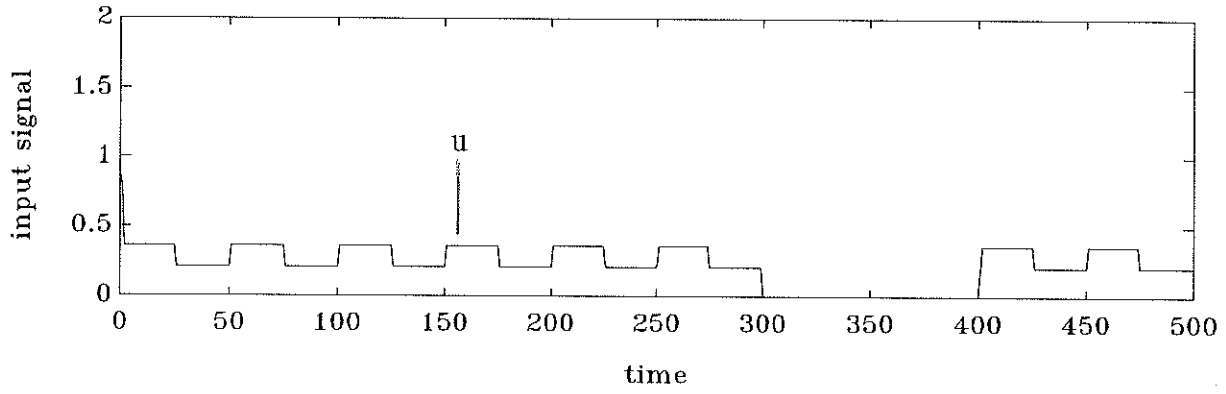

Figure 6.5.8 Control signal in experiment no. 3. The value of $S_i(t)$ is low for $300 \leq t \leq 400$.

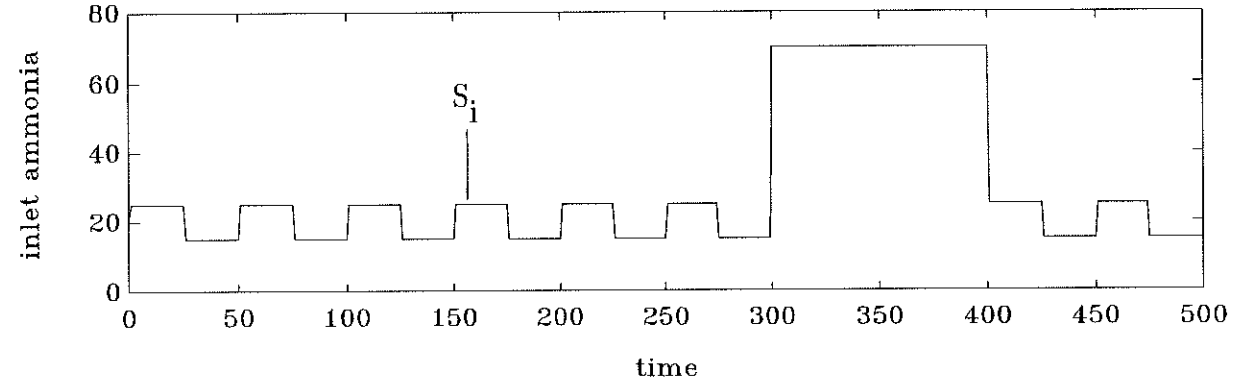

Figure 6.5.9 Inlet ammonia concentration in experiment no.4.

