The Voronoi diagram of circles made easy

François Anton
Department of Informatics and Mathematical Modelling
Technical University of Denmark, Richard Petersens Plads
DK-2800 Kongens Lyngby, Denmark
E-mail: fa@imm.dtu.dk

Darka Mioc
Department of Geodesy and Geomatics Engineering
University of New Brunswick, P.O. Box 4400
Fredericton, New Brunswick, Canada, E3B 5A3
E-mail: dmioc@unb.ca

Christopher Gold
School of Computing, University of Glamorgan
Pontypridd, CF37 1DL, UK
E-mail: cmgold@glam.ac.uk

Abstract

Proximity queries among circles could be effectively answered if the Delaunay graph for sets of circles could be computed in an efficient and exact way. In this paper, we first show a necessary and sufficient condition of connectivity of the Voronoi diagram of circles. Then, we show how the Delaunay graph of circles (the dual graph of the Voronoi diagram of circles) can be computed exactly, and in a much simpler way, by computing the eigenvalues of a two by two matrix.

1. Introduction

The proximity queries among circles could be effectively answered if the Delaunay graph for sets of circles could be computed in an efficient and exact way. This would require the embedding of the Delaunay graph and the location of the query point in that embedded graph. The embedded Delaunay graph and the Voronoi diagram are dual subdivisions of space, which can be stored in a quad-edge data structure [GS85]. The generalisations of Voronoi diagrams correspond to extending the possible sites from points to other geometric objects [Auren88]. Generalisations of the Voronoi diagram involving circles are the additively weighted Voronoi diagram [AKM02] and the Voronoi diagram for circles (set of sites comprising circles, see Figure 1.1) [KKS01b, KKS01a]. The definition of the weighted Voronoi diagram differs from the definition of the ordinary one in that the Euclidean distance is replaced by a weighted distance. The weighted distance between a point and a generator is the Euclidean distance between the point and the generator minus the weight of the generator, but since it must be a distance, it is not defined in the interior of the weight circles (circles centred on a generator and of radius the weight of the generator). The certified computation of the Voronoi diagram of spheres has been addressed in [GR03]. In 1995, Anishchik and Medvedev [AM95] were the first to provide the solution for Appollonius problem in 3D.

The exact computation of the Additively Weighted Voronoi diagram has not been addressed until Anton et al. [AKM02]. In their independent work, Karavelas and Emiris [EK06, KE03] provided several exact predicates for achieving the same “in-circle/orientation/edge-conflict-type/difference of radii” test as we do in a single conflict locator presented in this paper. Their work is more limited in scope than ours, because they compute the Additively Weighted Voronoi di-

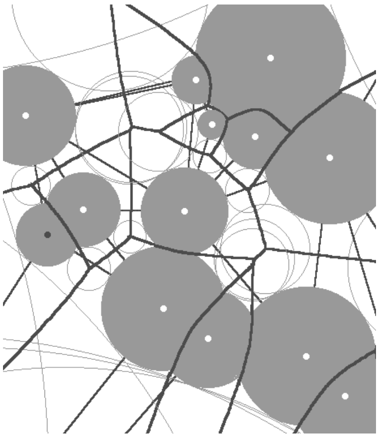

Figure 1.1. The Voronoi diagram (hyperbolas), the Delaunay graph (straight edges) and the empty circumcircles of circles produced by this algorithm

agram (or Appolonius diagram) rather than the Voronoi diagram of circles, and they assume the circles never intersect (they mention this assumption could be lifted, but they provide no justification), and they also assume no three circles can have a common tangent, or equivalently, no empty circle has infinite radius. We do not take such an assumption, and by using Gröbner bases, we can check whether the empty circle is a straight line. The approach adopted in [EK06, KE03] is also more complex than ours, because they compute exactly not only the Delaunay graph, but also the Additively Weighted Voronoi diagram. Indeed, only the exact computation of the Delaunay graph of circles is required for practical applications, because the Delaunay graph gives the topology of circles. Finally, our approach is much simpler, because we obtain the output of the predicate (in fact a Delaunay graph conflict locator) by computing the sign of the eigenvalues of a simple two by two matrix.

2. Preliminaries

Voronoi diagrams are irregular tessellations of the space, where space is continuous and structured by discrete objects [Auren88]. Sites were points for the first historical Voronoi diagrams, but in this paper we will explore sets of circles. We will recall now the formal definitions of the Voronoi diagram and of the Delaunay graph of circles. For this purpose, we need to recall some basic definitions. Let $\mathcal{C} = \{C_1, ..., C_N\}$ be the set of generators or sites, with all the C_i being circles in $\mathbb{R}^2$. Let p_i be the centre of C_i and r_i be the radius of C_i. Let the distance $d(M, C_i)$ between a point M and a circle C_i be the Euclidean distance between M and the closest point on C_i from M, i.e. $d(M, C_i) = |\delta(M, p_i) - r_i|$, where δ is the Euclidean distance between points.

Definition 2.1. (Influence zone) For $c_i, c_j \in \mathcal{C}, c_i \neq c_j$, the *influence zone* $D(c_i, c_j)$ of c_i with respect to c_j is: $D(c_i, c_j) = \{x \in M | d(x, c_i) < d(x, c_j)\}$.

Definition 2.2. (Voronoi region) The *Voronoi region* $V(c_i, \mathcal{C})$ of $c_i \in \mathcal{C}$ with respect to the set $\mathcal{C}$ is: $V(c_i, \mathcal{C}) = \bigcap_{c_j \in \mathcal{C}, c_j \neq c_i} D(c_i, c_j)$.

