Integrated Mortgage and Pension Portfolio Management for Households

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July 27, 2006

Informatics and Mathematical Modelling

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To the happiness of friends...

Abstract

The varieties of financial services and innovation in financial products have an increasing impact on households across global and local markets. Individuals need to make personal finance decisions upon choices of pension, savings and pure investment plans. At the same time, they may need a mortgage portfolio to fund a real estate purchase domestically or overseas, as well as a personal scheme to supplement life-long consumption. These are essentially dynamic portfolio optimization problems. Much has been accomplished in solving these from the corporate perspective. In particular, one of the approaches is asset liability modeling - a key instrument used in the financial services industry. However, the research effort of similar problems from the household standpoint is rather new, hence close study by means of mathematical modeling and risk management methodology could prove lucrative.

The main goal of this research is to achieve a high level of integration between pension and mortgage portfolio problems typical to an average household that have traditionally been solved separately. Such integration should yield portfolio strategies that perform effectively in terms of household objectives and are highly robust in the ever-changing markets. Hence, the desired model should optimize household utility whilst managing the risk exposure and fulfilling policy requirements.

Acknowledgements

Writing this thesis has been a memorable journey full of numerous bright ideas and challenging tasks. It would not have been the same without insightful discussions and educative advice of my supervisors Jens Clausen and Kourosh Marjani Rasmussen whom I greatly thankful to.

This journey has been supported and encouraged by the friends who kindly assisted me in reviewing this writing at its different stages and inspired me to express my creativity in research. I would like to offer my sincere gratitude to David Barnabey, Anna Lomova, Christian Lanzani, Omri Ross, Peter Darling, Elly Nkya, Jinghua Zhang and Stuart Glasson for invaluable care and dozens of moments of joy and thought. Thank you very much.

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1 Preface

This thesis fulfills the final requirement to obtaining a Master of Science degree in Computing and Mathematics at the Technical University of Denmark (DTU). It has been carried out at the Section of Operations Research of the Informatics and Mathematical Modelling department during the period from February, 1st 2006 to August, 1st 2006 under the supervision of Professor Jens Clausen and PhD student Kourosh Marjani Rasmussen.

Reflecting the actual project flow, this report is structured in the following manner. First, the research motivation and main concepts are presented, setting the groundwork for defining the integrated pension and mortgage portfolio management for households problem. Financial risk exposure associated with such integrated portfolios is studied. Next, the household pension and mortgage products are presented as applicable for modeling their characteristics. These include policy contribution, dealing, interest accrual, cost structure, and etc. Also, the underlying investment and credit links in these products are outlined and the stochasticity associated with them (i.e. market prices, interest rates, returns, and etc.) is presented. Utility optimization methods are combined with risk management techniques in defining the integrated portfolio objectives. To effectively manage risk exposure of the portfolio, capture uncertainties of the products integrated in it, and optimize the utility objectives, a multi-stage stochastic programming approach is taken. The challenge of modeling arises from the need to correlate the investment trust returns and interest rates in order to generate valid scenarios for the integrated portfolio management problem. The proposed solution to this is followed by formulation of the complete integration of pension and mortgage portfolios in a multistage stochastic programming model. This model optimizes expected utility of the portfolio using either of two risk measures: Conditional Value at Risk (CVaR) or Condition Drawdown at Risk (CDaR). Both CVaR and CDaR versions of the integrated portfolio management problem are tested. Lastly, their performance, sensitivity and robustness are analyzed. Conclusion of the research findings and contributions finalizes this Master Thesis report and briefly describes future aspirations of its author.

2 Introduction

2.1 Research Motivation

When Otto von Bismarck established the first retirement system in Germany in the nineteenth century, he set the retirement age at seventy. By the end of the 1930s, most of the major economies of the world had national systems of one sort or another, and in most of them, the eligibility age was lower. Although many national retirement systems were originally structured to be funded, most of them moved to pay-as-you-go financing during the baby boom periods after World War II. Fertility rates rates fell in most developed countries by the mid 1960s, and twenty years later the number of new workers stabilized or started to decline. Some forty years after entering the workforce, the baby boom generation would become the "elder boom" of the twenty-first century and aged dependency under pension systems would skyrocket, as reflected in the Table 1. The percentages in this table are computed based on the estimated statistics in [19].

	Ratio (%) of 60+ population to 15-59 age group, 2005	**Ratio (%) of 60+ population to 15-59 age group, 2050**
Australia	32.86	71.61
Denmark	42.10	66.30
France	42.50	85.74
Germany	48.68	94.21
Italy	50.82	124.18
Japan	52.09	126.95
Netherlands	36.42	77.97
Spain	39.66	112.80
Sweden	48.48	76.60
Switzerland	42.63	90.75
United Kingdom	42.04	70.45
United States	32.48	59.86
World	19.05	44.75

Table 1: Now and in 50 years: elderly vs. work-force population dependency. These are the ratios (%) of 60+ population to 15-59 age group, by country, 2005 and 2050 (Medium Variant)

By today, the phenomenon of aging populations and their implications for pension costs is relatively well studied, especially in developed countries. Although their approaches have varied, many of these countries have enacted public policies to stimulate greater funding of their pension systems or reduce the benefits paid out by their public pension programs. It remains a subject of controversy whether the current and future pension systems are beneficial to all the stakeholders and whether an average household may fully rely on them.

The real estate market is constantly exhibiting volatility as the demand and supply for houses balance in response to local and nation-wide economical factors. These include prevailing inflation trends, interest rates set by central banks, and etc. The recent boom in property prices around the globe has highlighted the importance of prudent and personalized mortgage planning. Homebuyers might not mind if they are building equity in an asset that is appreciating but if house prices fall, as looks possible in the overvalued markets, new owners will find themselves further out of the pocket.

Such economical issues inherently impact not only the governments and institutions but households facing complicated financial problems. They are planning their life-long consumption style, human capital and financial wealth investment whilst setting strategies to meet their retirement goals in the uncertain markets. At the same time, households may need to fund their property mortgage, school tuition for their children or car purchase, and etc. Having such multidimensional needs, household portfolio planning is essentially an integrated problem. Hence, it is natural to ask for an integrated solution that copes with achieving the financial targets set by a household under uncertainty. This question is the cornerstone of the thesis work carried out.

2.2 Main Concepts Involved

Before formalizing the problem addressed in this work, the main concepts used in the research are presented to the reader.

2.2.1 Asset Liability Modeling

Asset Liability Modeling (ALM) has evolved into a number of enterprise-wide, specialized, and integrated applications in the financial services industry [21]. Investors, be they corporate professionals or financially conscious individuals, face challenging problems allocating their asset holdings. These are caused by multiple uncertainties of market dynamics and time. The ALM assists fund managers, asset and wealth professionals et al. in achieving specialized investment goals, covering liabilities and managing risks of their customers, operations and financial markets. These models consider various scenarios of underlying portfolio securities to realize future and present financial decisions with the anticipation of uncertainty. In reality, the ALM is used on an ongoing basis. Essentially, this reflects that, as time elapses the behaviour of financial markets, investment preferences, internal and external conditions change. Hence, the decisions made in the past need to be readjusted.

2.2.2 Household Finance

Household finance, by analogy with corporate finance, asks how households use their financial instruments to attain their objectives [3]. There are certain features that define the character of household financial problems:

- Households plan over a long but finite period of time: they set their financial goals over years, i.e. retirement age or mortgage maturity.
- Households have important non-traded assets, namely their human capital: they receive labour income but cannot sell claims to it.
- Households own illiquid assets, in particular their property which makes it costly to adjust their consumption of housing services in response to economic events.
- Households face tight constraints in their ability to borrow: their future consumption may be determined not only by their wealth and investment opportunities, but also by their net income.

2.2.3 Life-Cycle Investing

Life-cycle investing is an area that currently receives plenty of attention in the light of upcoming global ageing and subsequent restructuring of the pension systems. According to the new paradigm of life cycle finance [1], the household welfare is measured by the lifetime consumption of goods and leisure rather than by wealth. The same work highlights that the time frame for financial planning consists of multiple periods. The main risks are managed by means of precautionary saving, diversification, hedging and insuring. Underlying quantitative modelling is no longer limited to the mean-variance efficiency and Monte Carlo simulation, but rather given preference to dynamic programming and contingency-claims analysis.

2.3 Problem Definition

Given a range of financial investment opportunities, a household needs to allocate their asset holdings into a life-cycle portfolio with maximum capital goals, at the same time meet the liability obligations consistent with their mortgage. Construction and management of such a portfolio should anticipate and minimize the risks associated with financial markets volatility, economic inflation, labour income and household dynamics.

This is essentially a request for a completely new financial product - a product that yields high returns for low risk, adjusts itself to changing market conditions, and to the changing risk profiles as the household progresses through its life. Such a product would smooth out the volatility, provide consistent inflation-beating returns, and last but not the least - take the detailed decision-making out of the investment.

Uncertainty about future economic events and conditions has a very important role in the portfolio management. In this context, multiple risk factors should be considered simultaneously and decisions about the effective portfolio composition and trading strategies should be applied. This thesis suggests an optimization approach suitable for the household portfolio management and control of associated financial risks. In particular, it develops multistage stochastic programming model, that can be further tailored for specialized use. Uncertainty in the input parameters of such a model is represented by means of discrete distributions (scenarios) that capture correlation of the stochastic variables, i.e. asset returns and interest rates. Such approach may be seen as a research in financial products innovation.

2.4 Research Flow

To plan the project activities and structure the study vs. modeling efforts accordingly, the high-level research flow was established as illustrated on the Figure 1. Firstly, the study of Risk Exposure and Universe of Products is carried out, resulting in the definition of main approaches: Scenario Generation, Utility Optimization and Products Modeling. These are used to accomplish formulation of the Integrated Pension and Mortgage Portfolio Management Model. The final part of the work is to Test and Analyze the model in order to prove its correctness and assess its diverse qualities.

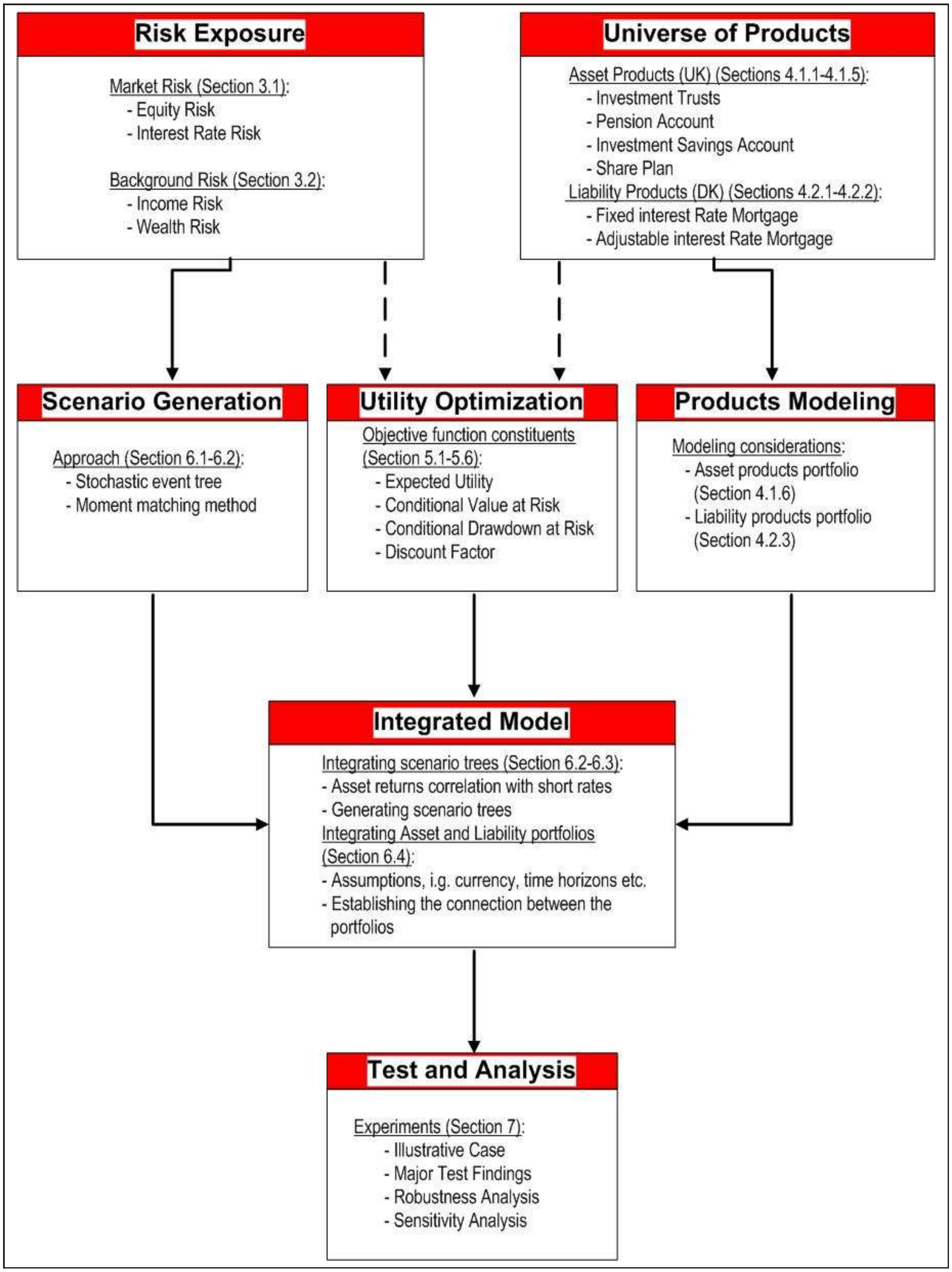


Figure 1: Research Flow of the Thesis

3 Risk Exposure

Risk factors affecting the prices of financial instruments under study and consequently the integrated portfolio value, vary from pure financial to non-systematic background risks. Most often, market risk is considered to be the most important risk to consider in the financial applications. From the household perspective, income risk is aknowledged to be the main determinant of the dynamic cash amount available for portfolio infusion.

3.1 Market Risk

The BIS[1] defines market risk as "the risk that the value of on- or off-balance-sheet positions will be adversely affected by movements in equity and interest rate markets, currency exchange rates and commodity prices". Accordingly, the main components of the market risk are:

- *Equity risk* - is the possible change of the financial instrument price over time due to adverse movements in the equity markets.
- *Interest rate risk* refers to the change in the price of the instrument due to the movements in the interest rates.
- *Currency rate risk* arises from the change in price of one currency against another.
- *Commodity risk* is the possible change in the price of the instrument due to the movements in the commodity markets.

Additionally, financial instruments are influenced by the residual risks, such as:

- *Spread risk* is the potential loss due to changes in spreads between two instruments (e.g. there is a credit spread risk between corporate and government bonds).
- *Basis risk* is the potential loss due to pricing differences between equivalent instruments, such as futures, bonds and swaps.
- *Specific risk* refers to the issuer specific risk (e.g. the risk of holding Company A stock vs. Company B bond).
- *Volatility risk* - is the risk that the price of an asset will change with time due to changes in volatility.

[1] Bank of International Settlements.

3.1.1 Equity Risk

Given the risk that the market price of the assets will change with time, Equity Risk takes different meanings depending on the asset type. Correspondingly, one may distinguish between stock market price risk, fixed income market price risk and various non-traditional instruments[2] market price risk.

Stock market price risk - encompasses the possibility of the stock price changing over time due to adverse movements of the stock market. When the stock market prices change, the present value of the investment portfolio has a risk of decreasing.

The risk measure that captures the sensitivity of the asset to the changes in the market index is β of this security when the market portfolio return changes:

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

where σ_{iM} is the covariance of the random variable asset rate of return $\tilde{r_i}$ and the market rate of return $\tilde{r_M}$, and σ_M^2 is the variance of the market rate of return.

Similarly, the *Fixed income market risk* - is the risk that the price of a fixed-income security will change with time due to adverse movements of the fixed-income market. The predominant risk of fixed income markets is the risk caused by movements in the overall level of interest rates on straight, default-free securities.

3.1.2 Interest Rate Risk

Interest Rate risk is the potential loss if the price of a security will change with the time due to movements of the general levels of interest rates. This risk effects fixed-income as well as all other securities with price dependencies on, among possibly other factors, the interest rates.

[2]For example, options, structured notes, and etc.

The general level of interest rates is determined by the interaction between supply and demand for credit. If the supply of credit from lenders rises relative to the demand from borrowers, the interest rate falls as lenders compete to find borrower for their funds. On the contrary, if the demand raises relative to supply, the interest rate will rise as borrowers are willing to pay more for increasingly scarce funds. The principal force of the demand for credit comes from the desire for current spending and investment opportunities. Supply of credit on the other hand, comes from willingness to defer spending. Besides, central banks are able to determine the levels of interest rates - either by setting them directly or by influencing the money supply - in order to achieve their economic objectives. For example, in the UK, the Bank of England sets the base rate charged to other financial institutions. When it is raised, these follow suit and raise rates to their customers, making it more expensive to borrow and slowing down economic activity. The base rate (also known as the official interest rate) will influence interest rates charged for overdrafts, mortgages, as well as savings accounts. Furthermore, a change in the base rate will tend to affect the price of property and financial assets such as bonds, shares and the exchange rate. The central bank influences the availability of money and credit by adjusting the level of bank reserves and by buying and selling government securities. These tools influence the supply of credit, but do not directly impact the demand for it. Therefore, central banks in general are not able to exert complete control over interest rates.

Inflation is also a factor. When there is an overall increase in the level of prices, investors require compensation for the loss of purchasing power, which means - higher nominal interest rates. As agents are supposed to base their decisions on real variables, it is the equilibrium between real savings and real investments that will determine the real interest rate. Hence, if this equilibrium remains the same, movements in the nominal interest rate should reflect movements in the prices or in expected future prices.

Another important factor is credit risk, which is a possibility of a loss resulting from the inability to repay the debt obligation. The larger the likelihood of not being repaid, the higher are the interest rate levels.

Time is also a factor of risk and it therefore has influence on the level of interest rates.

It is common to distinguish between short-term rates - for lending periods shorter than one year - and long-term rates for longer periods. Long-term rates are typically decomposed into two factors: the expected future level of short-term rates and a risk premium to compensate investors for holding assets over a longer timeframe. As a result, yields on long-dated securities are in general higher than short-term rates.

Figure 2 captures all the detrimental risk factors influencing the interest rate levels, summarizing the above study in accordance.