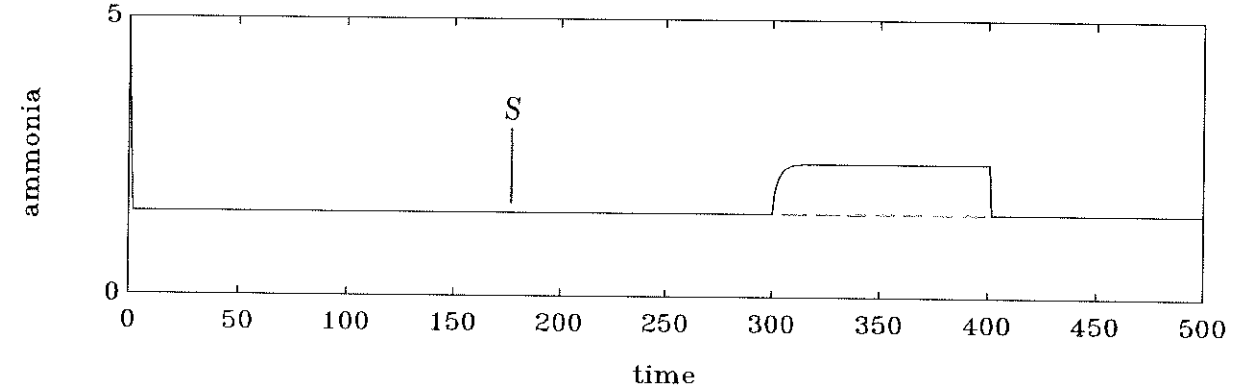

Figure 6.5.10 Reactor ammonia concentration in experiment no. 4. The value of $S_i(t)$ is high for $300 \leq t \leq 400$.

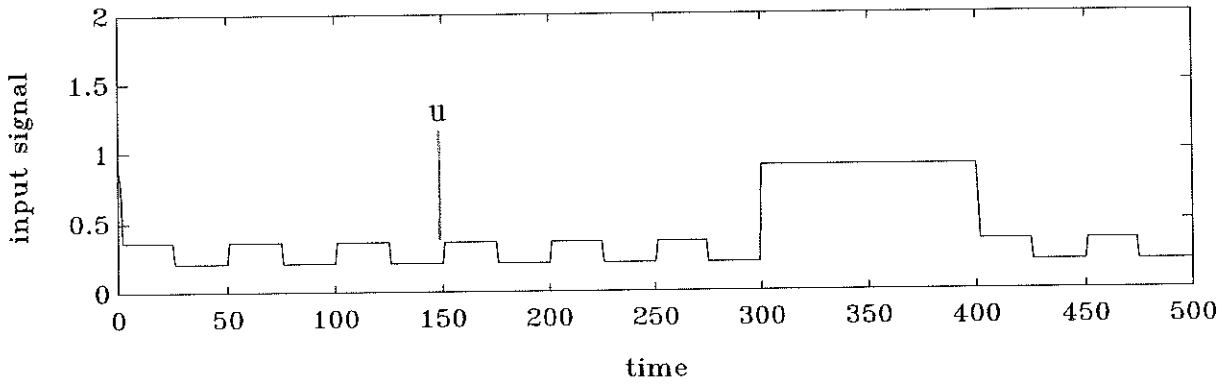

Figure 6.5.11 Control signal in experiment no. 4. The value of $S_i(t)$ is high for $300 \leq t \leq 400$.

∇

Example 6.5.2

Now let data be generated by the extended model

$$S(t+1) = S(t) + T \left(-\frac{\mu_m}{Y_a} X(t) \frac{S(t)}{K_1 + S(t)} u(t) + \frac{q(t)}{V} (S_i(t) - S(t)) \right)$$

$$X(t+1) = X(t) + T \left(\frac{\mu_m}{Y_a} X(t) \frac{S(t)}{K_1 + S(t)} u(t) - b_a X(t) + \frac{q(t)}{V} (X_i(t) - X(t)) \right)$$

where

$$T = 1 / 144 \text{ day} = 10 \text{ min}$$
$$K_1 = 0.8 \frac{g}{m^3}$$
$$V = 20.0 \text{ m}^3$$
$$\mu_m = 0.9 \text{ day}^{-1}$$
$$Y_a = 0.24$$
$$b_a = 0.1 \text{ day}^{-1}$$

$X(t)$ is the concentration of nitrifying bacteria, and as mentioned previously the value of this variable cannot be measured on–line.