Definition 2.3. (Voronoi diagram) The *Voronoi diagram* of $\mathcal{C}$ is the union $V(\mathcal{C}) = \bigcup_{c_i \in \mathcal{C}} \partial V(c_i, \mathcal{C})$ of all region boundaries (see example on Figure 1.1).

Definition 2.4. (Delaunay graph) The *Delaunay graph* $DG(\mathcal{C})$ of $\mathcal{C}$ is the dual graph of $V(\mathcal{C})$ defined as follows:

- the set of vertices of $DG(\mathcal{C})$ is $\mathcal{C}$,
- for each edge of $V(\mathcal{C})$ that belongs to the common boundary of $V(c_i, \mathcal{C})$ and of $V(c_j, \mathcal{C})$ with $c_i, c_j \in \mathcal{C}$ and $c_i \neq c_j$, there is an edge of $DG(\mathcal{C})$ between c_i and c_j and reciprocally, and
- for each vertex of $V(\mathcal{C})$ that belongs to the common boundary of $V(c_{i_1}, \mathcal{C})$,. . . ,$V(c_{i_4}, \mathcal{C})$, with $\forall k \in \{1, ..., 4\}, c_{i_k} \in \mathcal{C}$ all distinct, there exists a complete graph K_4 between the $c_{i_k}, k \in \{1, ..., 4\}$, and reciprocally.

The one-dimensional elements of the Voronoi diagram are called Voronoi edges. The points of intersection of the Voronoi edges are called Voronoi vertices. The exact computation of the Delaunay graph is important for two reasons. By exact computation, we mean a computation whose output is correct. First, unlike the Voronoi diagram, the Delaunay graph is a discrete structure, and thus it does not lend itself to approximations. Second, the inaccurate computation of this Delaunay graph can induce inconsistencies within this graph, which may cause a program that updates this graph to crash. This is particularly true for the randomised incremental algorithm for the construction of the Voronoi diagram of circles. In order to maintain the Delaunay graph after each addition of a circle, we need to detect the Delaunay triangles that are not empty any longer, and we need to detect which new triangles formed with the new circle are empty, and thus valid. The algorithm that certifies whether the triangle of the Delaunay graph whose vertices are 3 given circles is empty (i.e. does not contain any point of a given circle in its interior) or not empty is used for checking which old triangles are not empty any longer and which new triangles formed with the new circle are empty, and thus valid. This algorithm is called the "*Delaunay graph conflict locator*" in the reminder of this paper.

When the old triangles are checked, its input is a 4-tuple of circles, where the first three circles define an old triangle, and the fourth circle is the new circle being inserted. When the new triangles are checked, its input is also a 4-tuple of circles, where the first three circles define a new triangle, the first two circles being linked by an existing Delaunay edge, and the fourth circle forms an old Delaunay triangle with the first two circles. Its output is the list of all the Voronoi vertices corresponding to the $1-$dimensional facets of the Delaunay graph having the first 3 circles as vertices whose circumcircles contain a point of the fourth circle in their interior, and a value that certifies the presence of each Voronoi vertex in that list. The fact that a circumcircle (the circle that is externally tangent to three given circles) is not empty is equivalent to the triangle formed by those three circles being not Delaunay, and this is called a conflict. Thus, it justifies the name of "Delaunay graph conflict locator". In the context of the ordinary Voronoi diagram of points in the plane, the concept that is analogous to the Delaunay graph conflict locator is the *Delaunay graph predicate*, which certifies whether a triangle of the Delaunay triangulation is such that its circumcircle does not contain a given point.

The exact knowledge of the Delaunay graph for circles may sound like a purely theoretical knowledge that is not central in practical applications. This is not always the case in some applications. These applications include material science, metallography, spatial analyses and VLSI layout. The Johnson-Mehl tessellations (which generalise the Voronoi diagram of circles) play a central role in the Kolmogorov-Johnson-Mehl-Avrami [Kol37] nucleation and growth kinetics theory. The exact knowledge of the neighbourliness among molecules is central to the prediction of the formation of particle aggregates. In metallography, the analysis of precipitate sizes in aluminium alloys through Transmission Electronic Microscopy [Des03, Section 1.2.2] provides an exact measurement of the cross sections of these precipitates when they are "rodes" with a fixed number of orientations [Des03, Section 1.2.2].

3. Conditions for the Delaunay graph of circles

In this section, we will examine how the Delaunay graph conflict locator can be used to maintain the Voronoi diagram of circles in the plane as those circles are introduced one by one. Finally, we will give a necessary and sufficient condition for the connectivity of the Voronoi diagram of circles in the projective plane that has a direct application in the representation of spatial data at different resolutions. Knowing the Voronoi diagram $V(\mathcal{C})$ of a set $\mathcal{C}=\{c_1,\ldots,c_m\}\subset\mathbb{R}^2$ of at least two circles ($m>1$) and its embedded Delaunay graph $DG(\mathcal{C})$ stored in a quad-edge data structure, we would like to get the Voronoi diagram $V(\mathcal{C}\cup\{c_{m+1}\})$, where c_{m+1} is a circle of $\mathbb{R}^2$. In all this section, we will say that a circle $\mathcal{C}$ *touches* a circle c_i if, and only if, $\mathcal{C}$ is tangent to c_i and no point of c_i is contained in the interior of $\mathcal{C}$.