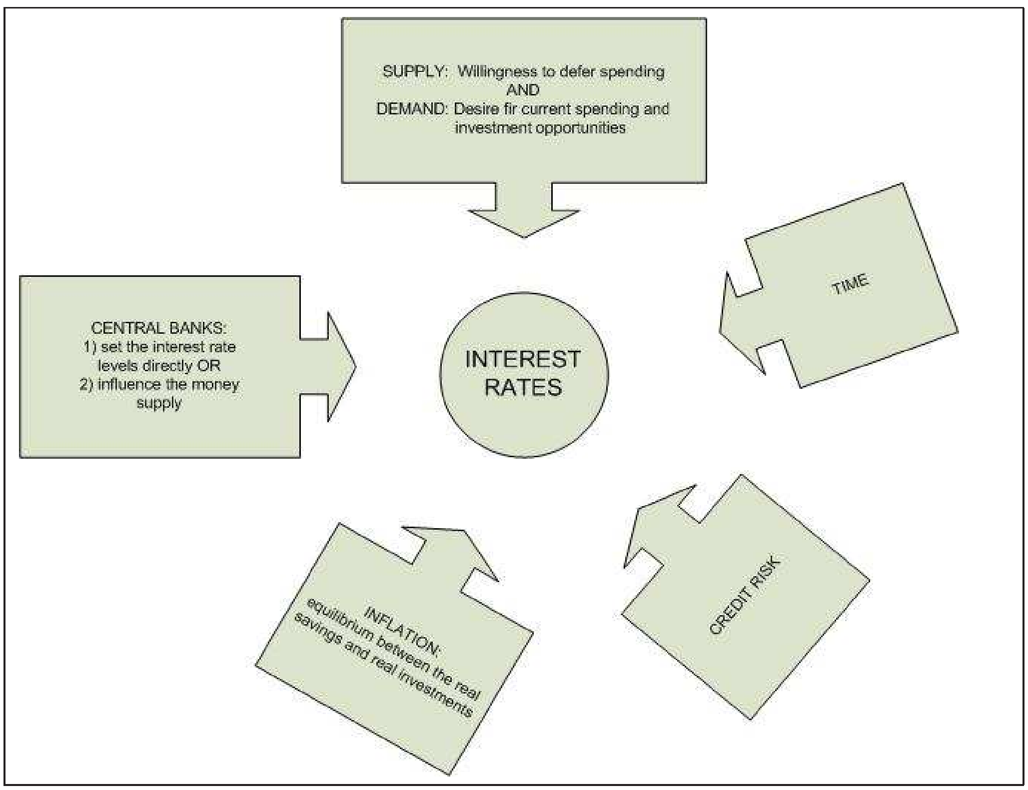


Figure 2: Detrimental Factors of Interest Rate Risk

3.2 Background Risk

3.2.1 Income Risk

In this thesis, the effect of labour income risk on the household portfolio managament is considered. The theoretical outlook to this background risk is based on the concept that a household with labour income has an implicit holding of a nontradable asset - human capital - which represents a claim to the stream of future income. It has been shown in [2] and [18] that such nontradable asset may "crowd out" explicit asset holdings in the following way.

> If labour income is totally riskless, then riskless asset holdings are strongly crowded out and the household will tilt its portfolio strongly towards risky assets. If the household is constrained from borrowing to finance risky investments, the solution may be a corner at which the portfolio is 100% risky assets. If labour income is risky but uncorrelated with risky financial assets, then riskless asset holdings are still crowded out but less strongly; the portfolio tilt towards risky assets is reduced. If labour income is positively correlated with risky financial assets, then those can actually be crowded out, tilting the portfolio towards safe financial assets.

Assuming that income dynamics is uncorrelated or only weakly correlated with risky asset returns, households with expected future income large relatively to their financial wealth should have the strongest desire to hold such assets. In a life-cycle model, an age-dependent individual profile of income is essentially represented by the tendency of increase relative to financial wealth in the youngest adulthood stage, and decline as the individual approaches retirement. This suggests that fairly young households are the most likely to be affected by borrowing constraints that limit their portfolio positions.

4 Universe of Products

One of the main challenges in this work has been to choose, study and model asset (pension) and liability (mortgage) products. Common knowledge about their structure, opportunities and limitations associated with their operation, along with main positions of uncertainty needs to be established. Main business parameters and relationships need to be chosen for modeling and appropriate assumptions to be made for capturing less important yet valuable elements of these products.

In the scope of this thesis, the asset and liability products illustrated on the Figure 3 are subject to study.

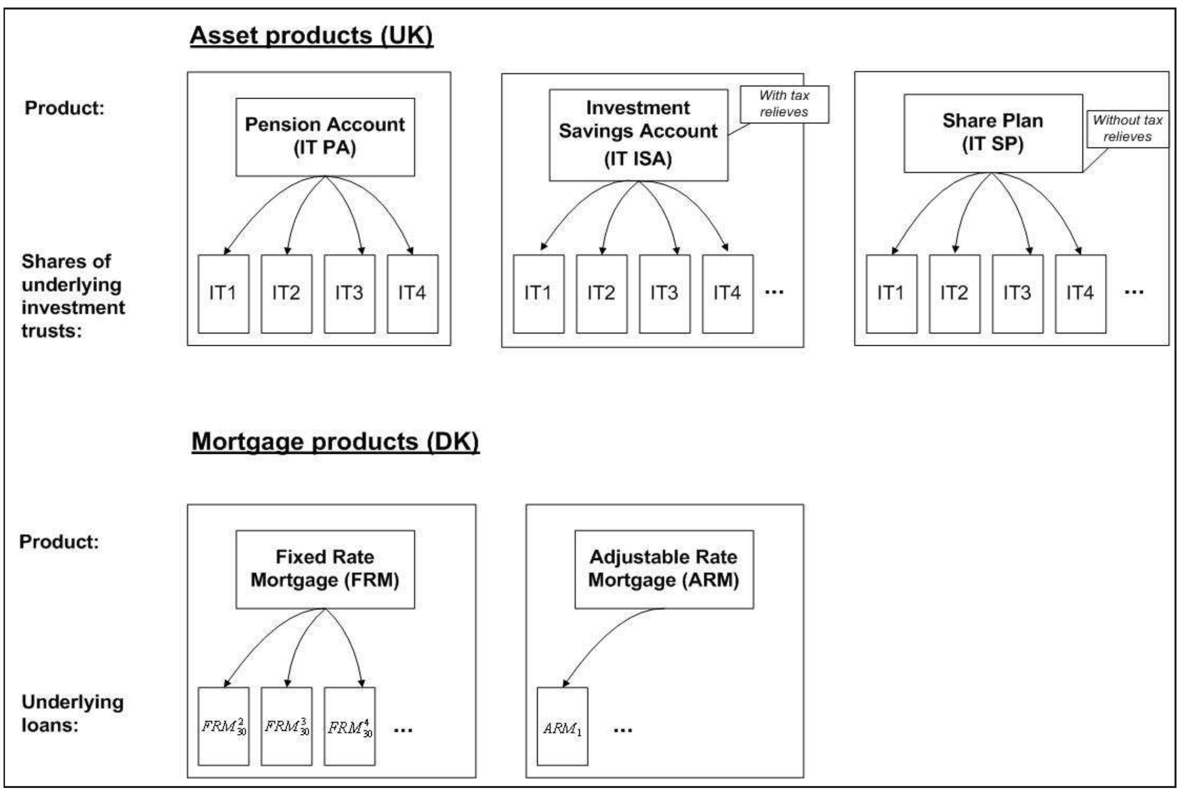


Figure 3: Asset and Liability Products Studied

In particular, these are three asset products available in the UK:

- Pension Account (IT PA),
- Investment Savings Account (IT ISA),

- Share Plan (IT SP),

and two liability products originated in Denmark:

- Fixed Rate Mortgage (FRM),
- Adjustable Rate Mortgage (ARM).

Such geographical position of product selection is based on the requirements associated with their appropriate modeling. Namely, the asset products are at a more mature level of operation and research in the UK, whereas the liability products are more mature in Denmark. Each of the asset products invests in shares of certain investment trusts (denoted by $IT1$, $IT2$, and etc.), in this way diversifying associated market risks. The liability products, on the other hand, are offered with a set of underlying loans (denoted by FRM_{30}^{2} and ARM_1 and etc.).

In the following the detailed product research and modeling considerations are presented to the reader.

4.1 Asset Products

4.1.1 Overview: Personal Pension, Savings and Investment Schemes

Private Pension Plans are defined contribution (DC) schemes which are sponsored by individual investments (bounded by a certain amount set by government), and depend on the performance of underlying securities. At retirement, its owner will take a tax-free lump sum (25%) and the rest is used to buy annuities (taxable guaranteed income). It is not possible to withdraw money from these schemes before the retirement age, however it is possible to refinance and change to a different provider.

Individual Savings Account (ISA) is an account into which an individual can save and invest without having to pay any capital gains tax on any profits made or on any income or interest received on his investment.

There are two kinds of ISA - Maxi and Mini. Each tax year, an investor can put up to £7,000 into either Maxi ISA or Mini ISA but he is not allowed to have both. With Mini ISA one can put up to £3,000 in cash and £4,000 in stocks and shares. With Maxi ISA, it is possible to invest up to

£7, 000 in stocks, shares and cash, although it still bound to the maximum amount invested in cash to £3, 000 (and in that case then, £4, 000 into stocks and shares). An investor has complete freedom over the way, amount and time of money withdrawal. In this thesis, only the Maxi ISA option is considered.

Individual Investment Plans are schemes that provide unlimited investment opportunities without any tax relieves.

4.1.2 Investment Trusts

The above described Private Pension Plan, ISA and Individual Investment Plan financial products are offered with the Investment Trusts rather than single securities underlying their policies.

Investment Trusts are companies that invest in the shares and securities of other companies. They pool investors' money and employ a professional fund manager to invest in the shares of a wide range of companies. This way even investors with small amounts of money can gain exposure, at low cost, to a diversified and professionally run portfolio of shares, spreading the risk of stock market investment. Investment trusts raise money for investing by issuing shares. Generally, this happens once - when the trust is created. This makes investment trusts close-ended: the number of shares the trust issues and therefore the amount of money raised to invest is fixed at the start.

The share prices of an investment fund are determined by supply and demand on the corresponding investment trust trade activity.

The equity and interest rate risks discussed in the *Section 3.1 Market Risk* are the major detrimental factors of uncertainty in the investment trust price dynamics. Following are investment trusts parameters that exhibit stochasticity.

Net Asset Value (NAV) of the investment trust - is the value of its assets available to shareholders after prior ranking charges have been deducted from total assets.

Net Asset Value (NAV) per share - is the value of shareholder funds expressed as an amount per ordinary share.

Bid/Sell is the price offered in the market to buy shares from an investor, also referred to as the selling price.

Offer/Buy/Ask is the price offered in the market at which shares are offered to investors also referred to as the buying price.

The *dealing spread* is the difference between the price at which the shares are sold (offer price) and purchased (bid price). The spread varies with the time and market conditions.

The *mid-market* price is calculated as the mid point between the bid and offer prices and is used to calculate the price related data (e.g. discount, yield and share price performance data).

The underlying investment product which an investor is buying is a share in a company listed on the London Stock Exchange. The price of its shares is determined by supply and demand. It is, therefore, not necessarily the same as the value of the underlying NAV per share.

Where the price of shares in an investment trust is lower than the NAV per share, the trust is said to be trading at a *discount*. When the price is higher than the NAV per share, it is said to be trading at a *premium*. The discount or premium varies depending on the demand for an investment trust's shares and represents an additional element of potential risk and reward.

Dividend yield expresses the dividend per share as a percentage of the market share price. Future dividends may be higher or lower than indicated by the current dividend yield depending on the performance of the trust.

Trusts specialize in what they aim to achieve for their shareholders. Some try and maximize income. Others aim exclusively for growth. Some trusts aim to provide a combination of income and capital growth.

Investment trusts, being companies, can borrow to purchase additional investments. This is called ’financial gearing’. It allows investment trusts to take advantage of a long-term view on a sector or to take advantage of a favorable situation or a particularly attractive stock without having to sell existing investments. Financial gearing works by magnifying the investment trust’s performance. If a trust ’gears up’ and then markets rise and the returns outstrip the costs of borrowing, the overall returns to investors will be even greater. But there is a downside to gearing too. If markets fall and the performance of the assets in the portfolio is poor, then losses suffered by the investor will be also magnified. Although the term ’gearing’ when applied to investment trusts usually describes the effect on the asset value, it also affects a trust’s revenue and dividend potential. Not all investment trusts use financial gearing and many of those that do only use it to a very limited extent. Other investment vehicles are unable to borrow to purchase additional investments to the same extent as investment trusts.

4.1.3 Investment Trusts Pension Account

Investment Trust Pension Account (ITPA) provides an exposure to investment trusts and lower risk cash fund, with low charges and flexible payment methods. It intends to provide an income in retirement and make the best use of available tax benefits.

Contribution

An investor may choose lump sum or regular monthly payments into any of the underlying investment trusts. There is a minimum lump sum investment of £1,000 gross per investment trust or cash fund in the ITPA. The minimum for regular saving is £100 gross per investment trust or cash fund. Most individuals are allowed to contribute up to £3,600 gross per annum without any reference to earnings[3].

> Monthly contributions can be made on either the 1st or the 15th of the month. The Dealing Day for Direct Debit contributions collected on the 1st of the month will be the 8th of the month.

[3] Contributions of over £3,600 are based on net relevant earnings in the ’basis’ tax year.

The Dealing Day for the contributions collected on the 15th of the month will be the 22nd of the month[4].

Furthermore, there is a limit on the maximum contribution per annum based on the net relevant earnings from the chosen basis year as presented in the Table 2.

The £3, 600 contribution limit and the maximum contribution as a percentage of earnings are

Age on the first day of tax year	Maximum % of earnings
35 or less	17.5%
36-45	20%
46-50	25%
51-55	30%
56-60	35%
61-74	40%

Table 2: Maximum Contribution per Annum Based on the Net Relevant Earnings.

total contribution as a percentage of earnings are total contributions in a tax year. Therefore, an investor must take account of any contributions being paid to any other personal pensions, retirement annuity contracts or trust schemes.

In the integrated pension and mortgage portfolio management problem modeled in this thesis only the lump sum contribution option is considered.

Investment Dealing

The shares of the investment trusts are owned on behalf of the Accountholders by the Trustee who is responsible for the investment dealing. This involves purchase of new shares, sales of the existing ones, contribution, switching among the investment links, all at the prevailing market prices. The dealing days on the account are the 8th, 15th, 22nd and 29th days of a month, or if these days fall at a weekend or bank holiday, the next working day. For simplicity of modeling, it is assumed that investment dealing takes place on the annual basis.

Dividends

Any dividends received will be reinvested into the additional shares of the same trust, unless an in-

[4]For simplicity purposes of modeling, it will be assumed that the 1st day of month is chosen by an investor.

vestor has switched out of that particular trust, then the dividend will be reinvested on the current allocation. A cash fund does not pay dividends. Some investment trust companies pay dividends on a quarterly or monthly basis. The majority pay dividends twice a year.

Charges

Charges describe the cost structure of the IT PA and are outlined in the following:

- There is an 0.3% dealing charge on purchases (including the cash fund) which is capped at £50, plus 0.5% Government stamp duty (excluding the cash fund).

- There is no dealing charge on the sale of shares.

- There is no annual charge on the pension account but the investment trusts and the cash fund have underlying expenses accumulated in the Total Expense Ratio.

Total Expense Ratio takes into account the *Annual Management Fee* paid to the manager and all other operating expenses such as audit fees and irrecoverable VAT. Where appropriate, tax relief allowable on expenses has been included. It represents the total net deductions (excluding interest payments) as a percentage of the trust's average net assets over a year.

Withdrawal

An investor cannot take his money out until receiving his pension benefits at retirement horizon. The withdrawal itself can be arranged in different ways. Two possibilities are:

- Up to 25% of the fund value can be paid as a tax free lump sum and the rest is used to purchase an annuity (which is treated as the earned income and therefore is assessable to tax).

- The withdrawal can be arranged through the income drawdown which is the facility that allows taking income at any time after the retirement horizon is reached and keeping the rest of the capital invested. Usually the income amount is limited to the 35%-100% of the income an investor would have if he bought a single-life level annuity.

As portfolio management for individuals after they have reached retirement age is not considered in this thesis, the simplified approach is taken: a withdrawal from ITPA is due to the investor retirement[5].

4.1.4 Investment Trusts Investment Savings Account

Investment Trusts Investment Savings Account (IT ISA) is a flexible Investment Trusts wrapper that protects from income or capital gain tax on the investment returns.

Contribution

Similarly to the IT PA, IT ISA is offered with the lump sum or regular monthly contribution options. The minimum for regular saving is £100 gross per investment trust and the minimum lump sum contribution is £1, 000 correspondingly. For consistency in the modeling of IT ISA, only the lump sum contribution is considered.

Investment Dealing

Shares that IT ISA invests in are purchased from a broker at the best offer price available at the time of the order. The selling of shares held in the IT ISA is made at the prevailing bid price through a withdrawal of shares to a value of at least £100. The value of shares remaining after the sales should be above £1, 000.

Dividends

Unless otherwise required by an investor, all his dividends shall be reinvested (minimum £10) in shares of the same trust.

Charges

The cost structure of IT ISA is presented in the following:

- There is a 1% transaction charge on purchases and sales in the IT ISA. This is subject to a maximum of £50 per trust. Moreover, 0.5% Government stamp duty applies to all purchases.

[5] For example, an investor may decide to invest the accumulated wealth into the income drawdown.

- Dividend reinvestments are subject to a 1% transaction charge (£50 maximum) plus 0.5% Government Stamp Duty.
- There is also a £25 annual account charge associated with the IT ISA.
- The underlying investment trusts in the IT ISA also bear expenses which are accumulated in the Total Expense Ratio[6].

Withdrawal

The withdrawal on the account is possible at any time, subject to:

- Minimum withdrawal amount is £100.
- The account value is not allowed to drop below £1,000 as a result of withdrawal operation.

4.1.5 Investment Trusts Share Plan

Investment Trusts Share Plan (IT SP) allows investing directly into shares of investment trusts, either on a lump sum or monthly basis. Holdings are subject to tax.

Contribution

Similarly to the IT PA and IT ISA, the IT SP is offered with the regular and lump sum contribution options. The minimum lump sum contribution is £500 gross per individual trust in the IT SP. The minimum for regular saving is £50 gross per investment trust. There is no maximum investment into the IT SP.

For consistency in modeling of IT SP among other asset products in the scope, only the lump sum contribution is considered.

Investment Dealing

The selling of shares held in the IT SP is made through a withdrawal of shares to a value of at least £50. The value of shares remaining after the sales should be above £500.

[6] See the **Charges** paragraph in the *Section 4.1.3 Investment Trust Pension Account.*

Dividends

- Unless agreed to be paid out, all dividends are reinvested (minimum of £10) in shares of the same trust.
- All uninvested cash balance is kept in the non-interest-bearing client account (subject to a flat-rate charge in accordance with Inland Revenue Regulations).

Charges

The cost structure of IT SP is described in the following:

- There is a 1% transaction charge on purchases and sales in the IT SP. This is subject to a maximum of £50 per trust. Moreover, 0.5% Government stamp duty applies to all purchases.
- Dividend reinvestments are subject to a 1% transaction charge (£50 maximum) plus 0.5% Government stamp duty.
- There are no annual charges on the IT SP itself, however the underlying investment trusts bear expenses accrued in the Total Expenses Ratio.

