Financial Giffen Goods:

Examples and Counterexamples*

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Abstract

In the basic Markowitz and Merton models, a stock's weight in efficient portfolios goes up if its expected rate of return goes up. Put differently, there are no financial Giffen goods. By an example from mortgage choice we illustrate that for more complicated portfolio problems Giffen effects do occur.

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1 Introduction and Summary

A Giffen good is one for which demand goes down if its price goes down. At first, it is counter-intuitive that such goods exist at all. But most introductory text-books in economics will tell you that they do; some with stories about potatoes and famine in Ireland, some with first order conditions for constrained optimization and partial derivatives. In this note we study similar effects — by which we mean a negative relation between expected return and demand — in financial models.

We first show that in the basic Markowitz mean/variance model, there are no Giffen goods; if a stock's expected rate of return goes up, its weight in any efficient portfolio goes up. This seems a text-book comparative statics result. We have, however, only been able to find it hidden in some recent articles, Stevens (1998) for example. So we give a simple proof. We then look at Merton's dynamic investment framework. In its basic version demand for any asset depends positively on its expected rate of return, but if a subsistence level is included, demand for the risk-free asset falls with the interest rate.

Skeptics would say that Giffen goods exist in *and only in* economic text-books. We end the paper by illustrating that it is not so. Our example uses the framework from Rasmussen & Clausen (2006) and shows that some – completely rational – mortgagors react to lower costs of long-term financing (reflecting a smaller market price of risk) by using more short-term financing.

2 The Markowitz Model

Consider a model with n risky assets with expected rate of return vector μ and invertible covariance matrix Σ. The mean/variance efficient portfolios (parametrized by risk-aversion γ) are found by solving

$$\max_w w^\top \mu - \frac{1}{2}\gamma w^\top \Sigma w \;\; \text{st} \;\; w^\top \mathbf{1} = 1,$$

where $\mathbf{1}^\top = (1, \ldots, 1)$. The optimal portfolios are

$$\widehat{w} = \gamma^{-1}\Sigma^{-1}\left(\mu - \eta(\gamma; \mu, \Sigma)\mathbf{1}\right)$$

where

$$\eta(\gamma; \mu, \Sigma) = \frac{\mathbf{1}^\top \Sigma^{-1}\mu - \gamma}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}}$$

can be interpreted as the expected rate of return on $\widehat{w}$'s zero-beta portfolio.

A sensible definition of a Giffen good in this context would be an asset, say the i'th, for which $\partial\widehat{w}_i/\partial\mu_i < 0$ for some γ, this meaning that when the asset's expected rate of return goes up, its weight in some optimal portfolio goes down. Let us show that there are no such assets. Look at the problem with the modified expected return vector $\mu + \alpha e_i$, where $\alpha \in \mathbb{R}$ and e_i is the i'th unit vector. This gives

$$\widehat{w} = \ldots + \alpha \underbrace{\gamma^{-1}(\Sigma^{-1}e_i - \frac{e_i^\top \Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1})}_{h_1},$$

where $\ldots$ represents a vector that does not depend on α. Showing that $\partial\widehat{w}_i/\partial\mu_i > 0$

amounts to proving positivity of the i'th coordinate of h_1, which we can write as

$$e_i^\top h_1 = \gamma^{-1} \left(e_i^\top \Sigma^{-1} e_i - \frac{(e_i^\top \Sigma^{-1} \mathbf{1})^2}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}} \right).$$

The positivity then boils down to

$$(e_i^\top \Sigma^{-1} e_i)(\mathbf{1}^\top \Sigma^{-1} \mathbf{1}) - (e_i^\top \Sigma^{-1} \mathbf{1})^2 \overset{?}{>} 0.$$

Because Σ^{-1} is strictly positive definite and symmetric, we may think of it as a covariance matrix of a random variable, say Y. But the left hand side in the inequality above is then the determinant of the covariance matrix of $(Y_i, \sum_{j=1}^n Y_j)$, and thus strictly positive.

The inclusion of a risk-free asset is handled in the same way with η replaced by the risk-free rate of return because the risk-free asset is any portfolio's zero-beta portfolio.