The Voronoi edges and vertices of $V(\mathcal{C})$ may or may not be present in $V(\mathcal{C}\cup\{c_{m+1}\})$. Each new Voronoi vertex w induced by the addition of c_{m+1} necessarily belongs to two Voronoi edges of $V(\mathcal{C})$, because two of the three closest circles to w necessarily belong to $\mathcal{C}$. The new Voronoi edges induced by the addition of c_{m+1} will clearly connect Voronoi vertices of $V(\mathcal{C})$ to new Voronoi vertices induced by the addition of c_{m+1} or new Voronoi vertices between themselves. Any of these later Voronoi edges e' must be incident to one of the former Voronoi edges at each extremity of e' (because the Voronoi vertex at each extremity of e' belongs to only one new Voronoi edge, i.e. e'). Any of the former Voronoi edges e must be a subset of a Voronoi edge of $V(\mathcal{C})$, since e must be a new Voronoi edge between circles of $\mathcal{C}$ (otherwise the Voronoi vertex belonging to $V(\mathcal{C})$ at one of the extremities of e by the definition of e would be a new Voronoi vertex). Thus, to get $V(\mathcal{C}\cup\{c_{m+1}\})$, we need to know which Voronoi vertices and edges of $V(\mathcal{C})$ will not be present in $V(\mathcal{C}\cup\{c_{m+1}\})$, which Voronoi edges of $V(\mathcal{C})$ will be shortened in $V(\mathcal{C} \cup \{c_{m+1}\})$ and which new Voronoi edges will connect new Voronoi vertices between themselves.

We can test whether each Voronoi vertex v of $V(\mathcal{C})$ will be present in $V(\mathcal{C} \cup \{c_{m+1}\})$. Let us suppose that v is a Voronoi vertex of c_i, c_j and c_k. v will remain in $V(\mathcal{C} \cup \{c_{m+1}\})$ if, and only if, no point of c_{m+1} is contained in the interior of the circle centred on v that touches c_i, c_j and c_k. This is a sub-problem of the Delaunay graph conflict locator that can be tested by giving c_i, c_j, c_k and c_{m+1} as input to the Delaunay graph conflict locator, and then retain only the solutions where the Voronoi vertex is v. We can test whether each Voronoi edge e of $V(\mathcal{C})$ will be present in $V(\mathcal{C} \cup \{c_{m+1}\})$. Let us suppose that e is a locus of points having c_i and c_j as closest circles. e will disappear entirely from $V(\mathcal{C} \cup \{c_{m+1}\})$ if, and only if, a point of c_{m+1} is contained in the interior of each circle centred on e and touching c_i, c_j and each common neighbour c_k to c_i and c_j in $DG(\mathcal{C})$ in turn. This can be tested by giving c_i, c_j, c_k and c_{m+1} as input to the Delaunay graph conflict locator and then retaining only the solutions where the Voronoi vertex belongs to e. e will be shortened (possibly inducing one or more new Voronoi edges) in $V(\mathcal{C} \cup \{c_{m+1}\})$ if, and only if, there exists Voronoi vertices of c_i, c_j and c_{m+1} on e and there is no point of any common neighbour c_k to c_i and c_j in $DG(\mathcal{C})$ in the interior of a circle centred on e and touching c_i, c_j and c_{m+1}. The centre of each one of such circles will be a new Voronoi vertex in $V(\mathcal{C} \cup \{c_{m+1}\})$. This can be tested by giving c_i, c_j, c_{m+1} and c_k as input to the Delaunay graph conflict locator and then retaining only the solutions where the Voronoi vertex belongs to e.

The Delaunay graph conflict locator is sufficient to maintain the Voronoi diagram of circles. Tests might be limited to edges and vertices on the boundaries of the Voronoi regions $V(c_i, \mathcal{C}), c_i \in \mathcal{C}$ that intersect c_{m+1} and of the Voronoi regions $V(c_j, \mathcal{C}), c_j \in \mathcal{C}$ adjacent to a Voronoi region $V(c_i, \mathcal{C})$. Indeed, a point (and thus a circle) can steal its Voronoi region only from the Voronoi region it belongs to and the adjacent Voronoi regions. We will finish this section with a necessary and sufficient condition for the connectivity of the Voronoi diagram of circles in the projective plane. This result allows the characterisation of dangling edges in the Delaunay graph corresponding to the presence of closed edges in the Voronoi diagram. In order to proceed, let us recall some notations used in point set topology: let $\overline{s}$ denote the closure of s, and $\overset{\circ}{s}$ denote the interior of s in the sense of the point set topology in $\mathbb{R}^2$. Note that if s bounds a closed domain then the interior of s is meant to be the interior of the closed domain bounded by s.

Proposition 3.1. *(Connectivity of the Voronoi diagram in the plane) The Voronoi diagram $V(\mathcal{C})$ of a set $\mathcal{C} = \{c_1, \ldots, c_m\} \subset \mathbb{R}^2$ of at least two circles ($m > 1$) considered in $\mathbb{P}^2$ is not connected if, and only if, there exist a subset I of $[1, \ldots, m]$ and one index j of $[1, \ldots, m]$ such that $\forall i \in I, c_i \subset \overset{\circ}{c_j}$ and $\forall k \in [1, \ldots, m] \setminus I, \overline{c_i} \cap \overline{c_k} = \overline{c_j} \cap \overline{c_k} = \emptyset$.*