Withdrawal

The withdrawal on the IT SP is possible at any time, subject to:

- Minimum withdrawal amount is £50.
- The IT SP value is not allowed to drop below £500 as a result of withdrawal operation.

4.1.6 Dynamics and Policy Constraints of Asset Products

Given a set of scenarios $l \in \Omega$ generated as described in the *Section 6.1 Scenario Generation*, a set of asset products $k \in \mathcal{K}$, a set of investment trusts $i \in \mathcal{I}$ underlying these products, and a set of time periods $t \in \{t_0, t_1, \ldots, t_T\}$, the following stochastic variables, being the main detrimental factors of uncertainty in the integrated pension and mortgage problem, are defined:

PO^l_{it} = Offer price (used in the purchase transactions) of the trust i at the time t, scenario l,

PB^l_{it} = Bid price (used in the sales transactions) of the trust i at the time t, scenario l,

PM^l_{it} = Midmarket price (used in the capital valuation) of the trust i at the time t, scenario l,

$r^l_{(Inv)it}$ = Return of the trust i at the time t, scenario l at the prevailing midmarket prices.

The policies of IT PA, IT ISA and IT SP products are further defined by a number of deterministic parameters:

APC_k = Annual Product Charge associated with the product k,

TER_i = Total Expense Ratio associated with the investment trust i,

PFR_k = Purchase Fee Ratio on the purchase dealing transactions of the product k,

$PFCap_k$ = Purchase Fee Cap on the Purchase Fee value associated with the purchase dealing transactions of the product k,

$GovStamp_k$ = Government Stamp Duty on the investment dealing transactions of the product k,

SFR_k = Sales Fee Ratio on the sales dealing transactions of the product k,

$SFCap_k$ = Sales Fee Cap on the Sales Fee value associated with the sales dealing transactions of the product k,

$C_{(Min)k}$ = Minimum lump sum contribution value of the product k,

$C_{(Max)k}$ = Maximum annual contribution value of the product k,

$W_{(Max)k}$ = Maximum withdrawal allowed at any time on the product k,

$W_{(Rem)k}$ = Minimum remaining amount required on the product k's account after the withdrawal.

To model the cash infusion dynamics into the portfolio, the following income parameter is used (represening a percentage of income available for financial investment per year):

$$ACI_t \quad = \quad \text{Available Cash for Investing at the time } t$$

The decision and auxiliary variables for modeling the IT PA, IT ISA and IT SP policies are:

C^l_{kt} = Contribution value into the product k at the time period t, scenario l,

U^l_t = Value of holding in the cash account at the time period t, scenario l,

V^l_{kt} = Value of holding in the product k account at the time period t, scenario l,

Z^l_{ikt} = Capital value of the investment trust i held within the product k at the time period t, scenario l,

z^l_{ikt} = Number of investment trust i's shares held via the product k at the time period t, scenario l,

W^l_{kt} = Withdrawal value from the product k at the time period t, scenario l,

X^{+l}_{ikt} = Value of the investment trust i purchased within product k at time t, scenario l,

x^{+l}_{ikt} = Number of the investment trust i's shares purchased via product k at time t, scenario l,

pf^l_{ikt} = Purchase dealing fee associated with the product k, underlying investment trust i, at the time t, scenario l,

Δpf^l_{ikt} = Positive or zero difference from the purchase cap, associated with the purchase fee paid on the dealing in the investment trust i's shares at the time t, scenario l,

X^{-l}_{ikt} = Value of the investment trust i sold within product k at time t, scenario l,

x^{-l}_{ikt} = Number of the investment trust i's shares sold via the product k at the time t, scenario l,

sf^l_{ikt} = Sales dealing fee associated with the product k, underlying investment trust i, at the time t, scenario l,

Δsf^l_{ikt} = Positive or zero difference from the sales cap, associated with the sales fee paid on the dealing in the investment trust i's shares at the time t, scenario l,

AW^l = Accumulated wealth of the portfolio at the optimization horizon, scenario l.

All variables are positive, hence the short selling is not allowed.

Available cash for investing at any time period t is used to finance the contribution in the IT PA, IT ISA and IT SP asset products. According to the cash equilibrium principle [20], at any time t, the available cash to invest, ACI_t is distributed among these products. Moreover, the same cash account is used to pay annual product charges:

$$ACI_t \geq \sum_{k\in\mathcal{K}}(C^l_{kt} + ATP_k), \quad \forall t \in \{t_0, \dots, t_T\}, \forall l \in \Omega \tag{1}$$

Once a contribution into an investment product is made, investments into its underlying trusts can be allocated. The cash dynamics at the product level also follow the equilibrium principle, distributing the contribution value into the purchase of underlying investment trusts, accounting for the sales of trusts, and justifying the cash flow for the required withdrawal from the product account. It is noteworthy to mention that the total expenses on the underlying investment trusts are paid on the holding value of the investment trust at the corresponding time period:

$$\begin{aligned} C^l_{kt} + \sum_{i\in\mathcal{I}} X^{-l}_{ikt} &= \sum_{i\in\mathcal{I}}(X^{+l}_{ikt} + TER_i \cdot Z^l_{ikt}) + W^l_{kt}, \quad \forall k \in \mathcal{K}, \forall t \in \{t_1, \dots, t_T\}, \forall l \in \Omega \\ C^l_{kt_0} &= \sum_{i\in\mathcal{I}}(X^{+l}_{ikt_0} + TER_i \cdot Z^l_{ikt_0}), \quad \forall k \in \mathcal{K}, \forall l \in \Omega \end{aligned} \tag{2}$$

The asset dynamics at the investment trust level underlying any of the IT PA, IT ISA or IT SP products is also based on the equilibrium principle. Namely, the total inbound value of assets at any time and scenario must be equal to the total outbound value. For the illustration of the asset dynamics equilibrium see Figure 4. Current value of purchased shares infused by the hold value from the previous month and reinvested dividends is used to hold and sell shares correspondingly at the present month:

$$\begin{aligned} x^{+l}_{ikt} + z^{anc(l)}_{ikt-1}(1 + r^{anc(l)}_{(Inv)it-1}) &= z^l_{ikt} + x^{-l}_{ikt}, \quad \forall i \in \mathcal{I}, \forall k \in \mathcal{K}, \forall t \in \{t_1, \dots, t_T\}, \forall l \in \Omega \\ x^{+l}_{ikt_0} &= z^l_{ikt_0}, \quad \forall i \in \mathcal{I}, \forall k \in \mathcal{K}, \forall l \in \Omega \end{aligned} \tag{3}$$

The above formulated dynamics at the cash account, product and investment trust levels (1)-(3) is connected into a network-like model which is illustrated on the Figure 5. The nodes of this network

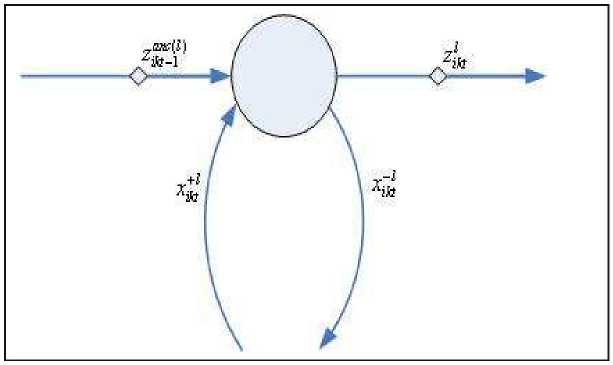


Figure 4: Asset dynamics equilibrium for the investment trust i underlying the product k, at time t, scenario l. Hold value from the previous time period ($z^{anc(l)}_{ikt-1}$) plus the purchased shares (x^{+l}_{ikt}) is equal to the current hold value (z^{l}_{ikt}) plus the shares sold (x^{-l}_{ikt}).

are positioned at three levels and are spanned over time and number of entities at each level. Bottom-level nodes signify the cash account state over time. The contribution is made from the cash account into the IT PA, IT ISA and IT SP products which is depicted by the bold arrows directed towards the middle-level nodes that represent states of these schemes over time.

Next, the shares in their underlying investment trusts i are purchased or sold which is signified by the directed arrows from the account-level nodes to the investment trust (top-level) nodes. The investment trust holding value is transfered to the corresponding node at the subsequent month and all the dividends are reinvested. At the first time period, only purchase operations take place. Withdrawal from a product account is represented by the bold arrows connecting the middle-level product and the corresponding bottom-level cash account nodes. Total amount of withdrawals is further matched to liability payments, as described in the *Section 6.4 Integrating the Pension and Mortgage Portfolios into a Multistage Stochastic Programming Model.*

This is a simplified version of the asset products network which visualizes only the concept behind the asset portfolio dynamics. For more details on the integrated pension and mortgage portfolio network, see *Section 6.4 Integrating the Pension and Mortgage Portfolios into the Multistage SP Model.*

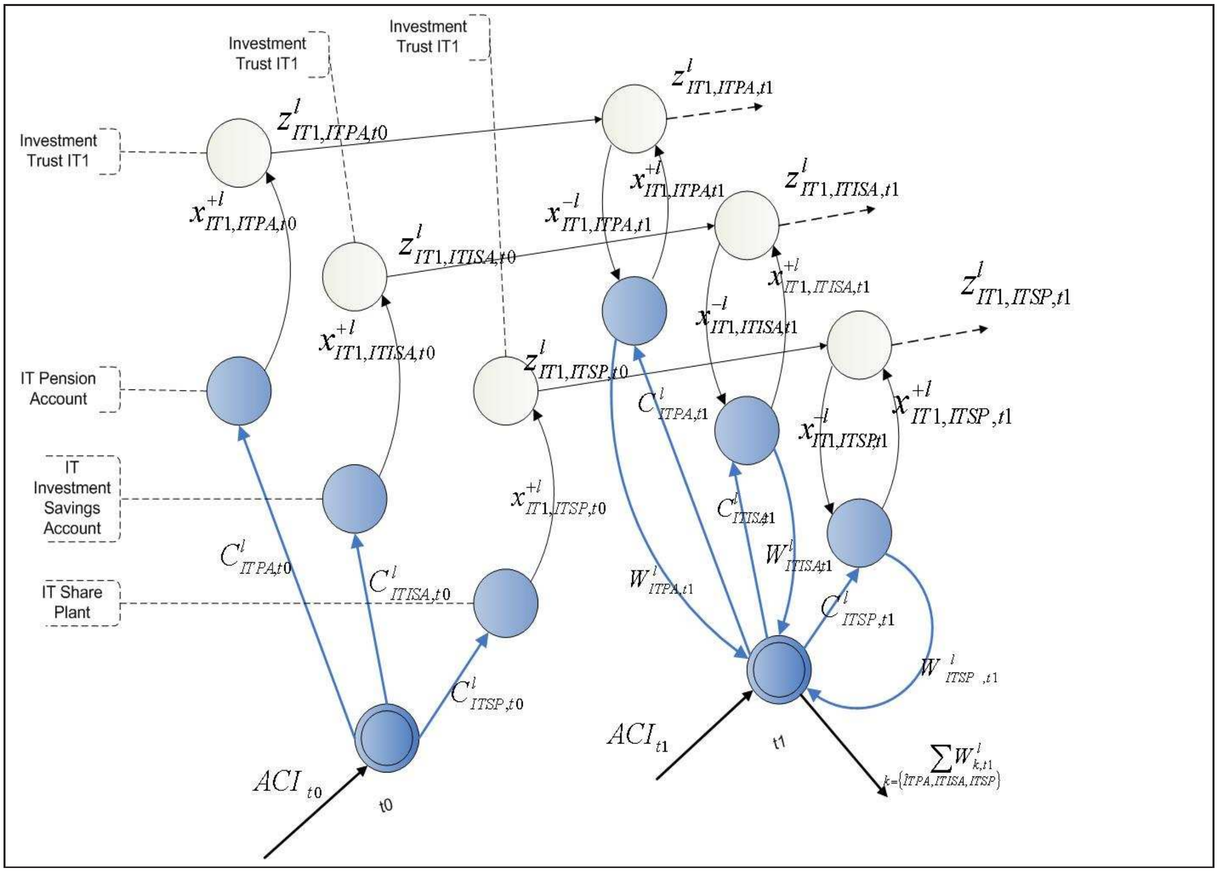


Figure 5: Asset side network. At any time t, scenario l available cash to invest (ACI_t) is distributed among the products k at values $C_{k,t}$. At the product level, this cash is used to purchase ($X^{+l}_{i,k,t}$) and sell ($X^{-l}_{i,k,t}$) shares of investment trusts i. The withdrawal ($W^l_{k,t}$) from the product k is accumulated in the cash account ($\sum_{\forall k \in \mathcal{K}} W^l_{k,t}$).

The lump sum contribution into the product k is required to be greater than a certain minimum $C_{(Min)k}$ and less than based on the net relevant earnings maximum $C_{(Max)k}$:

$$C_{(Min)k} \leq C^l_{kt} \leq C_{(Max)k}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_0, \ldots, t_T\}, \quad \forall l \in \Omega \tag{4}$$

The scenario-specific purchase value of the investment trust is composed of its shares acquired at the corresponding offer price, adjusted by the government stamp duty, and the purchase fee calculated on the purchase dealing value:

$$\begin{aligned}
X^{+l}_{ikt} &= PO^l_{it} \cdot x^{+l}_{ikt}(1 + GovStamp_k) + pf^l_{ikt} \\
&\quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_0, \ldots, t_T\}, \quad \forall l \in \Omega \\
\Delta pf^l_{ikt} &= PFCap_k - PFR_k \cdot PO^l_{it} x^{+l}_{ikt} \\
&\quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_0, \ldots, t_T\}, \quad \forall l \in \Omega \\
pf^l_{ikt} &= PFCap_k - \Delta pf^l_{ikt} \\
&\quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_0, \ldots, t_T\}, \quad \forall l \in \Omega
\end{aligned} \tag{5}$$

Similarly, the shares sold at the current bid price, adjusted by the sales fee calculated on the dealing value constitute the sales value of the investment trust:

$$\begin{aligned}
X^{-l}_{ikt} &= PB^l_{it} \cdot x^{-l}_{ikt} + sf^l_{ikt} \\
&\quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega \\
\Delta sf^l_{ikt} &= SFCap_k - SFR_k \cdot PB^l_{it} x^{-l}_{ikt} \\
&\quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega \\
sf^l_{ikt} &= SFCap_k - \Delta sf^l_{ikt} \\
&\quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega
\end{aligned} \tag{6}$$

Withdrawal from the investment products is limited by the maximum amount allowed to be cashed and the remaining capital value in the product account not falling below a certain level:

$$\begin{aligned}
W^l_{kt} &\leq W_{(Max)k} \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega \\
W^l_{kt} &\leq \textstyle\sum_{i \in \mathcal{I}} Z^l_{ikt} - W_{(Rem)k} \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega
\end{aligned} \tag{7}$$

At any time period t, the scenario l dependent capital value of the investment trust i (underlying product k) is calculated using the corresponding midmarket price:

$$Z_{ikt}^{l} \quad = \quad PM_{it}^{l} \cdot z_{ikt}^{l} \quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_0, \ldots, t_T\}, \quad \forall l \in \Omega \tag{8}$$

Finally, the accumulated wealth of the asset portfolio is determined by summing up capital values of all investment trusts held in it at the time horizon T:

$$AW^{l} = \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} Z_{ikt_T}^{l}, \quad \forall l \in \mathcal{N}_{\mathcal{T}} \tag{9}$$

4.2 Liability Products

4.2.1 Overview: Mortgage Products

There are two main types of mortgages: Repayment and Interest-only.

Repayment is the traditional type of mortgage. On a regular basis, its buyer will be paying an interest to the mortgage lender, together with a small portion of the initially borrowed amount. Over time a greater proportion of the regular payments will be used to repay the capital. After the agreed length of the mortgage, the buyer has completely repaid the loan.

With the *interest-only* mortgage, the buyer pays interest on the amount borrowed on a regular basis and does not make any inroads into the loan itself until the end of the mortgage term. Then, the mortgage is expected to be paid back in full.

Danish mortgage bond products have been actively studied from the financial optimization perspective, e.g. in [13], [12] and [14]. For practical reasons of modeling and scenario generation, loans on property offered in Denmark are assumed to be available to an English investor whose asset portfolio consists of the IT PA, IT ISA and IT SP, with underlying investment trusts. Danish mortgage products possess certain features, i.e. an early prepayment option with respect to prevailing market prices, caps on interest rates of the ARMs, etc. which protect the mortgagor from the market and interest rate risks. These features make Danish mortgage products an attractive choice for the study in the scope of this thesis. The UK mortgage market, on the other hand, is characterized by products without such protection and moreover - would demand more dedicated resource than available for quality modeling. In addition, holding the financial assets in different countries offers international diversification benefits by reducing the total risk of the portfolio.

4.2.2 Fixed Interest Rate Mortgage and Adjustable Interest Rate Mortgage loans

Fixed Rate Mortgage (FRM) loan, when issued must be prepaid at the fixed interest rate for its duration, usually 10-30 years. The principal prepayment and the costs associated with this loan are calculated on the outstanding debt face value at any time until the time horizon of the mortgage. Therefore, although the interest rate level is fixed, the real payment is dynamic due to the changes in the loan price following the base interest rates set by the central bank[7].

Further, when issuing an FRM, the mortgagor is granted with the buy-back delivery option[8], meaning that he has a right, not an obligation, to fully prepay his loan at any time before its maturity at the prevailing market prices. If compared with the non-callable bond, this option is more expensive in the yield terms of the callable bond, reflecting a risk premium to the mortgage provider, due to the uncertainty of the future yields of his invesments.

Adjustable Rate Mortgage (ARM) loans are funded by means of the non-callable bullet bonds with a short maturity from 1 to 11 years. The priciple behind the ARM loan is that the borrower takes out a 20- or 30-year annuity loan where the interest rate is adjusted at regular intervals - usually 1 year. When the remaining debt of the loan needs to be refinanced, new bonds are issued at the new interest rate which is determined based on the prices of the new bonds. Hence, the prepayment of the ARM loan is based on variable interest rate and variable prices, both dependent on the base interest rates CIBOR set by the National Bank of Denmark. As mentioned above, the ARM loan does not offer any embedded call options. Moreover, its price has a risk of increasing to such level that a mortgagor is not able to prepay his loan, if he decides to withdraw from it before the mortgage maturity.

In this Master Thesis work, the set of FRM loans with thirty years to maturity and one-year ARM are available to issue at the consequent years along the portfolio time span as described in the Table 3.

Loan	**Description**
ARM_1	One-year adjustable rate loan
FRM_{30}^{2}	30-years to maturity, fixed 2% coupon
FRM_{30}^{3}	30-years to maturity, fixed 3% coupon
FRM_{30}^{4}	30-years to maturity, fixed 4% coupon
FRM_{30}^{5}	30-years to maturity, fixed 5% coupon
FRM_{30}^{6}	30-years to maturity, fixed 6% coupon
FRM_{30}^{7}	30-years to maturity, fixed 7% coupon

Table 3: Mortgage Loans Considered in the Scope

[7]In this case, Danmarks Nationalbank - National Bank of Denmark.