3 The Merton Model

Another classic portfolio model is Merton's dynamic investment framework. In its most simple case, an agent invests his wealth, W, in either a risk-free asset with rate of return r or a risky asset whose price, S, is a Geometric Brownian motion,

$$dS(t) = \mu S(t) dt + \sigma S(t) dZ(t),$$

where Z is a Brownian motion. Suppose the agent maximizing expected utility cares only about terminal wealth, $W(T)$, and has a utility function with constant relative risk aversion,

$$U(W(T)) = \frac{W(T)^{1-\gamma}}{1-\gamma}.$$

It is then well-known that it is optimal to invest a fixed fraction, π, of wealth in the risky asset, specifically

$$\pi = \frac{\mu - r}{\gamma\sigma^2}.$$

So if the expected rate of return of an asset (be that risky or risk-free) goes up, that asset gets higher weight in any agent's portfolio. Further, by combining 2-fund separation with the Markowitz analysis from the previous section, the same conclusion is reached in a model with n rather than just one risky asset.

A popular extension is a utility function of the form

$$\widetilde{U}(W(T)) = \frac{(W(T) - \bar{W})^{1-\gamma}}{1-\gamma},$$

where $\bar{W}$ is some minimal required wealth level; a subsistence level. Assuming initial wealth is greater than the present value of of $\bar{W}$, the optimal strategy is to buy $e^{-rT}\bar{W}$ units of the risk-free asset and invest the rest of the wealth according to the Merton-rule from above. Thus the optimal fraction invested at time t in the risky asset is

$$\widetilde{\pi}(t) = \frac{W(t) - e^{-r(T-t)}\bar{W}}{W(t)} \frac{\mu - r}{\gamma\sigma^2},$$

so that

$$\frac{\partial\widetilde{\pi}(0)}{\partial r} = \frac{1}{\gamma\sigma^2}\left(\frac{e^{-rT}\bar{W}}{W(0)}(T(\mu - r) + 1) - 1\right).$$

Reasonably $\mu - r > 0$ and we see that if the subsistence level is sufficiently large compared to initial wealth then $\partial\widetilde{\pi}(0)/\partial r > 0$, so the percentage of initial wealth invested in the risky asset goes up, and hence the investment in the risk-free asset goes down when the risk-free rate of return goes up. The intuition behind is that if the return of the risk-free asset goes up, you need less of it to ensure survival, and you have more money to do what you like, rather than what you have to.

4 A Mortgage Choice Model

A way to quantify mortgage planning — for many people the largest financial decisions, they ever make — as a portfolio optimization problem is to study

$$\text{minimize}_\theta \ \gamma \times \mathbf{E}(X(\theta)) + (1-\gamma) \times \mathbf{ES}_\beta(X(\theta)),$$

where:

— $X(\theta)$ is the (cumulative discounted) payments from following the dynamic strategy θ; the loan portfolio the mortgagor chooses.

— $\mathbf{ES}_\beta(X) = \mathbf{E}(X|X \geq q_\beta)$ is the expected shortfall based on the β-quantile q_β. Expected shortfall is a coherent risk-measure and, as shown by Rockafeller & Uryasev (2000) it gives a piece-wise linear objective function.

This is an extension of the problems considered in Rasmussen & Clausen (2006), who look only at fully linear or min-max type objective functions; ie. only risk-neutrality or extreme risk aversion.

The minimization must be performed subject to

- a stochastic interest rate model discretized by paths through trees, each node having a universe of securities

- balance and cash-flow equations including transaction costs

- portfolio and cash-flow constraints such as non-negativity, liquidity

This is a multi-stage stochastic programming problem that can be solved numerically using for instance `GAMS` and `CPLEX`, see Rasmussen & Clausen (2006) for details.

To model the stochastic behaviour of interest rates, we use a Vasicek model

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dZ(t).$$

To specify the full yield curve dynamics, a market price of risk is needed. We parameterize this by λ, that technically shifts the stationary mean of r to $\theta + \lambda$ under the risk-neutral measure, but more tellingly, determines the typical difference between long and short rates. This represents the fundamental trade-off in the mortgagor's problem: Short rates are typically lower than long rates, but with short-term financing, he doesn't know how much he'll have to pay.