Proof. If: Assume there exist a subset I of $[1, \ldots, m]$ and one index j of $[1, \ldots, m]$ such that $\forall i \in I, c_i \subset \overset{\circ}{c_j}$ and $\forall k \in [1, \ldots, m] \setminus I, \overline{c_i} \cap \overline{c_k} = \overline{c_j} \cap \overline{c_k} = \emptyset$. Let $c_l \in \mathcal{C}$ with $l \in [1, \ldots, m] \setminus I$. Let $S = \bigcup_{i \in I} c_i$. Since $S \subset \overset{\circ}{c_j}$, any circle touching both a $c_i, i \in I$ and c_j must be contained in $\overline{c_j}$. Since $\overline{S} \cap \overline{c_l} = \overline{c_j} \cap \overline{c_l} = \emptyset$, no circle can touch each of an $c_i, i \in I$, c_j and c_l. Thus, there is no point that has a $c_i, i \in I$, c_j and c_l as nearest neighbours. Thus, there is no Voronoi vertex of a $c_i, i \in I$, c_j and c_l. Since there is no Voronoi vertex of a $c_i, i \in I$, c_j and an c_l with $l \in [1, \ldots, m] \setminus I$, there are no Voronoi vertices on the bisector of S and c_j. Since $\overline{S} \cap \overline{c_l} = \overline{S} \cap \overline{c_l} = \emptyset$, any circle centred on the bisector of S and c_j and touching both S and c_j does not intersect any circle c_k with $k \in [1, \ldots, m] \setminus I$. Thus, the bisector of S and c_j is contained in $V(\mathcal{C})$. Since c_j is connected and $S \subset \overset{\circ}{c_j}$, the bisector of S and c_j is a closed curve. Thus, the Voronoi diagram of $\mathcal{C}$ is not connected in $\mathbb{P}^2$.

Only if: Assume the Voronoi diagram of $\mathcal{C}$ is not connected in $\mathbb{P}^2$. Then, $V(\mathcal{C})$ has at least two connected components. Thus, at least one of these connected components does not have points at infinity. Let us consider the connected component (let us call it C_1) that does not have points at infinity. Since C_1 is composed of Voronoi edges (a one-dimensional component of the Voronoi diagram, which is also the locus of points having two nearest circles), each edge in C_1 must end at either a Voronoi vertex or a point at infinity. Since C_1 does not have any point at infinity, all Voronoi edges in C_1 connect Voronoi vertices. Thus C_1 is a network of vertices and edges linking those vertices. The regions that this network defines are Voronoi regions. Let $\mathcal{D}$ be the union of the closure of those Voronoi regions. $\mathcal{D}$ is a closed set by its definition. Let us consider now the circles

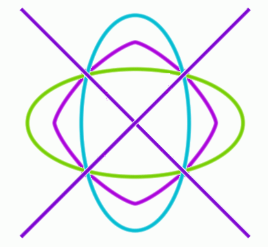

Figure 3.1. The relative position with respect to the bisector must be constant

$c_l, l \in L$ whose Voronoi regions are contained in $\mathcal{D}$. Let $S = \bigcup_{l \in L} c_l$. Thus S is a union of circles. We will now consider S as a circle instead of each one of the $c_l, l \in L$. The influence zone of $S = \bigcup_{l \in L} c_l$ is clearly $\mathring{\mathcal{D}}$, because the influence zone of a union of circles is clearly the closure of the union of the Voronoi regions of those circles. Let $e = \partial\mathcal{D}$. It is a portion of the bisector of S and another circle. Let us call it c_j. If not all the bisector of S and c_j was contained in $V(\mathcal{C})$, then e would end at Voronoi vertices (a point on the Voronoi diagram has at least two closest circles) or the point at infinity, a contradiction with e not being connected. Thus, the bisector of S and of c_j is contained in $V(\mathcal{C})$, and it is equal to e. By the definition of e, e must be a closed curve. Assume the positions of S and c_j with respect to e are not always the same. Then, S and c_j must intersect. The bisector of S and c_j must have two branches near the intersection points (see Figure 3.1). Since e is a closed curve and S is contained in the interior of e, c_j must be closed, and the other branches must be unbounded (a contradiction with e not being connected in $\mathbb{P}^2$). Thus, the positions of S and c_j with respect to e are always the same along e. Since c_j is connected, S is contained in the interior of e and the positions of S and c_j with respect to e are always the same along e, $S \subset \mathring{c}_j$. Since e is the bisector of S and c_j and belongs to $V(\mathcal{C})$, any circle centred on e and touching both S and c_j does not intersect any circle c_k with $k \in [1, \ldots, m] \setminus I$. Thus, $\forall k \in [1, \ldots, m] \setminus I, \overline{c_i} \cap \overline{c_k} = \overline{c_j} \cap \overline{c_k} = \emptyset$. $\square$

The only cases of disconnected (considered in $\mathbb{P}^2$) Voronoi diagrams correspond to one or more sites (circles) contained in the interior of another site. This property has a direct application in Geographic Information Systems. When the same region $\mathcal{R}$ bounded by a circle S is represented at different scales, the representation of the details inside $\mathcal{R}$ does not change the Voronoi diagram outside $\mathcal{R}$. The edges of the Delaunay graph corresponding to a disconnected Voronoi diagram (considered in $\mathbb{P}^2$) are respectively dangling edges or cut edges (the Delaunay graph is not bi-connected and removing a cut edge induces two connected components). It is possible to detect if there exists one or more circles $c_i, i \in I$ contained in the interior of another circle c_j by checking that there exists no Voronoi vertex of c_i, c_j and any $c_k \in \mathcal{C}$ distinct from c_i and c_j. This is again a subproblem of the Delaunay graph conflict locator.

4. The Delaunay graph conflict locator for circles

The conflict locator for the Delaunay graph of circles is related to the Apollonius Tenth problem (the circles that are tangent to three given circles are called Apollonius circles).