The reference signal is $S^* = 1.5 \frac{g}{m^3}$, and $u_{max} = 0.9$. The inlet concentration $X_i(t)$ is simulated as a ramp signal. The true values of $S_i(t)$ and $q(t)$ are generated as in example 6.5.1, experiment no. 2. However, the measured signals are now

$$q_m(t) = q(t) + e_1(t)$$
$$S_{i,m}(t) = S_i(t) + 0.1\ e_2(t)$$
$$S_m(t) = S(t) + 0.1\ e_3(t)$$

where $\{e_1(t)\}$, $\{e_2(t)\}$ and $\{e_3(t)\}$ are N(0,1) white noise processes.

Hence, the adaptive controller is implemented exactly as in example 6.5.1, but now the measured signals are polluted with noise and the true system is more complex than the assumed model.

Results from the simulation are shown below. It can be seen that the SF1 algorithm is able to provide a reliable on–line estimate of the concentration of nitrifying bacteria. It is also clear that efficient control of the ammonia concentration is obtained in spite of the disturbances and time–variations.

It can be added that good results also have been obtained in similar experiments where the IAWPRC model developed in Henze et al. (1986) has been employed to imitate the true system. This model includes phenomena such as carbon oxidation, nitrification and denitrification. It contains 13 coupled, nonlinear first order differential equations, and gives a realistic description of the dynamics of the activated sludge processes. The identifiability of this model was studied in Parkum and Poulsen (1990).

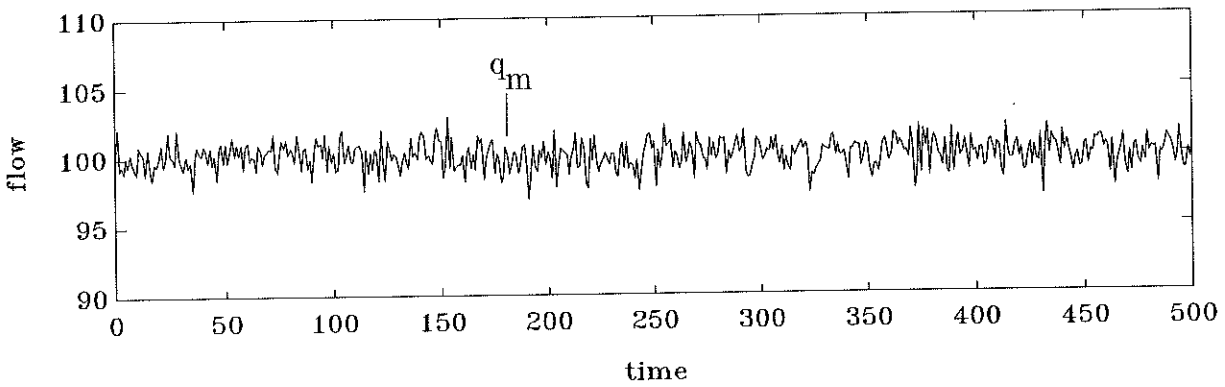

Figure 6.5.12 Measured inlet flow.

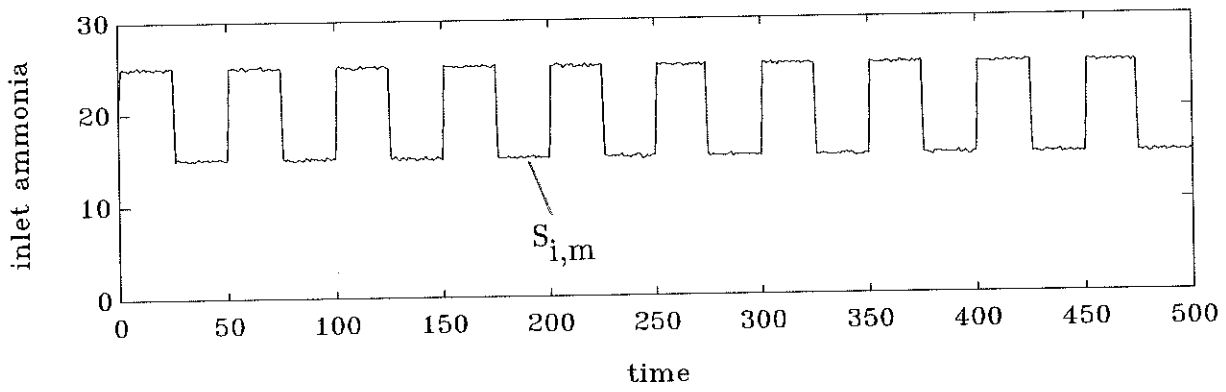

Figure 6.5.13 Measured inlet ammonia concentration.