[8]This option is a distinguishing feature available only on the Danish mortgage market [14].

It is noteworthy to mention that market risk associated with Danish mortgage bond is often hedged by means of early prepayment options and caps on ARMs. Alternatively FRMs, although offered at the higher interest rates than ARMs, also protect against interest rate risk. In this way they trade-off the possibility that if market rates indeed fall, the initial contractually agreed interest rate will still be required. There is evidence [14] that FRM and ARM sensitivities to the interest rate changes are negatively correlated and thus, the associated risks can be diversified by combining these products into one portfolio underlying the mortgage agreement.

4.2.3 Dynamics and Policy Constraints of Mortgage Products

Given a set of scenarios $l \in \Omega$ generated as described in the *Section 6.1 Scenario Generation*, a set of mortgage loans $j \in \mathcal{J}$, and a set of time periods $t \in \{t_0, t_1, \ldots, t_T\}$, the following parameters capture the uncertainty in the mortgage portfolio:

$$
\begin{aligned}
r^l_{(M)j} &= \text{Mortgage interest rate on the loan } j \text{ in the scenario } l,\\
K^l_j &= \text{Loan } j\text{'s price in the scenario } l,\\
CallK^l_j &= \text{Loan } j\text{'s call price in the scenario } l,
\end{aligned}
$$

The mortage policies are further defined by the following deterministic parameters:

$$
\begin{aligned}
\gamma &= \text{Tax reduction fee rate (\% of the outstanding face value)},\\
b &= \text{Administration fee rate (\% of the oustanding debt)},\\
\beta &= \text{Tax reduction from the administration fees},\\
\varrho &= \text{Fixed cost of refinancing},\\
\eta &= \text{Transaction fee rate (on sales and purchases of loans)}.
\end{aligned}
$$

Market prices of the property financed by the mortgage portfolio are important input parameters:

$$
\begin{aligned}
IA &= \text{Initial amount needed by the mortgagor},\\
HP^l_T &= \text{Market price of the house at the mortgage time horizon } T, \text{ scenario } l.
\end{aligned}
$$

To model the mortgage product life-cycle and policy constraints, the following stochastic variables are defined:

$$
\begin{aligned}
y_{jt}^{-l} &= \text{Prepayment value of the loan } j\text{, at the time } t\text{, scenario } l,\\
y_{jt}^{+l} &= \text{Issuance value of the loan } j\text{, at the time } t\text{, scenario } l,\\
RG_{jt}^{l} &= \text{Outstanding debt of the loan } j\text{, at the time } t\text{, scenario } l,\\
A_{jt}^{l} &= \text{Principal payment of the loan } j\text{, at the time } t\text{, scenario } l,\\
B_{t}^{l} &= \text{Total payment on the mortgage portfolio required at the time } t\text{, scenario } l,\\
PP_{l} &= \text{Prepayment amount at the mortgage horizon, scenario } l,\\
SB^{l} &= \text{Total payment incurred over time in the scenario } l,\\
Profit_{(T)}^{l} &= \text{Profit in the scenario } l \text{ at the mortgage horizon when all debt is prepaid.}
\end{aligned}
$$

All the above listed variables are positive, meaning that short selling does not occur.

The initial amount is financed by issuing loans at the first time period. At any subsequent time moment, the principal of cash flow conservation stipulates the total value of issued loans being spent on prepaying or holding them:

$$
\begin{aligned}
\textstyle\sum_{j\in\mathcal{J}} K_j^l \cdot y_{jt_0}^{+l} &\geq IA, \quad \forall l \in \Omega\\
\textstyle\sum_{j\in\mathcal{J}} K_j^l \cdot y_{jt}^{+l} &= \textstyle\sum_{j\in\mathcal{J}} CallK_j^l \cdot y_{jt}^{-l} \quad \forall t \in \{t_1,\ldots,t_T\}, \quad \forall l \in \Omega.
\end{aligned} \tag{10}
$$

The liability flow dynamics follows the equilibrium principle. This ensures that for any loan at any time between the initial and the final moments, its outstanding debt consists of its outstanding debt from the preceding time period plus the newly issued debt minus its prepaid value and relevant principal payment, as illustrated on the Figure 6. At the initial time period, the outstanding debt in any loan originates from its issued value. At the final time period, on the contrary, it is not allowed to issue any loans:

$$
\begin{aligned}
RG_{jt}^{l} &= RG_{jt-1}^{anc(l)} + y_{jt}^{+l} - y_{jt}^{-l} - Ajt^{l}, \quad \forall j \in \mathcal{J}, \quad \forall t \in \{t_1,\ldots,t_T\}, \quad \forall l \in \Omega\\
RG_{jt_0}^{l} &= y_{jt_0}^{+l}, \quad \forall j \in \mathcal{J}, \quad \forall l \in \Omega\\
y_{jt_T}^{+l} &= 0, \quad \forall j \in \mathcal{J}, \quad \forall l \in \Omega.
\end{aligned} \tag{11}
$$

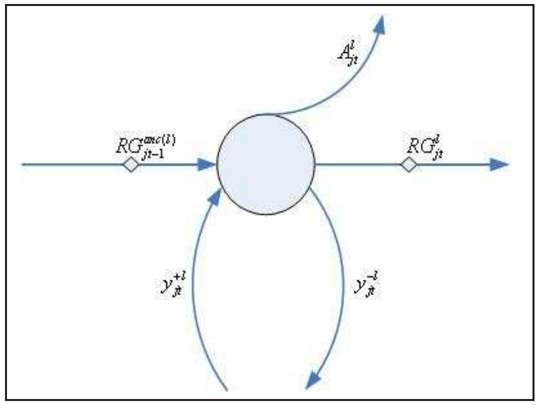


Figure 6: Liability dynamics equilibrium for the mortgage loan j at time t, scenario l. Hold value from the previous time period ($RG^{anc(l)}_{jt-1}$) plus the newly issued debt (y^{+l}_{jt}) is equal to the current hold value (RG^l_{jt}) plus the prepaid value (y^{-l}_{jt}) and relevant principal payment (A^l_{jt}).

The principal payment is determined on the annuity basis, in this way amortizing total principal in the form of regular payments until the time horizon:

$$A^l_{jt} = RG^{anc(l)}_{jt-1}\left(\frac{r^l_{(M)j}}{1-(1+r^{anc(l)}_{(M)j})^{-T+t-1}} - r^{anc(l)}_{(M)j}\right), \quad \forall j \in \mathcal{J}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega \tag{12}$$

Whilst the loans are being issued, prepaid and held in the portfolio, the annuity payment, interest rate tax, administration and transaction fees must be met by a mortgagor. By accumulating these into a total value, the payment amount on the liability products is determined in the following manner. It is worth mentioning that at the initial time only the transaction costs on issuing the loans must be paid.

$$\begin{aligned} B^l_{jt} &= A^l_{jt} + r^{anc(l)}_{(M)j}(1-\gamma)RG^{anc(l)}_{jt-1} + b(1-\beta)RG^{anc(l)}_{jt-1} + \eta(y^{+l}_{jt} + y^{-l}_{jt}), \\ & \quad \forall j \in \mathcal{J}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega \\ B^l_{jt_0} &= \eta(y^{+l}_{jt_0}) + \varrho, \quad \forall j \in \mathcal{J}, \quad \forall l \in \Omega \\ B^l_t &= \textstyle\sum_{j\in\mathcal{J}} B^l_{jt}, \quad \forall t \in \{t_0, \ldots, t_T\}, \quad \forall l \in \Omega \end{aligned} \tag{13}$$

The total prepayment made at the mortgage time horizon is accumulated from the final loan value:

$$PP^l_T = \sum_{j\in\mathcal{J}} CallK^l_j \cdot y^{-l}_{jT}, \quad \forall l \in \Omega \tag{14}$$

Portfolio profit at the mortgage time horizon is determined on the basis of the prevailing house prices[9]:

$$
\begin{aligned}
Profit^{l}_{(T)} &= HP^{l}_{T} - PP^{l}_{T}, \quad \forall l \in \Omega \\
PP^{l}_{T} &\leq HP^{l}_{T}, \quad \forall l \in \Omega
\end{aligned} \tag{15}
$$

[9]For simplicity of the model and clarity of the strategies it generates, the house price is a static parameter which is always larger or equal to the total prepayment amount.

5 Utility Optimization

Aside from modeling the asset (pension) and liability (mortgage) products, it is important to formulate the objectives an integrated portfolio should achieve. On one hand, an investor may be interested in accumulating the maximum possible wealth of the portfolio, and on the other - incurring minimum possible losses or other underperformance indicators, matching his risk aversion. The approach suggested in this thesis combines the utility function with risk management methods, i.e. Conditional Value at Risk or Conditional Drawdown at Risk. The reseach and modeling details behind these are presented in the following.

5.1 Expected Utility

Financial optimization problems often employ concepts associated with the Expected Utility Theory, which introduces a *Utility Function* as an integrating measure, assigning a value (utility) to each point of the possible outcomes distribution.

In general, optimization of the expected utility function is formulated as:

$$\begin{aligned} &max \quad \mathcal{E}[\mathcal{U}(\mathcal{R}(x,\tilde{r}))] \\ &\text{s.t.} \quad x \geq 0, \quad x \in \mathcal{K} \end{aligned}$$

where $\mathcal{R}(x,\tilde{r})$ is the expected return of the portfolio with asset positions x and underlying instrument returns $\tilde{r}$ which are uncertain; $\mathcal{U}$ is the real-valued concave risk averse function, and convex set $\mathcal{K}$ consists of problem-specific constraints.

Some of the popular utility functions are ($\mathcal{R}$ is used without parameters nor indexes for simplicity):

- Quadratic utility function

  $$\mathcal{U}(\mathcal{R}) = \overline{\mathcal{R}} - \lambda(\mathcal{R} - \overline{\mathcal{R}})^2, \quad \lambda \geq 0$$

  where parameter λ measures investor's risk aversion ($\lambda = 0$ represents a risk-neutral investor).

- Isoelastic utility function

$$\mathcal{U}(\mathcal{R}) = \frac{1}{\gamma}(1+\mathcal{R})^{\gamma}, \quad \gamma \leq 1$$

where parameter γ is a measure investor's risk tolerance ($\gamma = 1$ represents a risk-neutral investor).

- Logarithmic utility function

$$\mathcal{U}(\mathcal{R}) = log(1+\mathcal{R})$$

An investor who ranks investment opportunities (portfolios) by their logarithmic utility, is said to follow a "growth optimal"[10] strategy.

- Bilinear utility function

$$\mathcal{R} = (1-\lambda)Reward(\mathcal{R}) - \lambda Risk(\mathcal{R})$$

where *Reward* and *Risk* are any of the reward and risk measures, and parameter λ is a measure of the investor's risk aversion ($\lambda = 0$ signifies a risk-neutral investor). One of the classical examples of the bilinear utility function is Markowitz mean-variance model.

5.2 Certainly Equivalent Expected Return on Equity

Commonly used in the asset liability applications, Certainly Equivalent Expected Return on Equity has been identified as a potential objective function in this thesis. The following notation is used for its study:

L_t^l = portfolio liability amount to be paid at the time period t, scenario l,

A_t^l = asset capital of the portfolio at the time period t, scenario l,

E_t^l = equity available for investment at the time period t, scenario l.

Then, at any time period t, scenario $l \in \Omega$ the reward of the portfolio can be assessed by the *Return on Equity (ROE)* measure, which essentially reveals how much profit a portoflio generates with the

[10] Also known as geometric mean or Kelly criterion.

money invested in it:

$$ROE_t^l = \frac{A_t^l - L_t^l}{E_t^l}$$

As ROE_t^l is a random variable, the portfolio ROE at the time t is defined in terms of its expected value. Suppose, the scenario $l \in \Omega$ is of p^l probability, then the expectation of ROE is determined in the following manner:

$$expROE_t = \sum_{l \in \Omega} p^l \frac{A_t^l - L_t^l}{E_t^l}$$

Expected utility values are useful in ranking and then selecting the best strategy, but their unit of measurement is not standardized and therefore is difficult to interpret in practice. To communicate the investment recommendation more efficiently, it is common to use the *Certainly Equivalent*.

As defined in [20], the Certainly Equivalent return r_c of an asset with the risky return $\tilde{r}$ is the one that satisfies:

$$\mathcal{U}(r_c) = \mathcal{E}[\mathcal{U}(\tilde{r})]$$

meaning that the certainly equivalent of the random return $\tilde{r}$ is the sure return r_c which has the same utility value as the random return does.

For the practical reasons[11], the use of CEexpROE in the model has been substituted by an accumulated wealth utility function as described in the *Section 5.5 Choice of Objective Functions in the Integrated Pension and Mortgage Portfolio Management Problem.*

5.3 Conditional Value at Risk Optimization

5.3.1 Value at Risk

Value at Risk (VAR) is one of the most important and widely used statistics measuring potential risk of financial losses.

Being defined as the predicted worst-case loss at a specific confidence level over a certain period

[11] GAMS implementation would require a nonlinear solver to optimize the model with CEexpROE formulation. This has been an insentive to outscope its implementation.

of time, in particular VaR answers the question: What is the minimum amount one can expect to lose with a certain probability over a given horizon?

In mathematical terms, VaR corresponds to a percentile of the distribution of portfolio profits and losses (P&L).

Suppose $f(x, y)$ is the loss[12] associated with the decision vector $x \in X$, $X \in \mathbb{R}^n$ (e.g. portfolio position) and the random vector $y \in \mathbb{R}^m$ of uncertainties (e.g. market prices). For each x, the loss $f(x, y)$ is a random variable having a distribution in $\mathbb{R}$ induced by y with the density $p(y)$[13]. The probability of $f(x, y)$ not exceeding a threshold ζ is given by:

$$\Psi(x, \zeta) = \int_{f(x,y) \leq \zeta} p(y) dy$$

Then, VaR at the confidence level α is the lowest possible value of ζ such that the probability of losses less or equal to VaR is greater or equal to α:

$$\zeta(x, \alpha) = min\{\zeta | \Psi(x, \zeta) \geq \alpha\}$$

Despite being a very popular risk measure, VaR has undesirable mathematical characteristics, such as being coherent only when based on the normal distribution with the following implication:

- Lack of subadditivity which means that VaR of the composite portfolio may not be equal to the sum of the VaR values of its components.
- Lack of convexity which implies that optimization of VaR may results in multiple local extrema rather than desired unique absolute minimum.
- Providing just the lower bound for losses in the P&L distribution, it fails to distiniguish between the case when losses which are worse are insignificantly worse and the case when such losses are enormously worse.

[12] If negative, it is the portfolio profit

[13] The assumption that the probabilty function has density is made for simplicity. Refer to [16] for the study on CVaR for general distributions

Example 1 (VaR)

Consider a hypothetical portfolio P&L distribution, shown on the Figure 7, with the expected P&L is 50 and the fifth percentile is -200. Hence, the 95% VaR of this portfolio is a loss from the current value of 200 or a loss from the expected value 250. The decision upon which of the two - current or expected values to base the VaR calculation on is arbitrary, but for the purpose of this thesis, consider expected value of the portfolio as the anchor for VaR statistics.

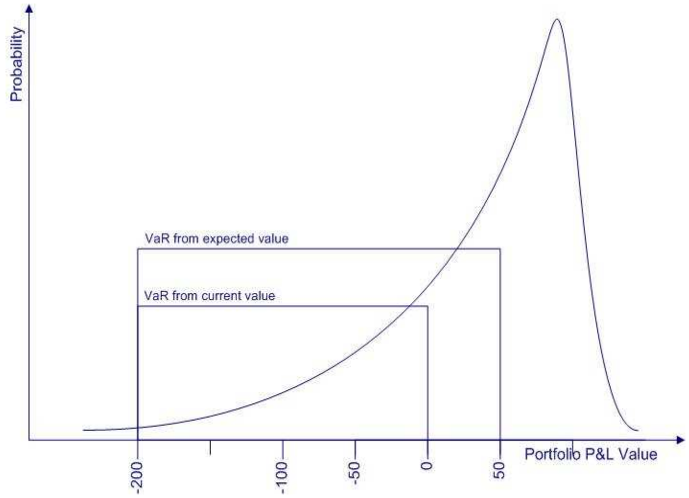


Figure 7: VaR of Portfolio Profits and Losses

5.3.2 Conditional Value at Risk

To overcome the disadvantages of the VaR, an alternative risk measure - *Conditional Value at Risk (CVaR)* was introduced by Uryasev et al. in [16]. For continuous P&L distributions, CVaR at a certain confidence level is the expected loss given that the loss is greater than or equal to the VaR at that level.

Example 2 (CVaR)

Recall the hypothetical portfolio case discussed in the prior Example 1. The expected loss in the (100-95)=5% worst scenarios is -217 from the expected portfolio value which describes the loss distribution beyond the VaR, as shown on the Figure 8.

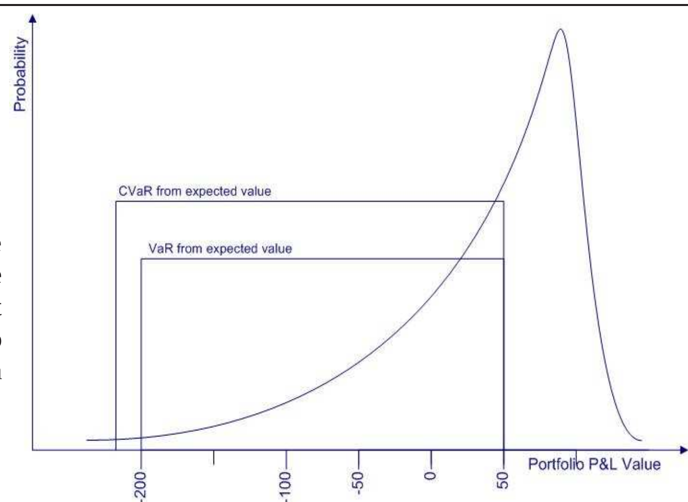


Figure 8: CVaR of Portfolio Profits and Losses

The $\alpha - CVaR$ value for the loss random variable associated with the decision x and confidence level

$\alpha \in (0, 1)$ is given by:

$$\phi_\alpha(x) = (1-\alpha)^{-1} \int_{f(x,y)\geq\zeta_\alpha(x)} f(x,y)p(y)dy$$

According to this formulation, $\phi_\alpha(x)$ is the conditional loss associated with x relative to that loss value being greater or equal to $\zeta_\alpha(x)$. To describe this measure, the function F_α is introduced:

$$F_\alpha(x,\zeta) = \zeta + (1-\alpha)^{-1} \int_{y\in\mathbb{R}^m} [f(x,y)-\zeta]^+ p(y)dy,$$

where:

$$[t]^+ = max(t, 0).$$

As shown in [15] such a function $F_\alpha(x,\zeta)$ is convex and continuously differentiable in respect to α. The $\alpha - CVaR$ of the loss associated with any $x \in X$ can be determined as follows:

$$\phi_\alpha(x) = \min_{\zeta\in\mathbb{R}} F_\alpha(x,\zeta)$$

Moreover, to find the decision x yielding the minimum $\alpha - CVaR$ value it is sufficient to minimize $F_\alpha(x,\zeta)$ over all (x,ζ) according to [15].