The motivation for an exact conflict locator lies in the fact that without an exact computation of the Delaunay graph of additively weighted points, some geometric and topologic inconsistencies may appear. This is illustrated with an example. The starting configuration is shown on Figure 4.1. The generators are three circles. The Delaunay graph is drawn in dashed lines. The Apollonius circles tangent to the circles have been drawn in dotted lines. The real configuration after addition of a fourth circle is shown on Figure 4.2. The configuration that might have been computed by an approximate algorithm is shown on Figure 4.3: the difference between real and perceived situations has been exaggerated to show the difference. The old Apollonius circles have been adequately perceived to be invalid with respect to the newly inserted circle. About the new Voronoi vertices, while on the right of the figure two new Voronoi vertices have been identified as valid with respect to their potential neighbours, on the left of the figure, only one Voronoi vertex has been identified as being valid with respect to its potential neighbours. While the new Voronoi edge between the middle and bottom circles can be drawn between the two new Voronoi vertices of the new, middle and bottom circles; the Voronoi edge between the top and the new circles cannot be drawn, because there is no valid Voronoi vertex on the left. There is an inconsistency within the topology: there is one new Voronoi vertex (the Voronoi vertex of the top new and middle circles) that cannot be linked by a new Voronoi edge to any other new Voronoi vertex and thus, that Voronoi vertex is incident to only two Voronoi edges. This additively weighted Voronoi diagram might have been computed by an approximative algorithm that is not an additively weighted Voronoi diagram. Thus, even if we perturbate the input circles, we will never get this additively weighted Voronoi diagram.

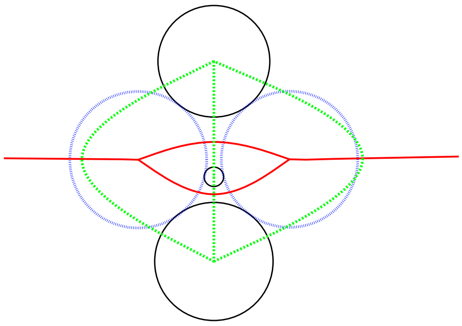

Figure 4.1. The starting configuration

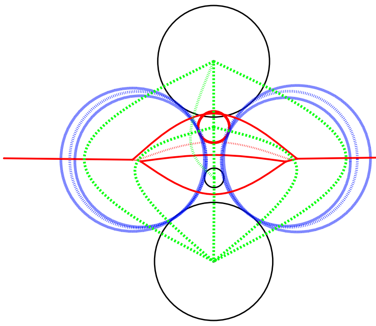

Figure 4.2. The real configuration after addition of the fourth circle (bold weight circle)

We consider the maintenance of the Delaunay graph of circles in an incremental way: we check first the validity of all the old triangles of the Delaunay graph whose vertices are a given triple of circles with respect to a given newly inserted circle. When old triangles are checked, four circles C_1, C_2, C_3 and C_4 are given: the first three are supposed to define one or more triangles in the Delaunay graph, and the last one is the newly inserted circle. Let (x_i, y_i) be the coordinates of p_i for $i = 1, 2, 3, 4$. There are two possible outcomes to the above test of validity. Either

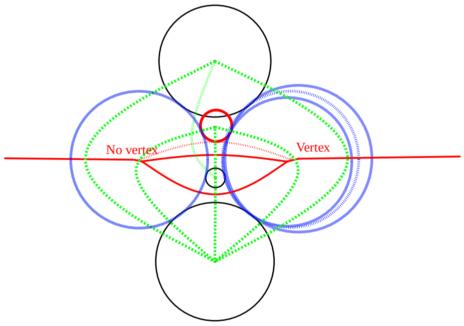

Figure 4.3. The configuration computed by an approximate algorithm

the triangles are valid with respect to the newly inserted weighted point and the triangles remain in the new Delaunay graph, or there is at least one triangle that is not valid with respect to the newly inserted weighted point and these triangles will not be present in the Delaunay graph any longer. We also need to check the validity of new triangles $C_1C_2C_3$ with respect to a circle C_4, where $C_1C_2C_4$ is an old Delaunay triangle and C_3 is the newly inserted circle. There are two possible outcomes to this test of validity. Either the triangles formed with an old Delaunay edge C_1C_2 and the newly inserted weighted point C_3 are valid with respect to any circle C_4, where $C_1C_2C_4$ is an old Delaunay triangle, and the triangles will appear in the new Delaunay graph, or there is at least one triangle that is not valid and these triangles will not be added in the Delaunay graph. In both cases, we check the validity of a triangle $C_1C_2C_3$ with respect to a circle C_4.

The Apollonius circles of C_1, C_2 and C_3 can be obtained algebraically by computing the common intersection of the three circles C_1', C_2' and C_3' expanding or shrinking from the three first circles C_1, C_2 and C_3 all with the same absolute value of the rate. The common unsigned expansion of the first three circles is denoted by r. The coordinates of the intersection I of C_1', C_2' and C_3' are denoted (x, y). The circle C'' centred on (x, y) and of radius r is tangent to the first three circles. Thus, the Apollonius circles are the solutions of one of the eight following systems (I) of three quadratic equations in three unknowns x, y, r:

$$\begin{cases} (x - x_1)^2 + (y - y_1)^2 - (r_1 \pm r)^2 = 0 \\ (x - x_2)^2 + (y - y_2)^2 - (r_2 \pm r)^2 = 0 \\ (x - x_3)^2 + (y - y_3)^2 - (r_3 \pm r)^2 = 0 \end{cases} .$$

By replacing r by $-r$ in one of the preceding systems of equations, we still get another one of the preceding systems of equations. Thus, let us suppose r is the signed expansion of C_1. Then, we can reformulate the preceding systems of equations as the following systems (II) of equations:

$$\begin{cases} (x-x_1)^2+(y-y_1)^2-(r_1+r)^2=0 \\ (x-x_2)^2+(y-y_2)^2-(r_2\pm r)^2=0 \\ (x-x_3)^2+(y-y_3)^2-(r_3\pm r)^2=0 \end{cases}$$

Now let us consider for each system (II) the set X of solutions of the system (II) in $\mathbb{C}^3$.