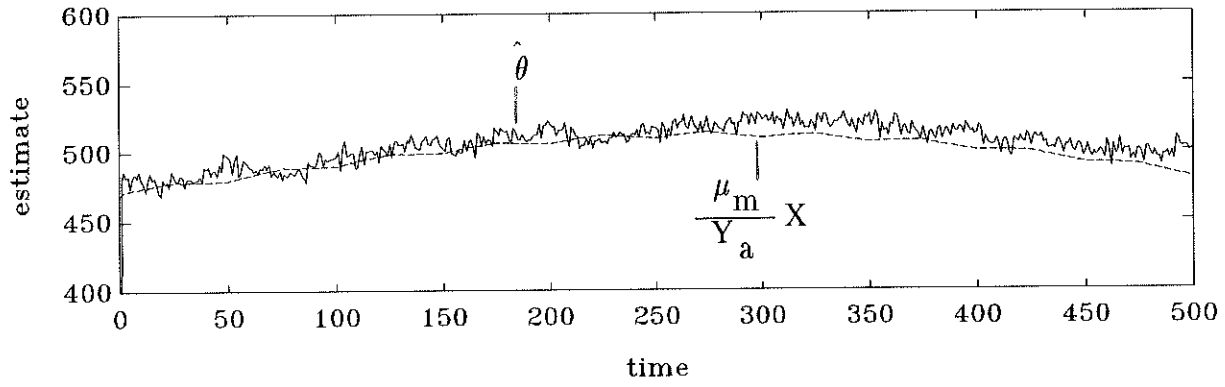

Figure 6.5.14 Parameter estimate and true value of $\frac{\mu_m}{Y_a} X(t)$.

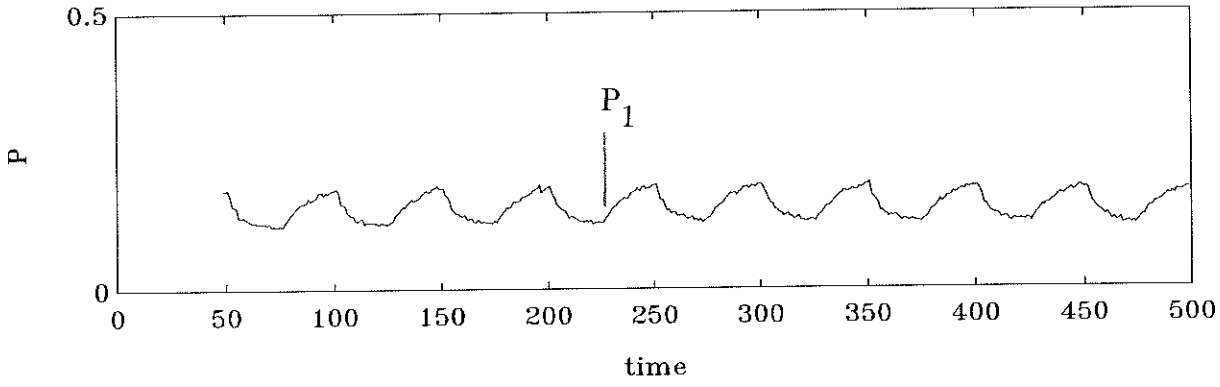

Figure 6.5.15 Parameter covariance $P_1(t)$.