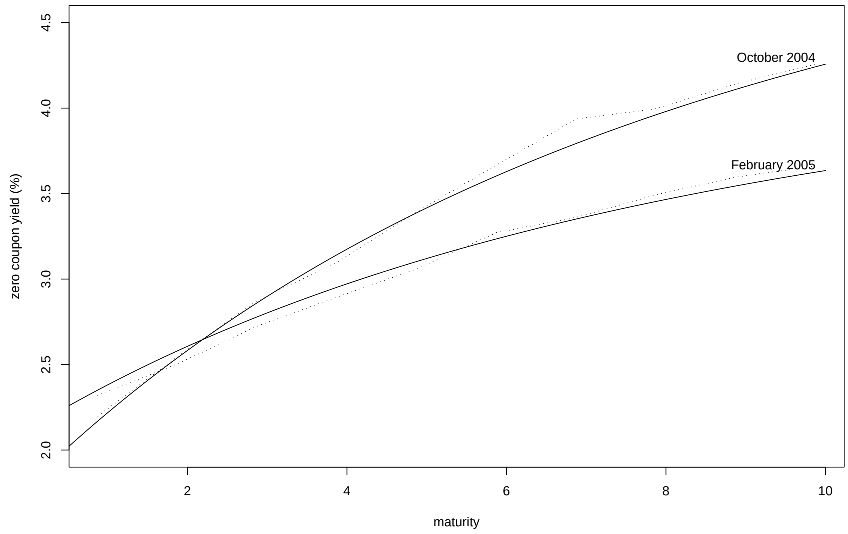

Figure 1: Danish yield curves from October 2004 and February 2005; full curves are calibrated model curves, dotted lines are observations. The estimated Vasicek model parameters $(\theta, \kappa, \sigma) = (0.042, 0.21, 0.01)$ are held fixed and only the calibrated market price of risk, λ, differs from October 2004 (0.023) to February 2005 (0.003).

Table 1 shows the composition of the initial optimal portfolios for two different values of the market price of risk. These two values correspond to calibration to observed Danish yield curves in October 2004 and February 2005, as depicted in Figure 1. (We use $\beta = 0.95$ and a 7-year horizon. Only the 1-year adjustable-rate bond and the 30-year callable, fixed-rate bond are used in the optimal portfolio, although the numerical algorithm allowed for a larger universe of mortgage products.)

Mortgagor risk aversion (γ)	Optimal initial loan portfolio compositions: October 2004		February 2005	
	Fixed rate callable	Full yearly refinancing	Fixed rate callable	Full yearly refinancing
0	0%	100%	0%	100%
1/4	16%	84%	4%	96%
1/2	48%	52%	49%	51%
3/4	48%	52%	58%	42%
1	48%	52%	60%	40%

Table 1: Optimal initial loan portfolio compositions for different mortgagors facing the different yield curves shown in Figure 1.

Row-wise comparisons in Table 1 give no surprises. The risk-neutral mortgagor uses full short-term financing and with higher risk-aversion (higher weight to expected shortfall) more long-term financing is used. Note, however, that because short rates were historically low and the yield curve quite steep, even very risk-averse mortgagors use a significant amount of short-term financing (52% for October 2004, 40% for February 2005). Comparing the columns tells us what a lowering of the market price of risk parameter can do to optimal portfolios. The very risk-averse mortgagor uses a larger proportion (60% compared to 48%) of fixed-rate loans, and the risk-neutral mortgagor does not care. But for a moderately risk-averse mortgagor ($\gamma = 1/4$), the lower λ-value, which makes short-term financing relatively less attractive, causes him to use more short-term financing (up to 96% compared to 84% before). Although a more complicated model, the intuition is again that this mortgagor uses long-term financing initially not because he wants to, but because he has to, and lower long rates — still higher than typical short rates — make the necessity cheaper; like the Irish potatoes.

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