Subtracting one of the equations from the remaining two results in a system of 2 linear equations, from which x and y may be expressed as linear functions of r. Substitution in the first equation then leads to a quadratic equation in r. This means that the unknown quantities x, y, r can be expressed with quadratic radicals as functions of the given centres and radii for each one of the systems of equations above. Though the simplest thing to do now would be to compute the two Voronoi vertices and use their computed coordinates and corresponding signed expansion in the computation of the values certifying the output of the Delaunay graph conflict locator, it is not desirable because this method would not guarantee the topology of the Voronoi diagram of circles. To get the exact Delaunay graph conflict locator in a more elegant and generalisable way, we evaluated the values certifying the conflict locator output without relying on the computation of the Voronoi vertices as an intermediary computation. This is done by evaluating the values taken by the polynomial function expressing the relative position of C_4 with respect to C'' on the set of solutions of the system (i.e. the common zeroes of the three polynomials c'_1, c'_2 and c'_3). This is possible due to the translation that exists between geometry and algebra.

More specifically, to the geometric set X of the set of common zeroes of the three polynomials c'_1, c'_2 and c'_3 in $\mathbb{C}^3$, we can associate the set of all polynomials vanishing on the points of X, i.e., the set of polynomials $f_1c'_1 + f_2c'_2 + f_3c'_3$ where the $f_i, i = 1, 2, 3$ are polynomials in the three variables x, y, r with coefficients in $\mathbb{C}$. This set is the *ideal* [GP02, Definition 1.3.1] $\langle c'_1, c'_2, c'_3\rangle$. The set of polynomials with coefficients in $\mathbb{C}$, forms with the addition and the multiplication of polynomials, a ring: the *ring of polynomials* [GP02, Definition 1.1.3]. A polynomial function $g(x, y, r)$ on $\mathbb{C}^3$ is mapped to a polynomial function on X if we recursively subtract from g any polynomial in g belonging to $\langle c'_1, c'_2, c'_3\rangle$ until no monomial in g can be divided by each one of the lexicographically highest monomials in c'_1, c'_2 and c'_3. The result of this mapping gives a canonic representative of the remainder of the Euclidean division of the polynomial g by the polynomials c'_1, c'_2 and c'_3. The image of the ring of polynomials by this mapping is called the *quotient algebra* [GP02] of the ring of polynomials by the ideal $\langle c'_1, c'_2, c'_3\rangle$. Moreover, $\langle c'_1, c'_2 - c'_1, c'_3 - c'_1\rangle = \langle c'_1, c'_2, c'_3\rangle$. Finally, if we recursively subtract from g any polynomial in g belonging to $\langle c'_1, c'_2 - c'_1, c'_3 - c'_1\rangle$ till the only monomials in g are 1 and r, we get the same result as the preceding mapping. The polynomials $c'_1, c'_2 - c'_1, c'_3 - c'_1$ constitute what is called a *Gröbner basis* [GP02, Definition 1.6.1] of the ideal $\langle c'_1, c'_2, c'_3\rangle$.

Gröbner bases are used in Computational Algebraic Geometry in order to compute a canonic representative of the remainder of the division of one polynomial by several polynomials generating a given ideal I. This canonic representative belongs to the quotient algebra of the ring of polynomials by the ideal I. The Gröbner basis for this system provides a set of polynomials that define uniquely the algebraic relationships between variables for the solutions of the system. The initial (largest with respect to some monomial order [GP02]) monomials of each one of the polynomials of the Gröbner basis form an ideal. The monomials that do not pertain to this ideal form a basis for the representatives of the equivalence class of the remainders of the division of a polynomial by the polynomials of the system in the quotient algebra. These monomials are called standard monomials. For the above Gröbner basis, the standard monomials are 1 and r if the system has solutions in $\mathbb{C}^3$, and only 1, if the system has no solutions in $\mathbb{C}^3$ (the empty circle is a straight line). If the only standard monomial is 1, we compute the relative position of the fourth circle with respect to the common tangent of the first three circles. The size of this basis equals the *dimension* [GP02, see definition on page 414] of the quotient algebra and the number of solutions of the system counted with their multiplicity [GP02]. In the case of the conflict locator for the additively weighted Voronoi diagram, there are two solutions.

The true Voronoi vertices are not all the solutions of one system of algebraic equations, but a subset of the solutions of four systems of algebraic equations. The solutions of the algebraic equations are the Apollonius circles, whose centres are generalised Voronoi vertices (a concept that was introduced in [Anton04]). We thus need to determine which Apollonius circles centres are potentially true Voronoi vertices (only the real Apollonius circles centres can be true Voronoi vertices). There are four possible determinations of the true Voronoi vertices from Apollonius circles centres of C_1, C_2 and C_3:

first case if C_1, C_2 and C_3 mutually intersect, then the real circles among the seven Apollonius circles that are not internally tangent to each of C_1, C_2 and C_3 correspond to true Voronoi vertices (their centres are true Voronoi vertices, see Figure 4.4), and reciprocally.