In the case of discrete scenario setting, the integral formulation of F_α is approximated by sampling the probability distribution of y with respect to its density $p(y)$ which generates vectors y^l, $l = \{1,\ldots,m\}$ each corresponding to a scenario l with probability p^l to happen. Then the corresponding approximation to F_α is established in the following manner:

$$\tilde{F}_\alpha(x,\zeta) = \zeta + \frac{1}{1-\alpha}\sum_{l=1}^{m} p^l [f(x,y^l)-\zeta]^+$$

Such $\tilde{F}_\alpha(x,\zeta)$ is convex, piecewise linear and can be minimized with respect to α.

Further, to linearize F_α, auxiliary variables z^l, $l = \{1,\ldots,m\}$ are introduced:

$$z^l \geq f(x,y^l) - \zeta, \quad z^l \geq 0, \quad l = \{1,\ldots,m\}, \quad \zeta \in \mathbb{R}.$$

Then, the function $\tilde{F}_\alpha(x,\zeta)$ can be replaced by the linear function $\zeta + (1-\alpha)^{-1}\sum_{l=1}^{m} p^l z^l$.

5.3.3 Modeling CVaR measure

In the context of this thesis, the loss of the portfolio $f(x^*, y^l)$ in the scenario l is the difference between the final wealth accumulated by following the optimal strategies x^* and the expected final accumulated wealth:

$$f(x,y^l) = AW^l - E(AW), \quad l \in \Omega$$

Introduce the following variables:

$$\begin{aligned}
Loss^l &= \text{loss of the portfolio in the leaf scenario } l,\\
VaR &= \text{VaR associated with the portfolio},\\
VaRDev^l &= \text{Auxiliary variable in the context of the } z^l \text{ above},\\
CVaR &= \text{CVaR associated with the portfolio.}
\end{aligned}$$

Then, the CVaR optimization of the integrated pension and mortgage portfolio is formulated in the following manner:

$$\begin{aligned}
&Minimize \quad CVaR\\
&s.t.\\
&Loss^l = AW^l - E(AW), \quad l \in \Omega\\
&VaRDev^l = Loss_i - VaR, \quad l \in \Omega\\
&CVaR = VaR + \frac{\sum_i p_i VaRDev^i}{1-\alpha}\\
&VaRDev^l \geq 0, \quad l \in \Omega
\end{aligned}$$

5.4 Conditional Drawdown at Risk Optimization

Portfolio drawdown is one of key performance indicators assessing the credibility and regulation concerns the portfolio management [5]. Being defined as the drop in the portfolio value compared to the maximum achieved in the past, the drawdown captures its performance insentives in sense that it can be used as an alarm when the portfolio returns are at risk.

The *Conditional Drawdown at Risk (CDaR)* function at a certain confidence level α, is the mean of the worst $(1-\alpha)\cdot 100\%$ drawdowns experienced over a time period. This way, it takes into account both the frequency and duration of portfolio drawdowns.

The concepts behind CDaR function are similar to those assiciated with the VaR and CVaR. In fact, CDaR may be presented as CVaR where the loss distribution is a distribution of portfolio drawdowns with discrete time moments on scenario tree paths. Accordingly, the optimization approach discussed in prior sections is adopted to the extent of modeling similarities between the CVaR and CDaR measure.

Assume that continuous time interval $[t_0, t_T]$ is sampled into N periods $t \in \{t_0, ..., t_T\}$. Given the discrete set of random events, $\Omega = \{\omega_l | l = \overline{1, m}\}$ with probabilities p^l, $W^l(x, t)$ - the uncompounded portfolio return at the time t with decision vector x defining the intrument positions constituting the portfolio, in the scenario l.

The drawdown function $D^l(x, t)$ is defined as a difference between the maximum return over all preceding $0 \leq \tau \leq t$ and current t time periods returns of the portfolio, in the scenario l, as illustrated on the Figure 9 and formulated in the following:

$$D^l(x,t) = \max_{0\leq\tau\leq t}\{W^l(x,\tau)\} - W^l(x,t)\}$$

Let $\xi_\alpha(x)$ be the treshold such that exactly $(1-\alpha)\cdot 100\%$ of all drawdowns exceed it. Then, the $\alpha - CDaR$ at the portfolio horizon T is the average of all drawdowns not exceeding this threshold:

$$G_\alpha(x,\xi) = \xi + \frac{1}{(1-\alpha)\cdot T}\sum_{t=t_0}^{T}\sum_{i=1}^{m} p^l \cdot [D^l(x,t) - \xi]^+$$

where $[g]^+ = max(0, g)$

To linearize the $\alpha - CDaR$ function, auxiliary variables v_t^l are defined:

$$v_t^l \geq D_t^l(x) - \xi, \quad v_t^l \geq 0, \quad i = \{1, \ldots, m\}, \xi \in \mathbb{R}$$

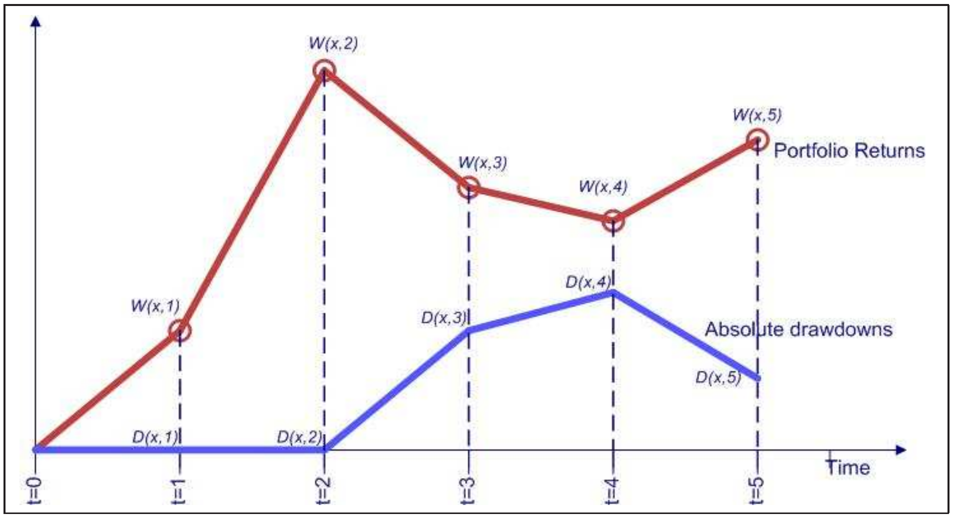


Figure 9: Concept of portfolio drawdown. Portfolio return $W^l(x,t)$ (in red) and corresponding absolute drawdown $D^l(x,t)$ (in blue).

Then, the $\alpha - CDaR$ function can be replaced by its linear equivalent:

$$G_\alpha(x,\xi) = \xi + \frac{1}{(1-\alpha)\cdot T}\sum_{t=0}^{T}\sum_{l=1}^{m} pi^l v_t^l$$

The CDaR measure is commonly used within maximization of expected portfolio returns at the final time moment subject to risk constraints:

$$\begin{aligned}
&\max_x E[W(x,T,\omega)]\\
&\text{s.t.}\\
&\xi + \tfrac{1}{(1-\alpha)\cdot T}\textstyle\sum_{t=0}^{T}\sum_{i=l}^{m} p^l v_t^l \leq \eta \sum_{l=1}^{m} p^l W^l(x,t)\\
&v_t^l \geq D_t^l(x) - \xi
\end{aligned}$$

with η denoting the largest tolerated drawdown (expressed as a percentage of the portfolio return).

5.5 Choice of Objective Functions in the Integrated Pension and Mortgage Portfolio Management Problem

In the Integrated Pension and Mortgage Portfolio Management problem, the bilinear utility function is given a role of the model objective. This utility function (see *Section Expected Utility*) aims at optimizing the reward with regard to the risk measure. The reward is expressed in terms of the accumulated wealth of the optimal portfolio at the time horizon. The risk measure is represented by either the $\alpha - CVaR$ or $\alpha - CDaR$ of this portfolio. Given a risk aversion parameter λ and the confidence level α, such an objective function is formulated in the following:

$$Maximize \quad (1-\lambda)\sum_{l\in\Omega} p^l AW_T^l - \lambda(CVaR_\alpha)$$

or:

$$Maximize \quad (1-\lambda)\sum_{l\in\Omega} p^l AW_T^l - \lambda(CDaR_\alpha)$$

where AW_T^l is the accumulated wealth of the integrated portfolio at the final time T, scenario $l \in \Omega$ with the probability p^l; $CVaR_\alpha$ and $CDaR_\alpha$ are its $CVaR$ and $CDaR$ measures at the confidence level α correspondingly.

5.6 Discount Factor

In order to depreciate future liability and assess the present value of the wealth achieved by the portfolio at its optimization horizon, the discounting factor g_t is used. It is a stochastic variable with dynamics which can be viewed from several perspectives. [6] suggests inflation rates I_t^l for adjusting the future payments, in the sense that the discount factor is defined as:

$$g_t^l = \frac{1}{(1+I_t^l)^t}.$$

In [4], the discounting factor is a sum of the guaranteed policy rate and extra dividend, both dependent on prevailing business conditions. In more sophisticated modeling sense, as in [17] discount factors are obtained both from preference-based and market-based (no arbitrage) approach. The preference-based approach is based on the assumption that an investor has certain preferences satisfying the expected utility theory. Alternatively, the market-based approach imposes the use of price to determine the discount factor and can be applied only in the complete markets. In this thesis, the investor's income (salary) is adjusted for inflation, hence the inflation rate is chosen to be a proper indicator for the discount factor. In general, discount factors are dependent on the individual investor's income dynamics and investment alternatives.

The inflation target of 2% set by the Bank of England is considered in this thesis[14]. This rate is expressed in terms of an annual rate of inflation based on the Consumer Prices Index (CPI). The remit is not to achieve the lowest possible inflation rate, as inflation below the target of 2% is viewed to be just as bad as inflation above this target. Hence, the inflation target is symmetrical. Indeed, the actual inflation is never kept at its target of 2%. To achieve such constancy in the inflation level, the interest rates would need to be changing all the time and by large amounts, causing unnecessary uncertainty and volatility in the economy. Even then, it would not be possible to keep the inflation at 2% in each and every month. Instead, the interest rates are set so that inflation can be brought back to target within a reasonable time period without creating undue instability in the economy.

[14]In contrast to the other Scandinavian central banks Danmarks Nationalbank does not pursue an inflation target but an exchange-rate target versus the euro. In 2005 the actual inflation rate was 1.7% which is close yet a bit lower than the inflation rate in the UK which is why the latter has been chosen in this thesis.

6 Multistage Stochastic Programming Model for the Integrated Pension and Mortgage Portfolio Management

Being a popular area for decision making, financial optimization is capable of encompassing a wide range of uncertainties: prices, interest rates, inflation, cash flows, liabilities etc. Modeling over time stages plays an important role in many financial problems. The multi-staged decisions are required e.g. when optimizing short-term vs. long-term objectives, dynamically rebalancing investment portfolio as the market and macroeconomical circumstances change, managing risks that occur over time, and etc.

Flexible and capable to cope with realistic market imperfections whilst solving decision-making problems under uncertainty over time periods, the multistage stochastic programming approach to modeling evolved as an optimal choice for this thesis. Such models allow both asset (pension) and liability (mortgage) sides of a portfolio to dynamically evolve over time following a certain probability distribution. Portfolio decisions are revised as the performance of its uncertain investment and credit constituents evolve, in the reversible manner.

The proposed multistage stochastic programming model is based on the concept of stochastic event tree and scenario generation capturing uncertainties in it. The main challenge is to integrate the scenarios of more risky investment trust returns and mortgage loan rates - less risky ones. This and further integration of pension and mortgage portfolios is subject of the following study.

6.1 Stochastic Event Tree

In multistage stochastic programming models the progressive evolution of random variables is commonly expressed by means of scenario trees. In this way, the scenarios are not restricted to follow any specific distribution or stochastic process meaning that any joint distribution of random variables can be reconciled. This adds special value to solving realistic financial problems that incur asymmetric distributions and heavy tails in the random variables.

Consider a finite probability space $(\Omega, \mathcal{F}, \mathcal{P})$ consisting of real-valued vectors of uncertain parameters, such as market prices, interest rates, and etc. over discrete time stages $t = \{t_0, t_1, \ldots, t_T\}$. According to [20], a stochastic event tree[15] is represented by a graph $\mathcal{G} = (\Sigma, \mathcal{E})$ where nodes Σ signify time and stochastic scenarios, whereas links $\mathcal{E}$ connecting these nodes - possible transitions as time evolves.

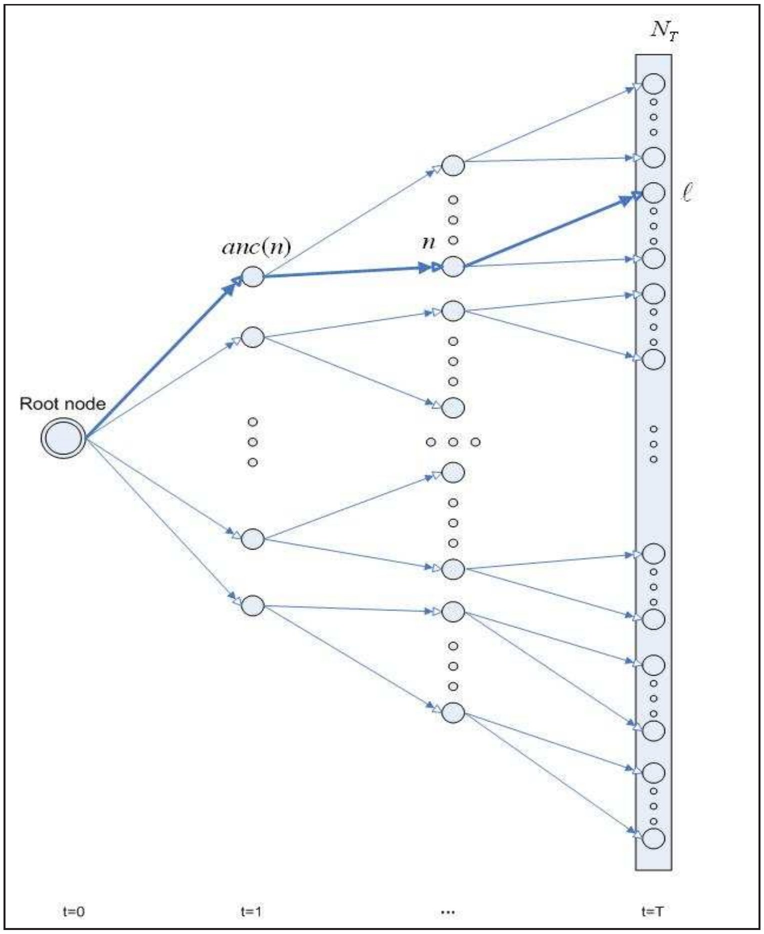


Figure 10: Stochastic event tree over T time stages with $\mathcal{N}_t$ nodes at each time stage t. Final stage nodes l are leaves of the tree. Each leaf is associated with a stochastic scenario spanning over a path that leads to it (an example of a scenario is given in bold).

Figure 10 illustrates the concept of the stochastic event tree. The root node of the tree (at time $t = t_0$) represents the initial state of the model. At any time $t_1 \leq t \leq t_T$, there are $\mathcal{N}_t$ of possible states. Scenarios correspond to the paths on the stochastic event tree, e.g. one scenario l is depicted in bold. Furthermore, every node $n \in \mathcal{N}_t$ for $t_1 \leq t \leq t_T$ has a parent node $anc(n) \in \mathcal{N}_{t-1}$ and every node

[15] Also refered to as the stochastic scenario or just scenario tree.

$n \in \mathcal{N}_t$ for $t_0 \leq t \leq t_{T-1}$ has a non-empty set of child nodes $C(n) \subset \mathcal{N}_t$. States at the last time period are represented by the leaf nodes and their number corresponds to the total number of scenarios generated by the tree.

Establishing the one-to-one matching between probability atoms $\omega \in \Omega$ and event tree $\mathcal{G}$, the probability distribution $\mathcal{P}$ is imposed on the event tree by weighting its leaf nodes $n \in \mathcal{N}_t$ with values $p_n > 0$, so that $\sum_{n \in \mathcal{N}_t} p_n = 1$. The probability at a certain state node n, denoted by p_n is a product of the conditional probabilities of the states preceding it on the same path starting from the root node. Indeed, the total probability at each time stage must sum up to 1 and the probabilty of any node which has children nodes must be equal to the sum of their probabilities:

$$p_n = \sum_{m \in C(n)} p_m, \quad \forall n \in \mathcal{N}_t, \quad t = \{t_0, t_1, \ldots, t_{T-1}\}$$
$$\sum n \in \mathcal{N}_t p_n = 1, \quad t = \{t_0, t_1, \ldots, t_{T-1}\}$$

6.2 Interest Rate and Investment Trust Returns Scenario Generation

Scenario generation is a crucial part of modeling the Integrated Pension and Mortgage Portfolio Problem. To correctly quantify the risks associated with the investment trusts underlying the asset products and mortgage loans considered at the liability side of the portfolio, realistic rates and prices are required as input parameters. The approach to assembling these is described in the following.

The short interest rates and mortgage prices are estimated using the Vasicek model as in [14] and approximative pricing method according to [12].

The asset return scenarios originate from the estimated historical prices of the investment trusts. These prices are adjusted to account for the dividend reinvestment and are further used to obtain the annualized returns over different time intervals. Denoting the price of the investment trust at the current time t by P_t, its annualized return $R_t(\tau)$ over the time period of τ years is determined in the following manner:

$$R_t(\tau) = (\frac{P_{t+\tau}}{P_t})^{1/\tau} - 1, \quad t = \{t_0, t_1, \ldots, t_T\} \tag{16}$$

Given the generated scenario tree of annualized returns $R_t(\tau)$ over fixed period τ, with the matching between its time stages and fixed periods (at the first time stage, the returns are annualized over one year, at the second time stage - they are annualized over two years, and at the third time stage - over three years, correspondingly), and the price P_t at the time period t, the price $P_{t+\tau}$ at time stage $t+\tau$ (corresponding to the period τ) is determined by reversing the equation (16):

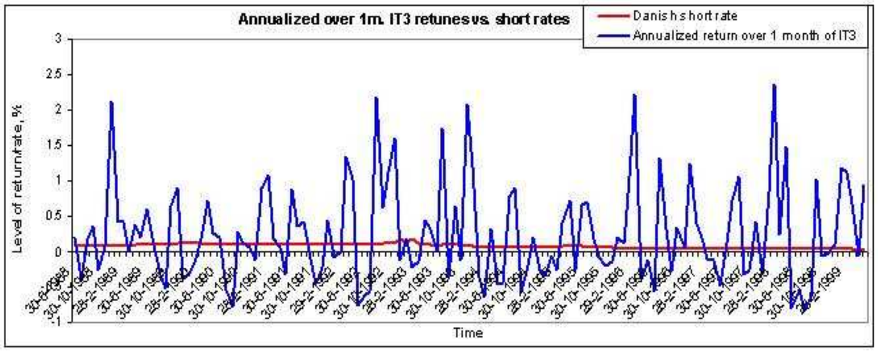
$$P_{t+\tau} = P_t(R_t(\tau) + 1)^{\tau} \tag{17}$$

To account for the interest rates while generating the investment trust return and dividend yield scenarios, the short rates annualized over three months[16] are correlated with the annualized investment trust returns. The most positively or negatively correlated, with the short rates, investment trust return variable is used by the scenario generating framework to combine the investment trust and interest rate uncertainties into a complete stochastic event tree.