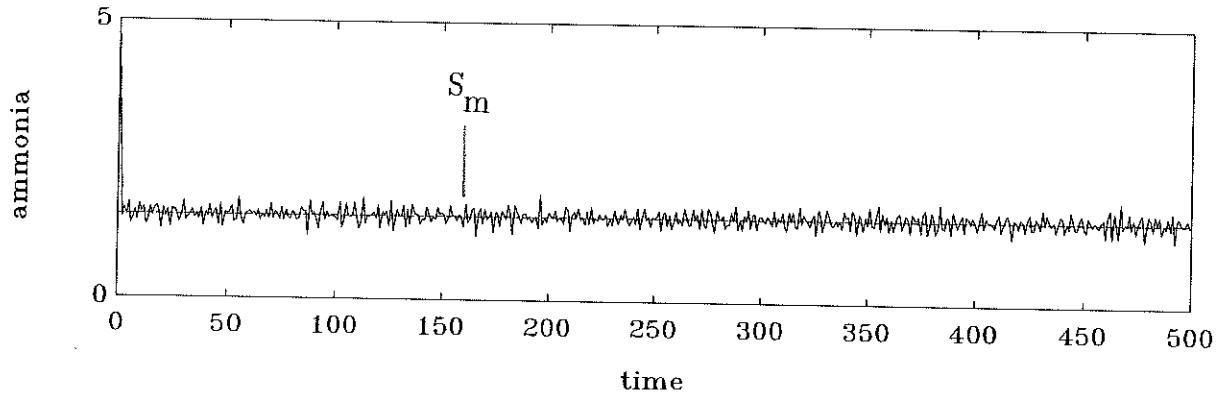

Figure 6.5.16 Measured reactor ammonia concentration.

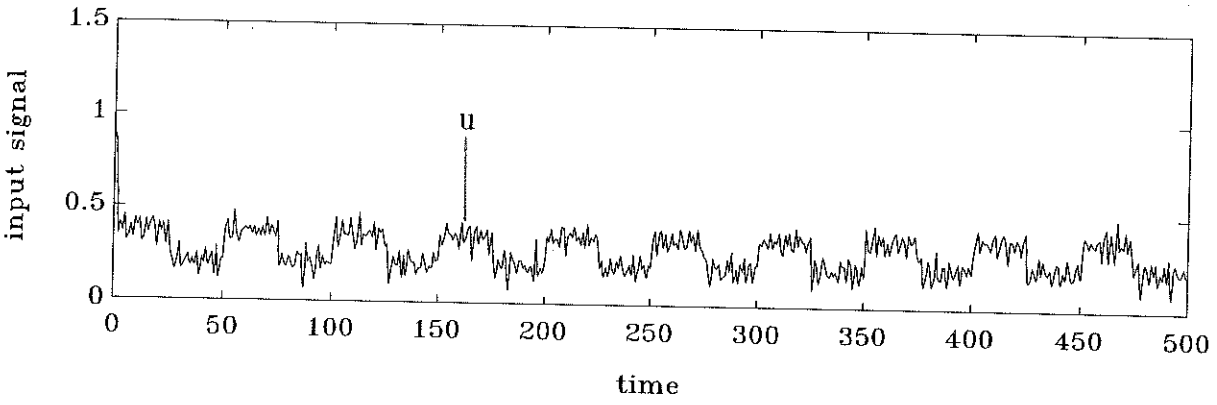

Figure 6.5.17 Control signal.

∇

6.6 Summary

In this chapter an application example has been studied. An adaptive controller for the nitrification process, which is important in connection with wastewater treatment, has been developed.

The nitrification process is coupled to several other processes taking place simultaneously in wastewater treatment. The modeling is problematic since a large number of equations, variables and parameters are required to describe the process accurately. Based on a literature study a simple non–linear model has been chosen. The ability of the model to explain a large part of the variation has been verified by comparing its predictions with real–life data from a wastewater treatment plant. One parameter in the model is actually proportional to the concentration of nitrifying bacteria, and it can therefore be expected to be slowly time–varying.

Assuming that the oxygen level in the wastewater treatment reactor can be directly manipulated via an aeration equipment, a control law has been developed. The control law is based on a modified version of the one step ahead principle described in chapter 2.

By combining the controller with an on–line identification algorithm, adaptivity can be obtained. Since the estimator has to be able to cope with time–variations, it is natural to apply one of the forgetting algorithms described in this thesis.

The adaptive controller has been examined theoretically. Under a set of strict assumptions it has been proved that asymptotically the controller manages to avoid that the ammonia concentration exceeds a certain prespecified level. The result assumes that the estimator has the basic RLS properties discussed in chapter 4.