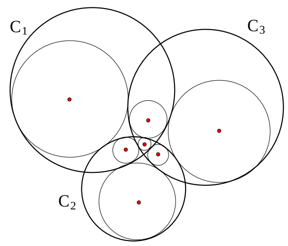

Figure 4.4. Seven Apollonius circles centres that are true Voronoi vertices (first case)

second case if one circle (say C_1) intersects the two others (C_2 and C_3) which do not intersect, then only the real Apollonius circles that are either externally tangent to each of C_1, C_2 and C_3, or internally tangent to C_1 and externally tangent to C_2 and C_3 correspond to true Voronoi vertices (their centres are true Voronoi vertices, see Figure 4.5).

third case if two circles (say C_1 and C_2) intersect the interior of the third one (C_3) and at least one of them (say C_1) is contained in the interior of C_3, then only the real Apollonius circles that are externally tangent to C_1 and C_2 and internally tangent to C_3 correspond to true Voronoi vertices (their centres are true Voronoi vertices, see Figure 4.6).

fourth case otherwise (if none of the three situations above apply), only the real Apollonius circles that are externally tangent to C_1, C_2 and C_3 correspond to true Voronoi vertices (their centres are true Voronoi vertices, see Figure 4.7).

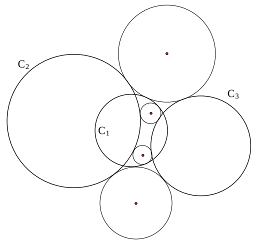

Figure 4.5. Four Apollonius circles centres that are true Voronoi vertices (second case)

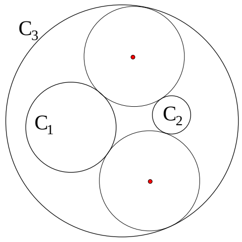

Figure 4.6. Two Apollonius circles centres that are true Voronoi vertices (third case)

When the old Delaunay triangles are checked, the case where one circle (say C_1) lies in the interior of a second circle (say C_2), which lies in the interior of the third circle (C_3), or only one circle (say C_1) lies within the interior of one of the other ones (say C_2) cannot happen because then, there would be no Voronoi vertices and the triangle $C_1C_2C_3$ would not exist in the Delaunay graph. If we check new triangles, we can check if the situation described just above happens by computing the sign of the determinant of the multiplication matrix for the fourth case.
Now that we have seen the different cases of true Voronoi vertices, we will see how we can test in which case we are and which solutions of the systems of equations (II) described above correspond to true Voronoi vertices.

first case C_1, C_2 and C_3 mutually intersect if, and only if, $d(p_1, p_2) - r_1 - r_2 \leq 0$ and

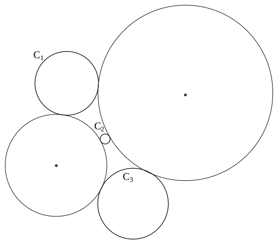

Figure 4.7. Two Apollonius circles centres are true Voronoi vertices (fourth case)

$d(p_1, p_3) - r_1 - r_3 \leq 0$ and $d(p_2, p_3) - r_2 - r_3 \leq 0$. The computation of this test can be done exactly, since the only variables that are not input to the Delaunay graph conflict locator are the distances, and these distances are expressed by radicals. Indeed, we need to test the sign of the difference of a radical and a number which do not depend on intermediary computations. The true Voronoi vertices are the real solutions of all the systems of equations (II) such that $r > 0$.

second case C_1 intersects C_2 and C_3, and C_2 and C_3 have no point of intersection if, and only if, $d(p_1, p_2) - r_1 - r_2 \leq 0$ and $d(p_1, p_3) - r_1 - r_3 \leq 0$ and $d(p_2, p_3) - r_2 - r_3 > 0$. The computation of this test can be done exactly for the same reasons as the previous case. The true Voronoi vertices are the real solutions of the system of equations:

$$\begin{cases} (x - x_1)^2 + (y - y_1)^2 - (r_1 \pm r)^2 = 0 \\ (x - x_2)^2 + (y - y_2)^2 - (r_2 - r)^2 = 0 \\ (x - x_3)^2 + (y - y_3)^2 - (r_3 - r)^2 = 0 \end{cases}$$

with $r < 0$.

third case C_1 lies in the interior of C_3 and C_2 intersects the interior of C_3 if, and only if, $d(p_1, p_3) + r_1 - r_3 < 0$ and $d(p_2, p_3) - r_2 - r_3 < 0$ and $(x_1 - x_3)^2 + (y_1 - y_3)^2 - r_3^2 < 0$. The computation of this test can be done exactly for the same reasons as the previous case. The true Voronoi vertices are the real solutions of the system of equations:

$$\begin{cases} (x - x_1)^2 + (y - y_1)^2 - (r_1 + r)^2 = 0 \\ (x - x_2)^2 + (y - y_2)^2 - (r_2 + r)^2 = 0 \\ (x - x_3)^2 + (y - y_3)^2 - (r_3 - r)^2 = 0 \end{cases}$$

such that $r > 0$.

fourth case this is the case if all the previous three tests failed. The true Voronoi vertices are the real solutions of the system of equations:

$$\begin{cases} (x - x_1)^2 + (y - y_1)^2 - (r_1 + r)^2 = 0 \\ (x - x_2)^2 + (y - y_2)^2 - (r_2 + r)^2 = 0 \\ (x - x_3)^2 + (y - y_3)^2 - (r_3 + r)^2 = 0 \end{cases}$$

with $r > 0$.