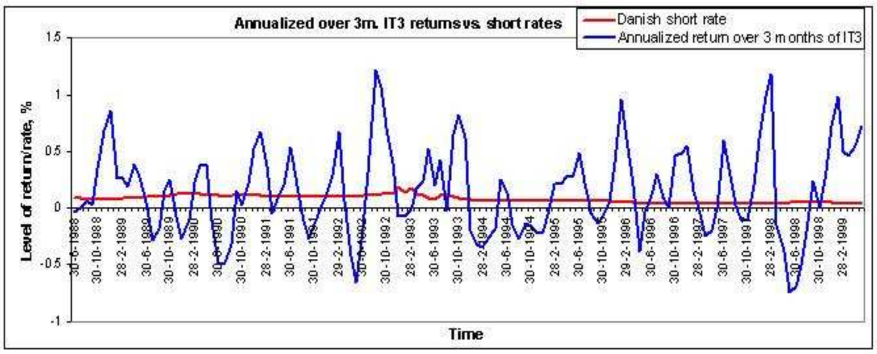
The process of correlating the annualized over different fixed intervals returns of the investment trusts with the short interest rates is illustrated in Figure 11 for the investment trust IT3. It is worth mentioning that the returns fixed over the short periods (e.g. one, three and six month returns are shown on the graphs 11(a), 11(b) and 11(c) accordingly) are rather weakly correlated with the short rates. However, as the fixed interval over which the returns are annualized is getting longer, the correlation between these and short rates is getting stronger (e.g. one, two, three, etc. year returns as represented on the graphs 11(d), to 11(h) correspongingly). Moreover, it is noteworthy that the investment trust returns over the long run are higher and riskier than the fixed income securities with returns based on the interest rates as illustrated on the graphs 11(d) - 11(h).

The moment-matching method has been introduced in [9] and [11]. This method generates scenario realizations of stochastic variables matching their statistical moments to the specific target values. In particular, the first four statistical moments and the correlation matrix of the annualized investment trust returns, adjusted for the dividend yield and mortgage interest rate variables are matched to the

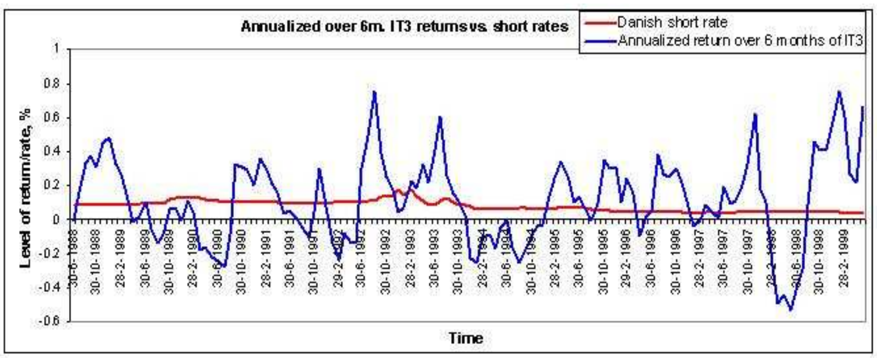
[16] CIBOR, three months. Source: Danmarks Nationalbank.



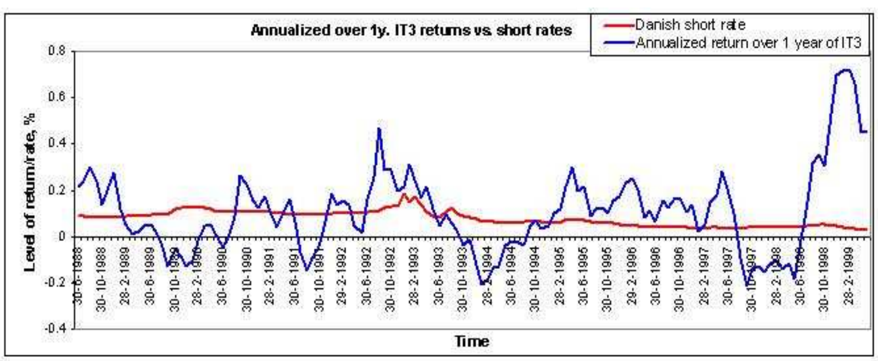
(a) Annualized midmarket return over 1 month of the investment trust IT3 and 3 months Danish short rate



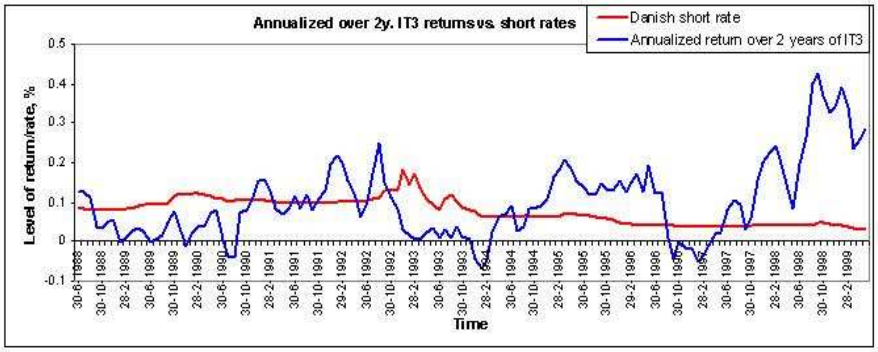
(b) Annualized midmarket return over 3 months of the investment trust IT3 and 3 months Danish short rate



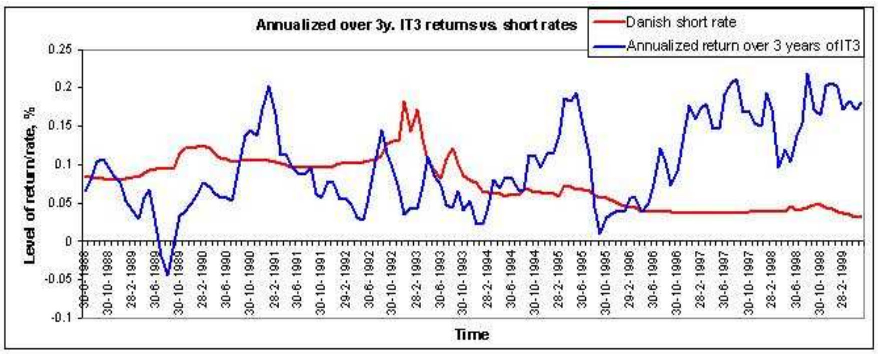
(c) Annualized midmarket return over 6 months of the investment trust IT3 and 3 months Danish short rate



(d) Annualized midmarket return over 1 year of the investment trust IT3 and 3 months Danish short rate

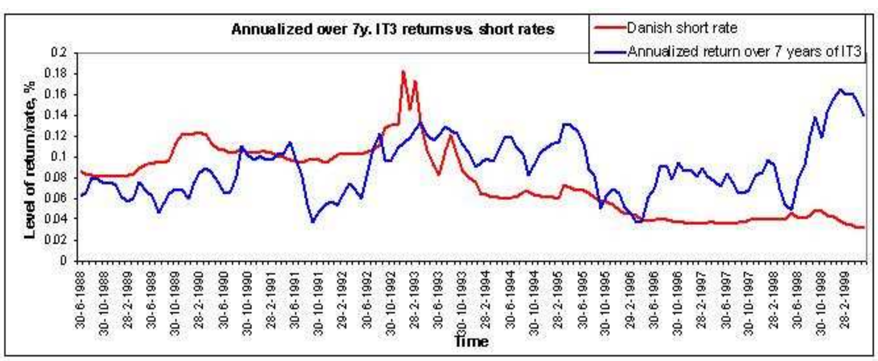


(e) Annualized midmarket return over 2 years of the investment trust IT3 and 3 months Danish short rate



(f) Annualized midmarket return over 3 years of the investment trust IT3 and 3 months Danish short rate

(g) Annualized midmarket return over 5 years of the investment trust IT3 and 3 months Danish short rate

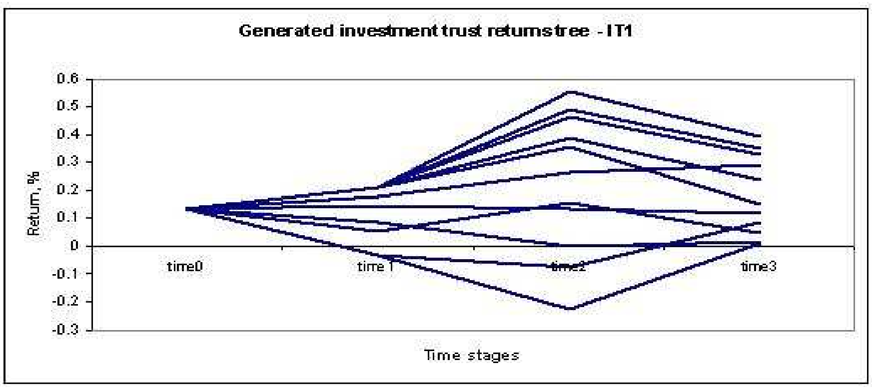


(h) Annualized midmarket return over 7 years of the investment trust IT3 and 3 months Danish short rate

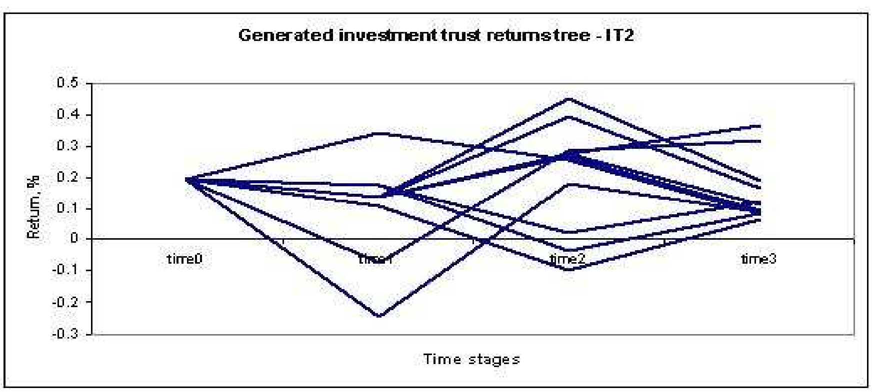
Figure 11: Correlation among investment trust IT3 returns (in red) annualized over different periods and Danish short rate annualized over three months (in blue).

corresponding target values estimated from the historical data.

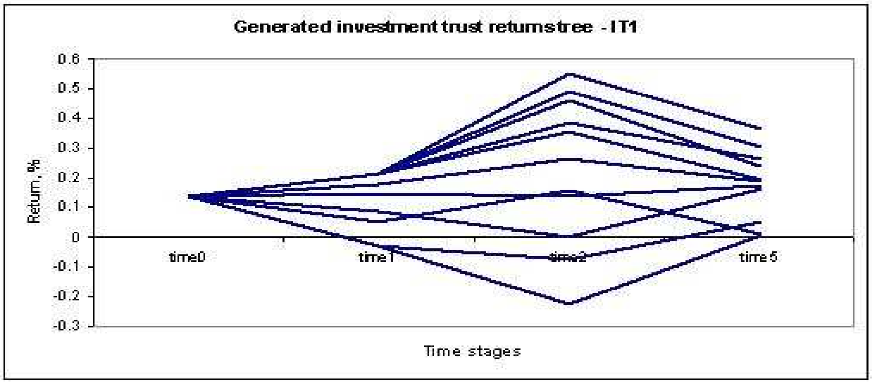
Figure 12 presents selected stochastic event trees generated to model the return dynamics of the sample investment trusts considered. Every path on the tree corresponds to a certain scenario *l* (chosen among all scenarios to represent general tree structure). The most correlated investment trust return series with the short rates - linking criteria being used in the scenario generation algorithm - is depicted by the most "regularly-constructed" tree, with the least number of branch intersections. Consequently, the less correlated investment trust returns with the short rates are represented by the higher portion of irregularities in their structure, such as overlapping branches, etc. For more details on the scenario trees used in the modeling, see Appendix B.



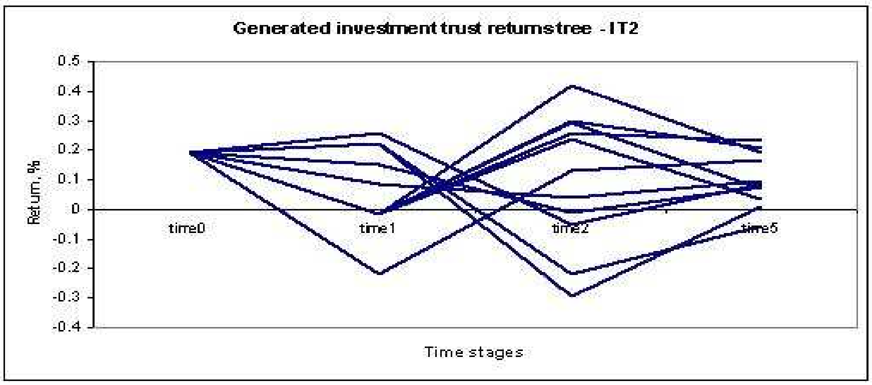
(a) The 0y-1y-2y-3y tree of the IT1 returns



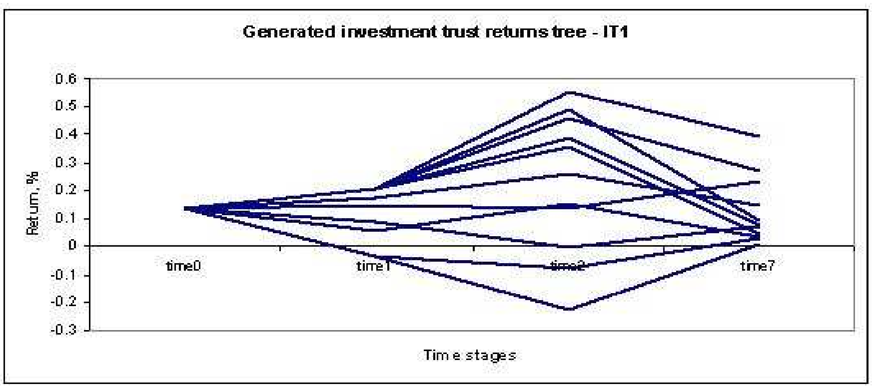
(b) The 0y-1y-2y-3y tree of the IT2 returns



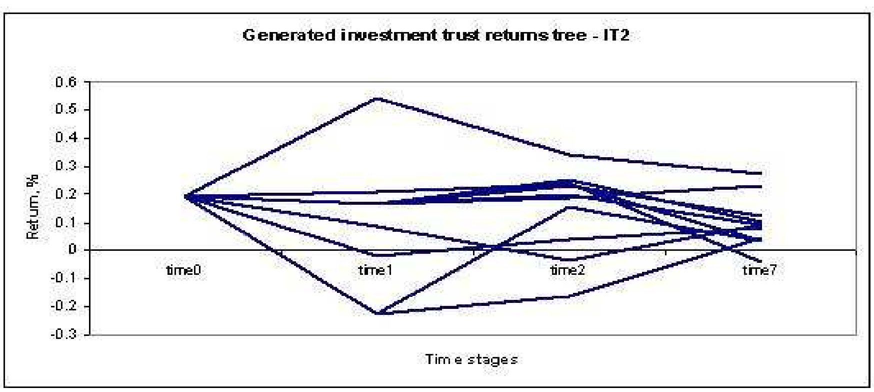
(c) The 0y-1y-2y-5y tree of the IT1 returns



(d) The 0y-1y-2y-5y tree of the IT2 returns



(e) The 0y-1y-2y-7y tree of the IT1 returns



(f) The 0y-1y-2y-7y tree of the IT2 returns

Figure 12: Stochastic Event Trees of the Investment Trust Returns

6.3 Integrating Scenario Trees of Investment Trust Returns and Mortgage Rates

To integrate the investment trust returns and mortgage rates scenarios, the problem of combining stochastic trees with different structures is posed. Essentially, the investment trust returns are characterized by more stochasticity in their dynamics if compared to the mortgage rates. In order to capture both, a unique stochastic event tree must be constructed. In the following an abstract instance of such problem is described.

Figure 13 illustrates an original three-stage trinomial tree of mortgage prices with time periods t_0, t_1, t_2, t_3, and a target one-stage tree $t_0 - t_3$ with 729 scenarios. To construct the trarget tree given the scenarios from the original one, its 27 nodes at the time stage t_3 are duplicated (by 27). In this way, nodes at the intermediate stages t_1 and t_2 are avoided. The target tree has 729 branchings from the stage t_0.