The practical performance of the adaptive controller has been studied through extensive simulation experiments, some of which have been presented in this chapter. In all cases the non–linear adaptive controller has behaved well. A reliable on–line estimate of the concentration of nitrifying bacteria has been

generated. The algorithm has been able to cope with varying inlet concentrations, varying flow rates and changing set points. Good results have been obtained with extended and realistic models imitating the true system.

7

Conclusion

The present thesis is addressed to methods for on–line identification of time–varying systems.

A survey of on–line estimation methods designed to track slowly drifting system parameters has been presented. The attention has been focused on forgetting methods. The perhaps best known representative for these is the conventional exponential forgetting method which discounts information uniformly in time and in parameter space.

In situations where no new information is received, the use of a constant forgetting factor will gradually lead to a total loss of the information stored in the estimator. This phenomenon is called covariance wind up since the elements of the parameter covariance matrix start to grow unboundedly. It results in very noise sensitive estimates. In order to improve the robustness of the method, variants with a time–varying forgetting factor have been developed in the literature. They typically turn off the forgetting in periods where the measurements do not provide a sufficient amount of information. However, when the information is non–uniformly distributed on the directions of the parameter space the exponential forgetting principle becomes less suitable. Instead it can be recommended to employ a method which adapts the forgetting profile to the current nature of the information flow. Loosely speaking, discounting should only take place in the directions where the information is continuously renewed. A directional forgetting method based on this principle has been discussed in the thesis.

Another problem occurs when the parameters are known to have different rates of variation. The exponential forgetting method applies the same forgetting factor to all directions and consequently gives no possibility to utilize the a priori knowledge about the parameter variations. Two possible solutions have been discussed. One is based on the update scheme formally derived by assuming random walk behaviour for the parameters and by applying the Kalman filter. Another employs a matrix of forgetting factors and connects individual discount rates to the parameters.

In the present thesis a general algorithm which contains most existing forgetting methods as special cases has been suggested. The algorithm is

written in a form which is known from the Kalman filter, i.e. with measurement and time update equations. The choice of time update for the parameter covariance matrix becomes the point which separates the individual methods. It represents the ad hoc discounting principle on which the method is based.

An important benefit resulting from the formulation of a general forgetting method is that it becomes possible to perform a unified theoretical analysis. Results for the deterministic, time–invariant special case have been presented. It has been proved that if the choice of time update guarantees boundedness from above for the parameter covariance matrix, then certain fundamental least squares convergence properties will be retained. These properties imply global stability and asymptotically perfect tracking for the adaptive controller obtained by combining the estimator with a one step ahead control law. If the covariance matrix is bounded from below as well as from above, then in addition the estimation algorithm will be exponentially convergent provided that data are sufficiently informative. Exponential convergence is important since it implies robustness for example in connection with slowly drifting system parameters.

The general analytical results simplify the convergence question. When examining a particular scheme the above mentioned properties can be established by simply verifying that the conditions on the parameter covariance matrix are fulfilled. This can often be checked without great difficulties. As an example, a convergence result has been derived for the matrix forgetting method.

As an attempt to solve some of the problems occurring when using the examined existing techniques, a new recursive estimation algorithm based on a principle of selective forgetting has been developed. The idea is to connect individual forgetting factors to stochastically independent parameters. Hereby it is ensured that each forgetting factor will act only on the corresponding parameter. The forgetting profile can now be adjusted in accordance with the rate of variation and the level of excitation for each parameter.

In its most general form the selective forgetting algorithm requires calculation of eigenvectors and eigenvalues for the parameter covariance matrix at each sampling instant. However, it has been shown how this can be avoided without giving up the selective update principle by implementing the time update as a matrix polynomial. The choice of a first order polynomial has been recommended. This version of the method has the same order of complexity as the exponential forgetting method.