We computed the Gröbner basis of the ideal of X for each one of the systems (II) encountered. Each one of these Gröbner bases consists of the earlier mentioned quadratic equation in r and linear equations in x, y and r if the system admits solutions in $\mathbb{C}^3$. For the above Gröbner basis, the standard monomials are 1 and r if the system has solutions in $\mathbb{C}^3$, and only 1, if the system has no solutions in $\mathbb{C}^3$ (the empty circle is an infinite circle). If the only standard monomial is 1, we compute the relative position of the fourth circle with respect to the common tangent of the first three circles. Evaluating the signs of a single polynomial ($g = (x_4 - x)^2 + (y_4 - y)^2 - (r + r_4)^2$) taken on the real points of X allows one to compute the relative position of C_4 with respect to the empty circle of C_1, C_2 and C_3. We can check for the existence of real solutions by evaluating the sign of the discriminant of the characteristic polynomial. We will suppose the real solutions to the systems (II) have been tested. We need to evaluate the signs taken by both g and r on each one of the points of X. Indeed, we need not only to check the relative position of C_4 with respect to each one of the Apollonius circles, but we need to check the conditions on r.

We considered the operation of multiplication of polynomials by the polynomial g, whose sign expresses the relative position of C_4 with respect to C''. We also considered the operation of multiplication of polynomials by the polynomial r, whose sign allows one to check whether the solutions correspond to true Voronoi vertices. These operations are linear mappings. The operations of these mappings on the canonic representative of the remainder of the Euclidean division of a polynomial by the three polynomials of the system are also linear mappings that can be expressed by a matrix. We need to be able to associate the signs of the values of g with the signs of the values of r taken on the (real) solutions of each system (II). For a given system (II), let M_g and M_r be the matrices of the result of the multiplication by g and by r respectively on the canonic representative of the remainder of the division of a polynomial by the three polynomials of the system. Since these multiplication maps commute, it is possible to use the transformation matrix obtained during the computation of the Jordan form of one of these matrices to triangularise the other matrix by a simple multiplication of matrices. Indeed, the computation of the Jordan form for M_g gives the triangular matrix $P^{-1}M_gP$ of the Schur form of that matrix where P is a unitary matrix called the transformation matrix; and $P^{-1}M_rP$ is triangular. Finally, we can obtain the solutions by reading the diagonal entries in turn in each one of the Jordan forms of these matrices (the diagonal entries of the Jordan form of a matrix are its eigenvalues). The row number on each one of these matrices corresponds to the index of the solution. By evaluating the signs of the diagonal entries in the Jordan forms of M_g and of M_r on the same line, we associate the signs of the values of g with the signs of the values of r taken on the solutions of each system (II).

5. Conclusions

We have provided a predicate for the incremental construction of the Delaunay graph and the Voronoi diagram of circles that amounts to computing the sign of the eigenvalues of a two by two matrix. Unlike other independent research, our work proposes a single predicate that can compute the Delaunay graph even in the case of one circle being entirely in another circle or intersecting circles. We have also provided an application of the Voronoi diagram of circles to the modelling and the visualisation of the growth of crystal aggregates. We have been also working on the Delaunay graph of conics and of semi-algebraic sets (see [Anton04]).

References

[AM95] Anishchik S. V., Medvedev N. N. Three-dimentional Apollonian packing as a model for dense granular systems ll *Phys. Rev. Lett.* 1995. Vol. **75, (23)**. P. 4314-4317.

[AKM02] François Anton, David Kirkpatrick, and Darka Mioc. An exact algebraic predicate for the maintenance of the topology of the additively weighted voronoi diagram. In *The Fourteenth Canadian Conference on Computational Geometry*, pages 72–76, Lethbridge, Alberta, Canada, 2002.

[Anton04] Anton, F.: *Voronoi diagrams of semi-algebraic sets,* Ph.D. thesis, The University of British Columbia, Vancouver, British Columbia, Canada, 2004.

[Auren88] Aurenhammer, F.: Voronoi diagrams - A survey, *Institute for Information Processing, Technical University of Graz, Report 263*, 1988

[Des03] Alexis Deschamps. *Handbook of Aluminum*, chapter Analytical Techniques for Aluminium Alloys, vol. 2, Alloy Production and Materials manufacturing, pages 155–192. Marcel Dekker, Inc., New York, USA, 2003.

[EK06] Ioannis Z. Emiris and Menelaos I. Karavelas. The predicates of the Apollonius diagram: algorithmic analysis and implementation. *Comput. Geom.*, 33(1-2):18–57, 2006.

[GP02] Gert-Martin Greuel and Gerhard Pfister. *A* **Singular** *introduction to commutative algebra*. Springer-Verlag, Berlin, 2002.

[GR03] Gavrilova, M. L. and Rokne, J. Updating the topology of the dynamic Voronoi diagram for spheres in Euclidean d-dimensional space. Computer Aided Geometric Design, 20(4): 231–242, 2003.

[GS85] Guibas, L., and Stolfi, J.: Primitives for the Manipulation of General Subdivisions and the Computation of Voronoi Diagrams, *ACM Transactions on Graphics*, Vol. 4, No. 2, pp. 74-123, 1985

[KE03] Menelaos I. Karavelas and Ioannis Z. Emiris. Root comparison techniques applied to computing the additively weighted Voronoi diagram. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms (Baltimore, MD, 2003)*, pages 320–329, New York, 2003. ACM.

[KKS01a] Deok-Soo Kim, Donguk Kim, and Kokichi Sugihara. Voronoi diagram of a circle set from Voronoi diagram of a point set. I. Topology. *Comput. Aided Geom. Design*, 18(6):541–562, 2001.

[KKS01b] Deok-Soo Kim, Donguk Kim, and Kokichi Sugihara. Voronoi diagram of a circle set from Voronoi diagram of a point set. II. Geometry. *Comput. Aided Geom. Design*, 18(6):563–585, 2001.

[Kol37] Andrei Nikolaevich Kolmogorov. A statistical theory for the recrystallization of metals. *Akad. nauk SSSR, Izv., Ser. Matem.*, 1(3):355–359, 1937.