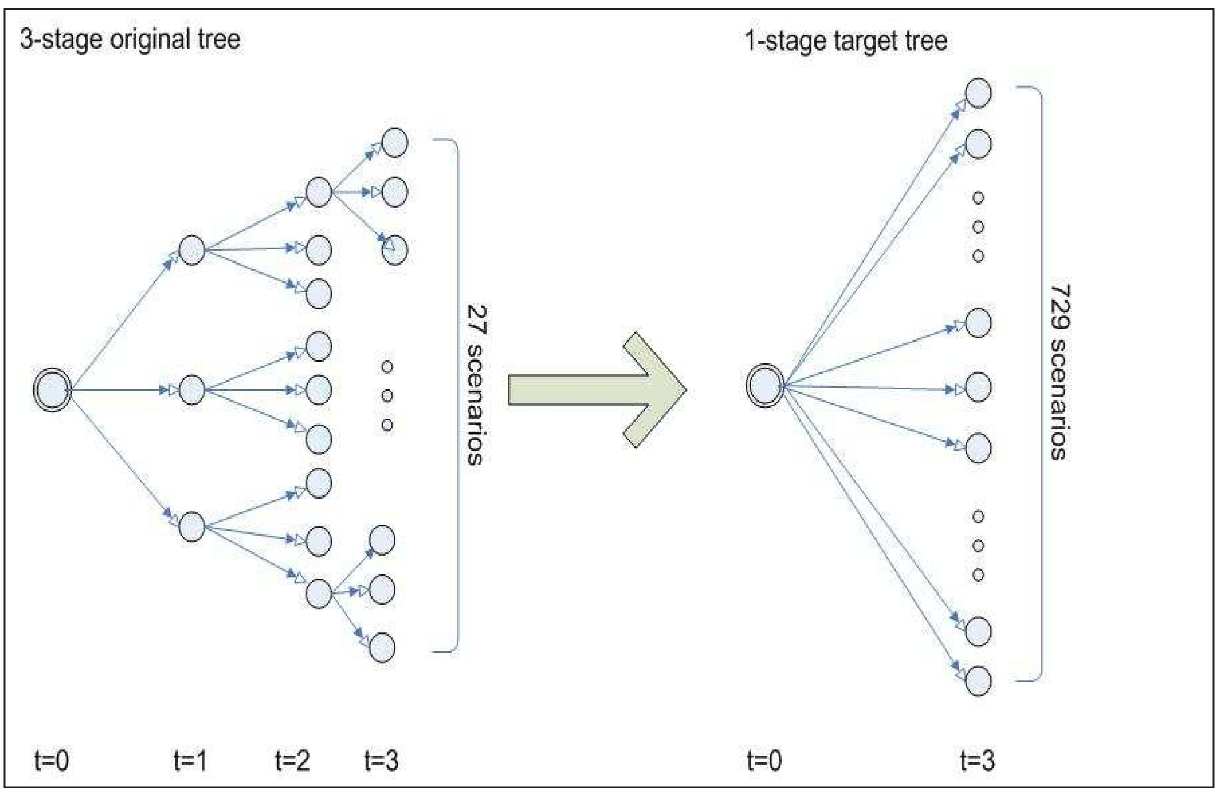


Figure 13: Integrating Investment Trust Returns and Mortgage Rates Scenario Trees

Similarly, to construct the two-stage tree $t_0 - t_1 - t_3$, the nodes from the original tree at the stage t_2

must be avoided, and its nodes from stage t_1 - duplicated so that there are 81 of them, and nodes from the stage t_3, so that there are 729 of them. The target tree has 81 branchings from the t_0 stage and 9 branchings from the t_1 stage.

Further, to construct the three-stage tree $t_0 - t_1 - t_2 - t_3$, all time stages from the original tree are used, the nodes at the stage t_1 are duplicated so that there are 9 of them, nodes at the time stage t_2 - so that there are 81 of them, and nodes at the time stage t_3 - so that there are 729. The target tree has 9 branchings at each stage t_0, t_1 and t_2.

6.4 Integrating the Pension and Mortgage Portfolios into a Multistage SP Model

The problem of integrated pension and mortgage portfolio management is viewed from the perspective of a UK investor with real estate liabilities in Denmark. Without loss of generality, no direct exchanges between foreign currencies are executed. In order to simplify the model formulation and reduce its data needs, all prices are in local currencies, but the asset-liability matching is performed in Euro[17].

Earlier considerations on modeling the investment and mortgage products[18] naturally lead to combining them into one integrated portfolio management model. The conceptual idea behind such an integration is visualized by the abstract portfolio network on the Figure 14.

This network is structured from two parts: investment products on the top of the figure and the mortgage loans - below. At the very first time period, the initial amount (IA) is financed by issuing the ARM and FRM loans and the available cash to invest (ACI_{t_0}) is contributed into the IT PA, IT ISA and IT SP products. Each of these products has underlying investment trusts as shown for the initial time stage (for clarity of visualization the investment trust level is not included further on this network; for corresponding details refer to the Figure 5). At every subsequent time period $t > t_0$, total

[17] The Euro exchange rate mechanism is based on the concept of fixed currency exchange rate margins, which allows regulating the currency exchange at rather high level of stability.

[18] See *Section 4.1.6 Dynamics and Policy Constraints of Asset Products* and *Section 4.2.3 Dynamics and Policy Constraints of Mortgage Products.*

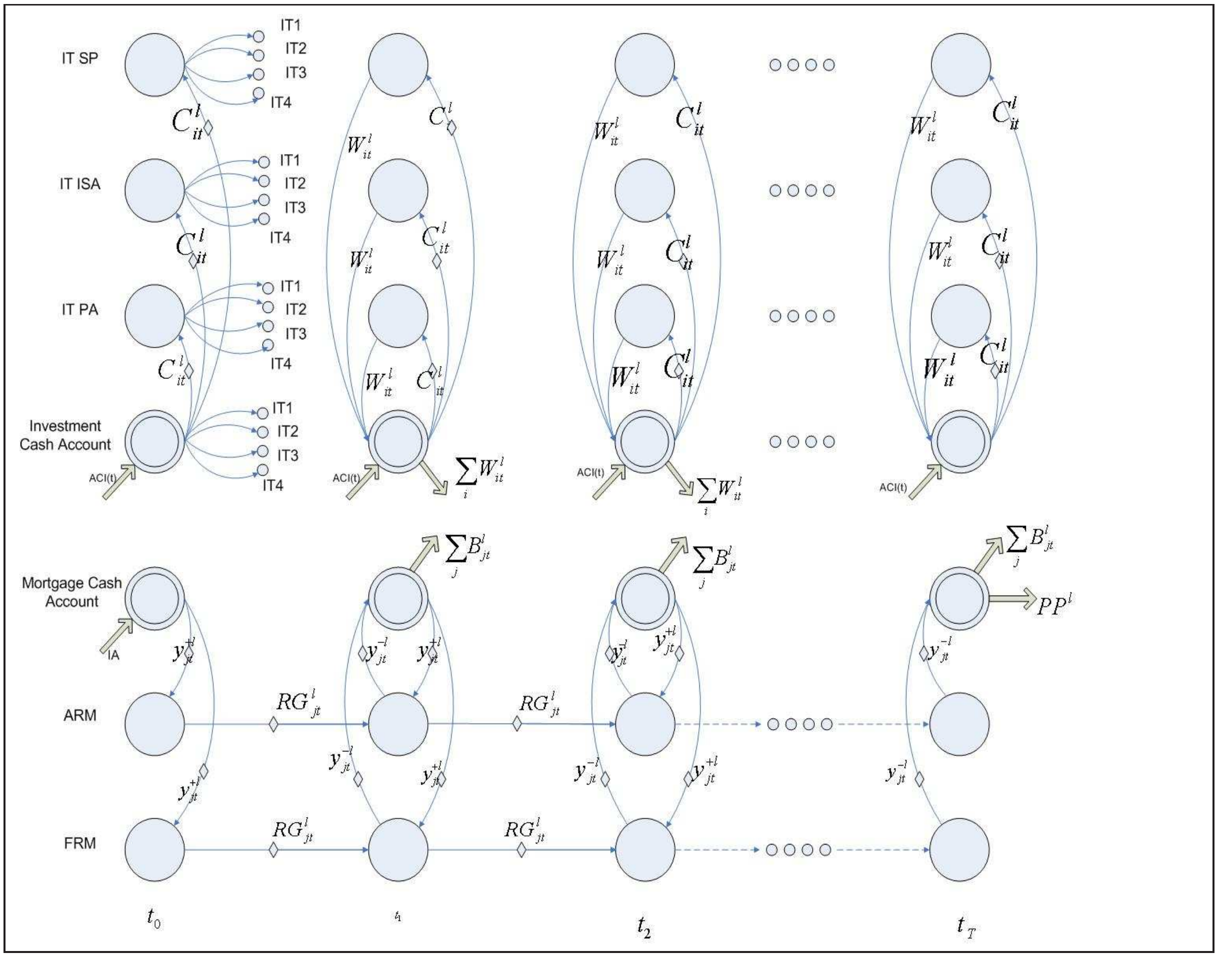


Figure 14: Integrated pension and mortgage portfolio network, arbitrary scenario l. The investment and mortgage portfolios are mached by means of assets (accumulated withdrawal value $\sum_{k\in\mathcal{K}} W^l_{k,t}$) dedicated to fund liability obligations (total payment value $\sum_{j\in\mathcal{J}} B^l_{jt}$).

payment on the mortgage side ($\sum_i B^l_{it}$) is funded by means of withdrawing investment capital from the asset side ($\sum_j W_{jt}$) and the available cash to invest is infused into the portfolio along the way. The investment transactions taking place at the asset side are represented only at the first - cash account and second - product levels. For the complete network representation of the investment portfolio see *Section 4.1.6 Dynamics and Policy Constraints of Asset Products.*

It is worth mentioning that the time horizons for the investment and mortgage products is not necessarily the same, yet for practical modeling purposes the objective horizon is shortened and assumed to be unique for the integrated portfolio.

The integrated model specifies decision, i.g. product contribution, withdrawal, investment and loan dealing, at discrete time moments in time (the lengths of time periods are generic, meaning they may be per month, per quarter, per annum etc. Moreover, the time periods do need to be equal - in this way allowing rather high flexibility in portfolio planning) with simulated knowledge about the future uncertainty of the financial markets in the form of the stochastic scenario tree. This tree represents the dynamic evolution of the random variables over the planning time frame. Along with the utility in the objective function of the model, the risk measure, confidence and averseness levels are accounted for whilst optimizing.

7 Fundamental Test and Analysis

Having modeled the integrated pension and mortgage portfolio management problem, it is crucial to uncover its capabilities and assess its qualities by employing a number of test and analysis techniques. This section attempts to accomplish such goals in the following manner.

First, the Illustrative Case is presented. It aims at establishing common knowledge about the solution structure of the model given certain parameters that define products involved and investor's profile. Next, the Test Metrics are defined, which assists at creating Test Cases used to evaluate the model and solution performance across different settings. The Sample Test aims at revealing the correctness and systematics of efficient frontiers yielded by the models. Consequently, the comparative Study of efficient frontiers in the Multiperiod sense is carried out to illustrate improvements in the portfolio performance as the number of decision stages increases. In order to evaluate solution stability of the CVaR vs. CDaR versions of the integrated pension and mortgage portfolio management model, the Robustness Analysis is carried out. It assists in proving that efficient frontiers yielded by the models are genuinely efficient by comparing them to the frontiers yielded by their counterparts (CDaR vs. CVaR and CVaR vs. CDaR correspondingly). Finally, to assess the responsiveness of the model to changes in the uncertain investment trust returns, the Sensitivity Analysis is conducted. In general, it is expected that efficient frontiers yielded by the models with perturbed input data are sensitive to the market volatilities, yet the solution structure is reasonably consistent whenever such changes happen.

7.1 Illustrative Case

This illustrative case reports an optimal solution of the integrated pension and mortgage portfolio management problem for a fictitious investor. Data with respect to nominal investment trust returns, interest rates, and short rates used in the scenario generation, is historically true[19], as well as certain policy characteristics, i.g. constribution ranges, investment dealing costs, Government Stamp Duty, Total Expense Ratios and etc. All other parameters are set by choosing a profile of a risk-averse household that has average level of dynamic income available for investment in a private pension and

[19] Extracted from the sources [24] and [23].

mortgage liabilities to a house with a presumed market price. Before presenting numerical results yielded by the integrated model, these parameter settings are defined.

The participation horizon of the integrated portfolio is three years, split into three decision stages: corresponding to the starting (t_0), first (t_1), second (t_2) and third (t_3) years. There are twelve branches at each stage of the stochastic event tree, having in total 1728 scenarios (or 1885 nodes).

The investor is offered IT PA, IT ISA and IT SP asset products with underlying investment trusts: IT1, IT2, IT3 and IT4. Further, the investor is requesting an initial position of 2,000,000.000 Dkk in the mortgage portfolio which can be financed by the loans listed in the Table 4 available to issue in the starting, first and second years accordingly.

Loan	**Description**
ARM_1	One-year adjustable rate loan
FRM_{30}^{2}	30-years to maturity, fixed 2% coupon
FRM_{30}^{3}	30-years to maturity, fixed 3% coupon
FRM_{30}^{4}	30-years to maturity, fixed 4% coupon
FRM_{30}^{5}	30-years to maturity, fixed 5% coupon
FRM_{30}^{6}	30-years to maturity, fixed 6% coupon
FRM_{30}^{7}	30-years to maturity, fixed 7% coupon

Table 4: Illustrative Case: Mortgage Loans

The mortgage contract agreement is due to prepayment in thirty years, however early prepayment is considered of interest to the investor, e.g. he may be planning to sell the house in three years. Hence, the portfolio optimization horizon T is set to three years.

Table 5 outlines the characteristics of products, their underlying links (investment trusts and mortgage loans), as well as investor profile parameters.

Scenarios of interest rates and short rates are generated based on the Vasicek model and approximative pricing technique. Scenarios of investment trusts returns are generated based on the historical prices from July, 30th 1988 to May, 30th 2006 correspondingly[20].

[20]These approaches were discussed in the *Section 6.2 Interest Rate and Investment Trust Returns Scenario Generation.*

Asset product parameters				
Asset product	IT PA	IT ISA	IT SP	
Annual Product Charge (APC_k), £	0.00	25.00	0.00	
Purchase Fee Ratio (PFR_k), %	0.003	0.010	0.010	
Purchase Fee Cap ($PFCap_k$), £	50.00	50.00	50.00	
Government Stamp Duty ($GovStamp_k$), %	0.005	0.005	0.005	
Sales Fee Ratio (SFR_k), %	0.000	0.010	0.010	
Sales Fee Cap ($SFCap_k$), £	0.00	50.00	50.00	
Minimum lump sum contribution ($C_{(Min)k}$), £	1,000.00	1,000.00	500.00	
Maximum annual contribution ($C_{(Max)k}$), £	3,600.00	7,000.00	1,000,000.00	
Maximum withdrawal value ($W_{(Max)k}$), £	0	1,000,000.00	1,000,000.00	
Minimum remaining assets after withdrawal($W_{(Rem)k}$), £	0	1000	500	
Investment trust parameters				
Investment trust	IT1	IT2	IT3	IT4
Total Expense Ratio (TER_i), %	0.0086	0.0042	0.0073	0.0047
Mortgage product parameters				
Tax reduction rate (%) on	interest, γ 0.32	adm. fee, β 0.32		
Fees, %	administration 0.005876	transaction 0.0025		
Fixed costs, Dkk	refinancing 2,500.00			
Investor profile parameters				
Risk Averseness, λ	0.8			
Available Cash for Investing (ACI_t, £)	ACI_{t_0} 15,000.00	ACI_{t_1} 15,500.00	ACI_{t_2} 16,000.00	ACI_{t_3} 17,550.00
Initial amount needed (IA), Dkk	2,000,000.00			
Market price of the house at the horizon (HP_T), Dkk	2,000,000.00			

Table 5: Illustrative Case: Parameters Used

Optimal solution of the CVaR model

Optimizing the integrated pension and mortgage portfolio with the parameters defined above and CVaR measure (with confidence level $\alpha = 0.9$) used for risk management purposes, yields the solution described in the following.

Development of the contribution rates

The optimal solution suggests contribution values shown on the Figure 15 for the starting, first and second years correspondingly. At each time stage, contribution in IT ISA and IT SP exceeds contribution in IT PA, which may be seen as a reasonable strategy given that IT PA cannot be used for

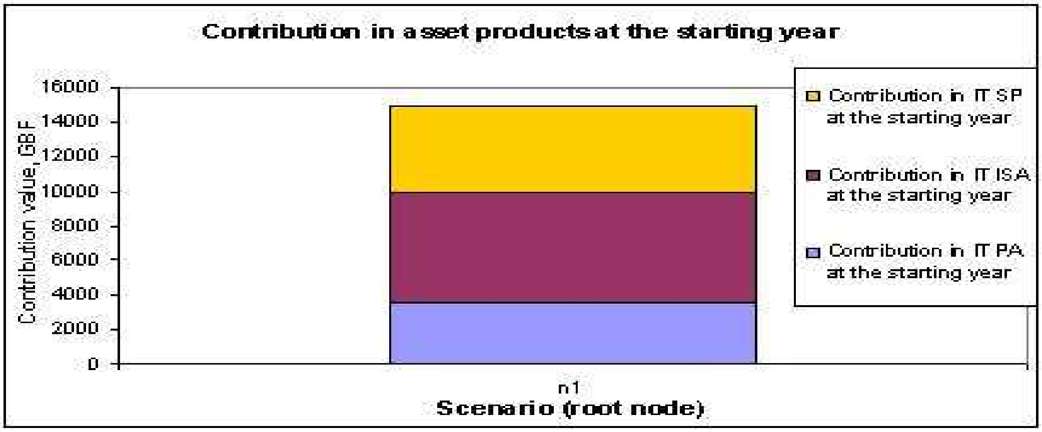
For complete visualization of generated scenarios of investment trust returns see Appendix B.

withdrawal and furthermore - has the lowest bound on the maximum annual amount allowed for contribution. Contribution in IT ISA is positively correlated with that of IT SP in the second year. This may be explained by similarities in the contribution vs. withdrawal policies. As IT ISA is bound in the maximum contribution sense, but IT SP is not - one may observe that the IT SP contribution values are generally larger than those of IT ISA correspondingly.

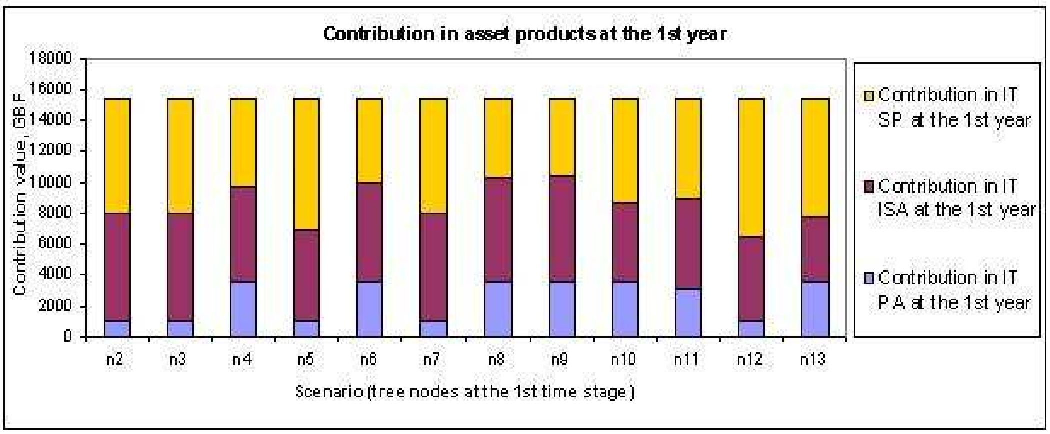
Investment dealing

Figure 16 illustrates the purchase of investment trusts shares at the starting, first and second years. Purchase amount is the total amount of shares purchased with the corresponding transaction fees, Government Stamp Duty included.