The theoretical properties of the selective forgetting algorithm have been studied. It has been shown that under certain conditions connected with the choice of forgetting factors, the parameter covariance matrix of the method will be bounded from above and from below. Hence, the covariance wind up phenomenon is completely avoided and the tracking ability is retained. The bounds on the covariance matrix are tuning parameters of the method, i.e. they do not depend on the nature of the experimental data. Applying the analytical results derived for the general forgetting method it can be concluded that the algorithm has the fundamental least squares properties and that it is exponentially convergent in case of persistently exciting data. In addition it has been proved i.a. that the algorithm smoothly resets itself and becomes a gradient algorithm in case the information flow turns off. In this situation the directional information stored in the parameter covariance matrix gradually becomes obsolete, and it is better to apply a gain proportional to the identity matrix.

The practical properties of the selective forgetting algorithm have been studied via simulation experiments. The method has generated reliable estimates in all the test situations. It is able to cope with slowly drifting parameters and periods of poor information.

As an application example an adaptive controller for the nitrification process has been developed. The nitrification process is important in connection with removal of ammonia from wastewater. A single completely mixed wastewater treatment reactor has been considered. It has been assumed that the oxygen level of the reactor can be perfectly manipulated. Based on a simplified non–linear model of the nitrification process, a modified version of the well known one step ahead control law has been suggested. A particular parameter

in the model can be expected to vary slowly in time. Consequently it has been suggested that an estimate of this parameter is updated on–line using one of the forgetting methods belonging to the general family formulated previously.

The performance of the resulting adaptive controller has been examined analytically assuming that the model is identical to the true system, that the inlet variables are bounded and that the environment is deterministic and time–invariant. It has been shown that if the estimator has the fundamental error properties, then the controlled process will eventually behave ideally in the sense that the ammonia concentration will become smaller than or equal to a prespecified acceptable level.

Results from simulation experiments indicate that the adaptive controller works as intended even in more realistic situations. Good performance has been observed in simulation experiments where the true system was imitated by an extended model, and where the measurements were polluted by noise.

The work presented in this thesis can be continued on several fronts. For example it would be interesting to extend the analysis of the general forgetting algorithm. It is probably possible to derive explicit expressions for the tracking ability as a function of the chosen time update. Another task might be to generalize the convergence analysis to the case where stochastic disturbances are present.

In connection with the ammonia controller it would be interesting to apply the technique in a pilot experiment. It is also worth noting that typical models for other wastewater treatment and fermentation processes have close resemblance to the one studied here. It is therefore likely that similar ideas can be applied for identification and control of those processes.

8

Appendices

Appendix 1: Constraining the estimates

Appendix 1: Constraining the estimates

This appendix describes how the forgetting algorithms discussed in the report can be extended by inclusion of a facility for constraining the estimates to a prespecified region. The technique is described in Goodwin and Sin (1984).

Let θ be a p–dimensional vector and let P be a symmetrical, positive definite (p x p)–matrix. Define

$$\rho = P^{-\frac{1}{2}} \theta \tag{A1.1}$$

where

$$P^{-1} \stackrel{d}{=} (P^{-\frac{1}{2}})^T P^{-\frac{1}{2}} \tag{A1.2}$$

Let D be a closed convex region in R^p and let $\overline{D}$ be the image of D obtained via the mapping (A1.1) . Define g(.) by

$$g(\rho) = \rho \quad \text{if } \rho \in \overline{D}$$

$$= \rho' \quad \text{if } \rho \notin \overline{D} \tag{A1.3}$$

where ρ' is the ortogonal projection of ρ onto the surface of $\overline{D}$

Now consider the following extention of the general forgetting algorithm:

Measurement update :

$$\hat{\theta}(t|t) = \hat{\theta}(t|t-1) + P(t|t)\ \varphi(t)\ \epsilon(t) \tag{A1.4}$$

$$P^{-1}(t|t) = P^{-1}(t|t-1) + \varphi(t)\varphi^T(t) \tag{A1.5}$$

Parameter constraint

(A1.6) $$\hat{\theta}'(t|t) = P^{\frac{1}{2}}(t|t)\, g(\, P^{-\frac{1}{2}}(t|t)\, \hat{\theta}(t|t)\,)$$

(A1.7) $$P'(t|t) = P(t|t)$$

Time update:

(A1.8) $$\hat{\theta}(t+1|t) = \hat{\theta}'(t|t)$$

(A1.9) $$P(t+1|t) = F\,\{P'(t|t),\ \gamma(t)\}$$

$\gamma(t)$ represents the tuning parameters of the estimator. The extension (A1.6) – (A1.7) ensures that

(A1.10) $$\hat{\theta}(t+1|t)\ \epsilon\ D$$

for all t.

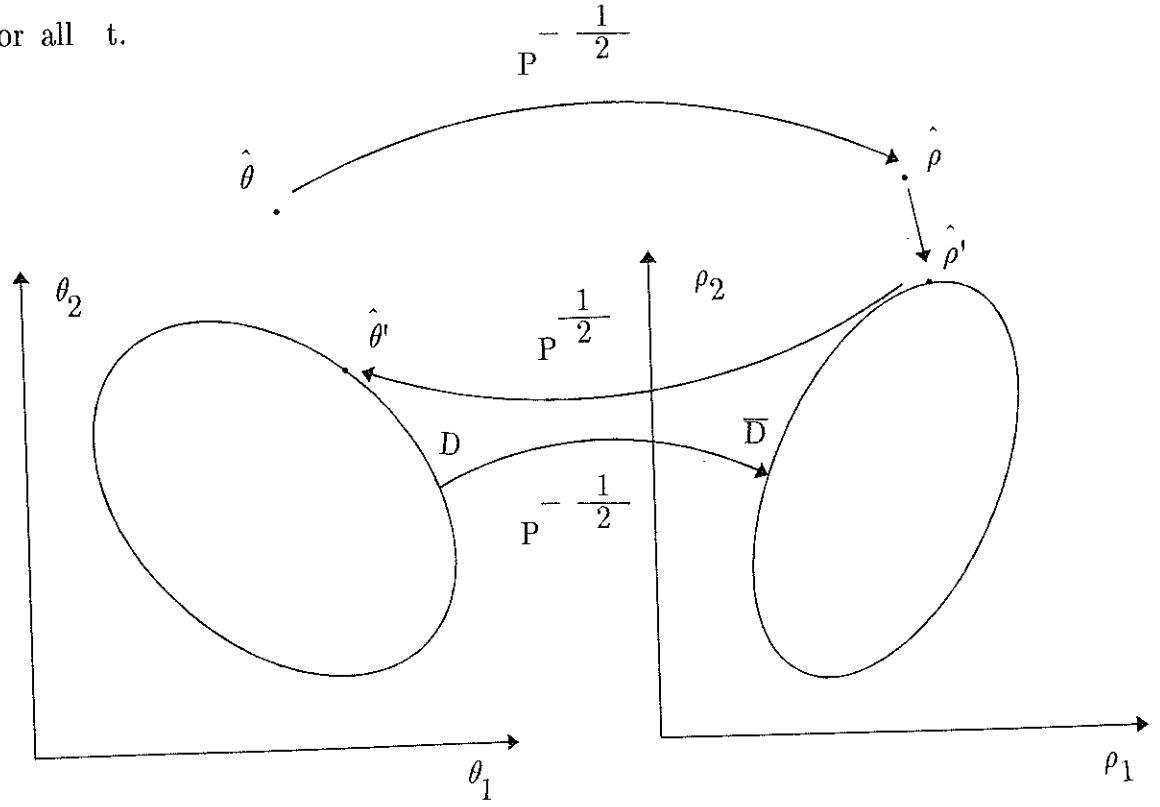

Figure A1.1 Parameter constraint algorithm

Notice that if $\rho^0 \in \overline{D}$ than

(A1.11) $$\| \rho^0 - g(\rho) \|^2 \leq \| \rho^0 - \rho \|^2$$

Using this inequality it can readily be seen that the modification acts only to decrease the Lyapunov function

(A1.12) $$V(t) = \tilde{\theta}^T(t)\ P^{-1}(t|t)\ \tilde{\theta}(t)$$

Consequently, convergence results relying on the non–increasing property of $V(t)$ are retained when including the parameter constraint equations.

9

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