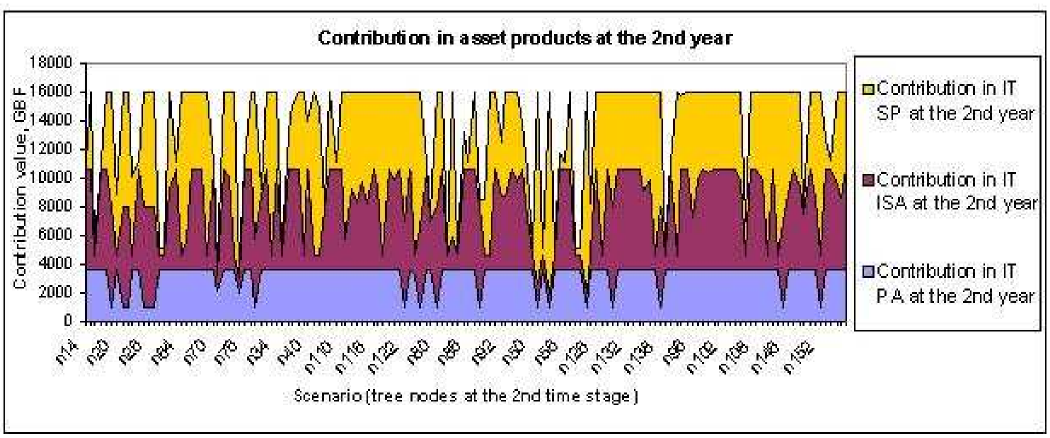
To understand the purchase dynamics, it is useful to consider scenarios of interest rates and investment trusts returns, which are the main determinants of market risks in the model. Figure 16 shows the dynamics of of interest rates and investment trust returns as used in the Illustrative Case. One may observe from the Figure 16(c) that purchase dynamics in the second year is dominated by the IT1 and IT2 shares. These investment trusts have the least risky returns, which given rather high value of the risk aversion parameter $\lambda = 0.8$, is the reasonable strategy to choose. Analyzing purchase decisions per product (e.g. IT1 shares purchased via IT ISA as shown on the Figure 17(b)), it is noteworthy to mention significant correlation between dealing the shares of the investment trusts and their return dynamics presented on the Figure 17(a). Lastly, IT1 returns are the most correlated with the interest rate (Figure 17(a)) which explains steadiness in this investment trust purchase decisions as shown on the Figures 16(c) and 17(b) correspondingly.



(a) Contribution over scenarios at the starting year

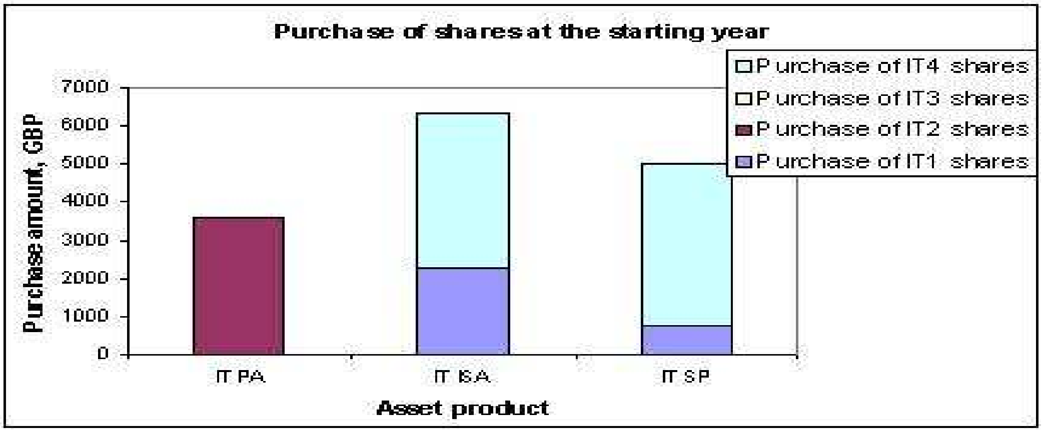


(b) Contribution over scenarios at the 1st year

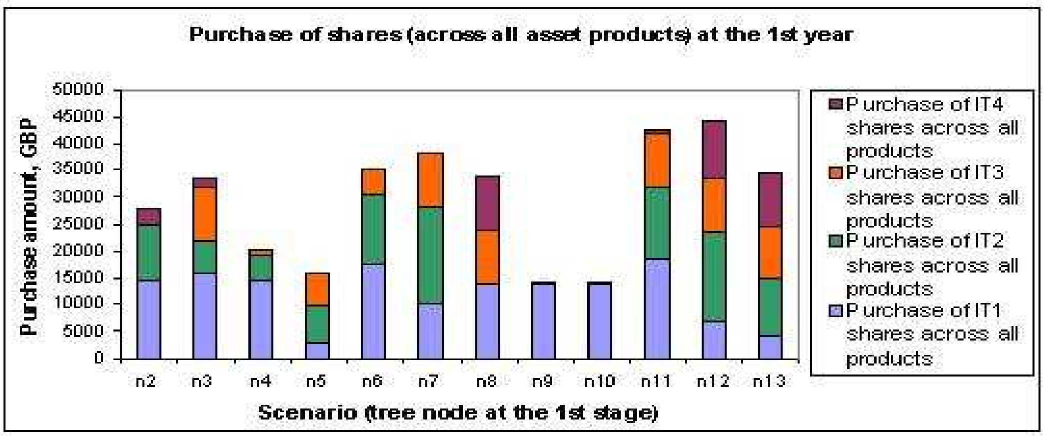


(c) Contribution over scenarios at the 2nd year

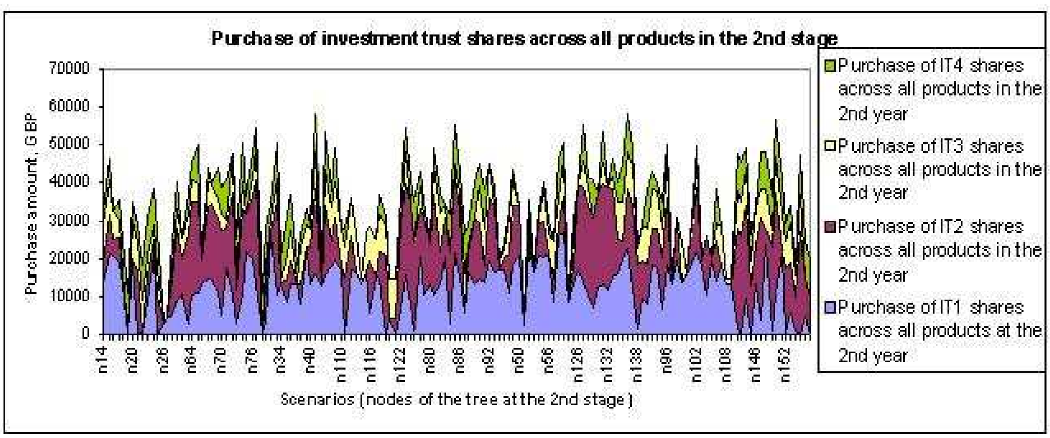
Figure 15: Illustrative Case: Contribution in the IT PA, IT ISA and IT SP over scenarios in the starting (15(a)), first (15(b)) and second (15(c)) years of the model solution. The strategy structure is dominated by the contribution in IT ISA and IT SP over that in IT PA.



(a) Purchase of shares at the starting year

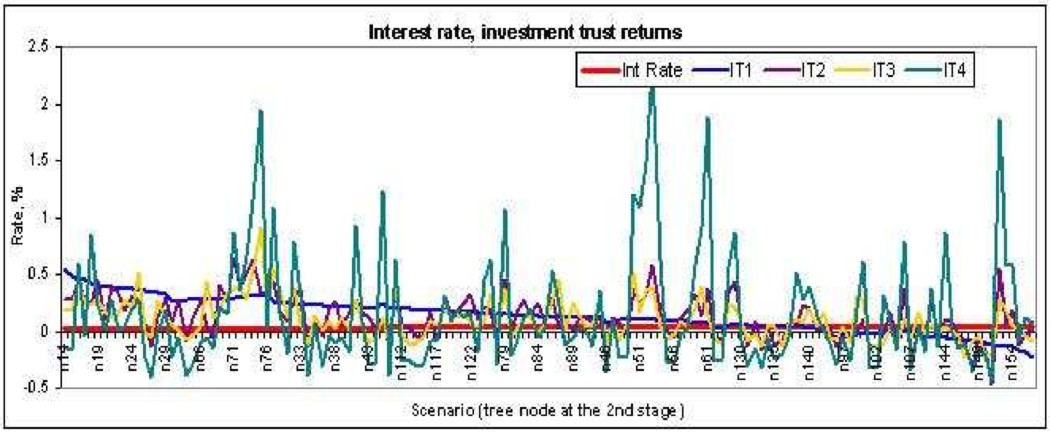


(b) Purchase of shares at the first year

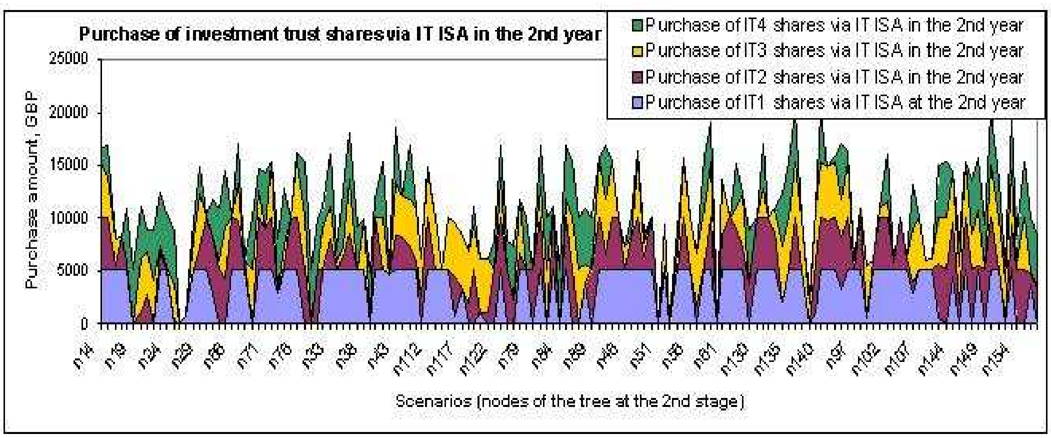


(c) Purchase of shares at the second year

Figure 16: Illustrative Case: Investment trust shares purchased via IT PA, IT ISA and IT SP products in the starting (16(a)), first (16(b)) and second (16(c)) years of the model solution.



(a) Interest rates and investment trust returns dynamics at the second year



(b) Purchase of shares via IT ISA at the second year

Figure 17: Illustrative case: Investment Returns, Interest Rates (17(a)) and IT ISA Purchase Decisions (17(b)).

Portfolio performance

Portfolio performance can be assessed e.g. in terms of the accumulated wealth it yields at the last decision stage (the third year in this case), total profit at the mortgage side (the margin above the essential principle prepayment), and the total payment on the scenario. Figures 18, 19 and 20 illustrate histograms of these performance indicators over all scenarios in the illustrative case accordingly.

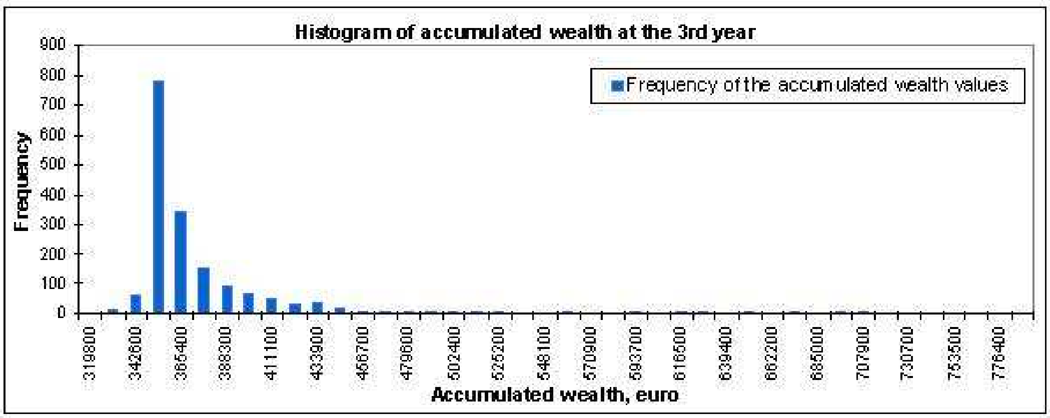


Figure 18: Illustrative Case: Accumulated Wealth Histogram.

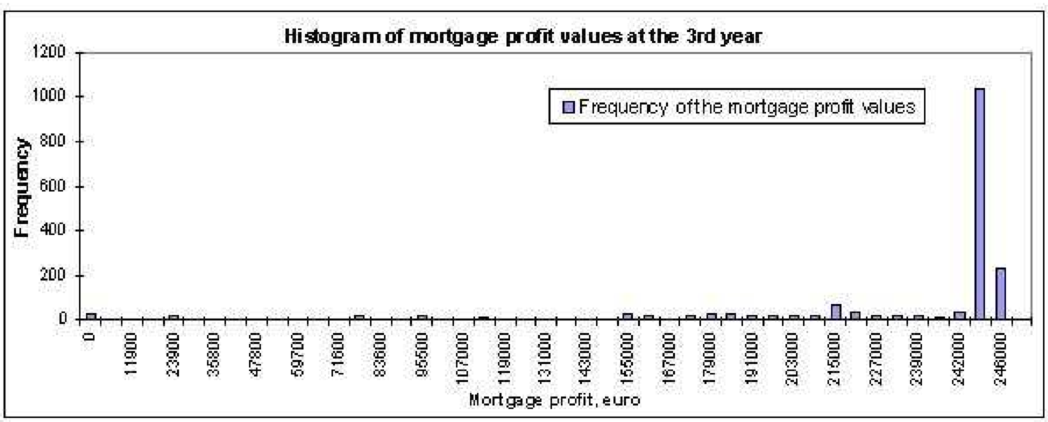


Figure 19: Illustrative Case: Mortgage Profit Histogram.

By analyzing general characteristics of these histograms, the following observations are noteworthy to mention:

1. They are well-centered at:

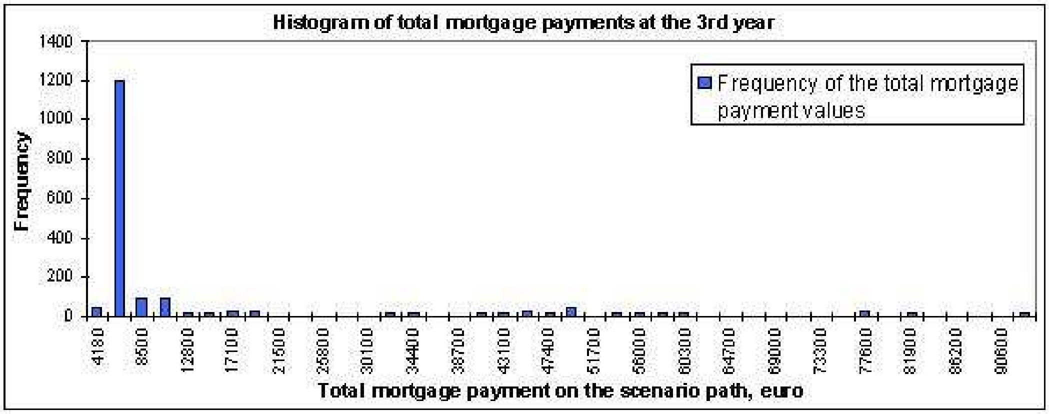


Figure 20: Illustrative Case: Total Mortgage Payment Histogram.

- Accumulated wealth (Figure 18): ≈ 354,000.000 Euro;

- Mortgage profit (Figure 19): ≈ 244,000.000 Euro;

- Total mortgage payment (Figure 20): ≈ 6,340.000 Euro.

Given the input parameters, relative to each other, such solution values imply successful outcome of the portfolio planning in the illustrative case.

2. These histograms are narrow which identifies the variabilities of accumulated wealth, mortgage profit and total mortgage payments as rather low. This provides a ground to a conclusion of the effective risk management accomplished in the illustrative case.

3. In general, having a normal curve, these histograms indicate the consistency in the solution structure over scenarios in the illustrative case.

Finally, the CVaR of such portfolio is 31,740.000 Euro with the confidence level $\alpha = 0.9$.

7.2 Test Metrics and Test Cases

In order to show that the integrated pension and mortgage portfolio planning model is correct and operationally robust, the following test metrics are considered:

- Optimization horizon of the portfolio is chosen among: the third, fifth or seventh years. Hence, the portfolio time frames can be of three, five or seven years long.
- Number of periods within the portfolio time frame: one, two or three periods.
- Time stages at which the decisions are made within the portfolio time frame: year 0 (starting), year 1, year 2, year 3, year 5, year 7.
- Risk measure used in the optimization model: CVaR or CDaR.

Combining different values of optimization horizon, number of periods, and time stages at which decisions are made within the portfolio time frame results in such a subset of test cases:

1. Three year optimization horizon:
    - 1 period: 0y-3y time stages,
    - 2 periods: 0y-1y-3y and 0y-2y-3y time stages,
    - 3 periods: 0y-1y-2y-3y time stages.
2. Five year optimization horizon:
    - 1 period: 0y-5y time stages,
    - 2 periods: 0y-1y-5y, 0y-2y-5y, 0y-3y-5y time stages,
    - 3 periods: 0y-1y-2y-5y, 0y-1y-3y-5y, 0y-2y-3y-5y time stages.
3. Seven year optimization horizon:
    - 1 period: 0y-7y time stages,
    - 2 periods: 0y-1y-7y, 0y-2y-7y, 0y-3y-7y time stages,
    - 3 periods: 0y-1y-2y-7y, 0y-1y-3y-7y, 0y-2y-3y-7y time stages.

7.3 Major Test Findings

Having conducted the test cases defined in the prior *Section 7.2 Test Metrics and Test Cases*, it is noteworthy to capitalize on the following major findings.

When analyzing portfolio management solutions, it is a standard practice to construct and compare efficient frontiers they yield. This approach has been used in the following test and analysis. Hence, it is important to revise the concept of efficient frontier.

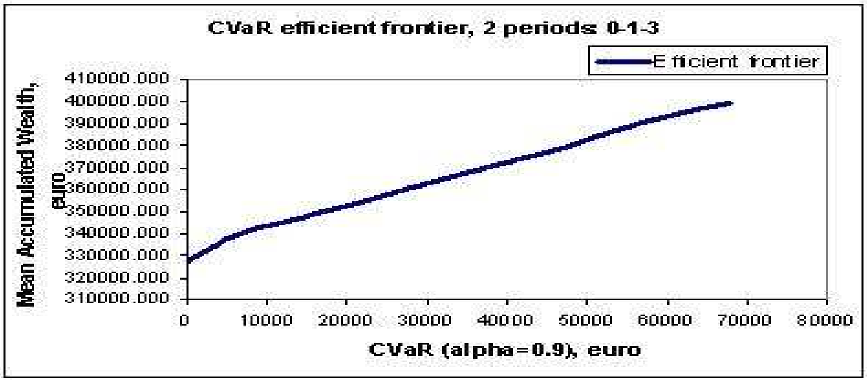
According to Modern Portfolio Theory, a portfolio is efficient if two conditions are met: no other portfolio has a greater expected return with no more risk, and no other portfolio has less risk with no less expected return. If one or both of these conditions are not true, a portfolio is said to be inefficient. When all portfolios are plotted on a graph of value vs. risk, the efficient portfolios form on a line called the "efficient frontier". There are no viable portfolios above this line.

7.3.1 Sample Test

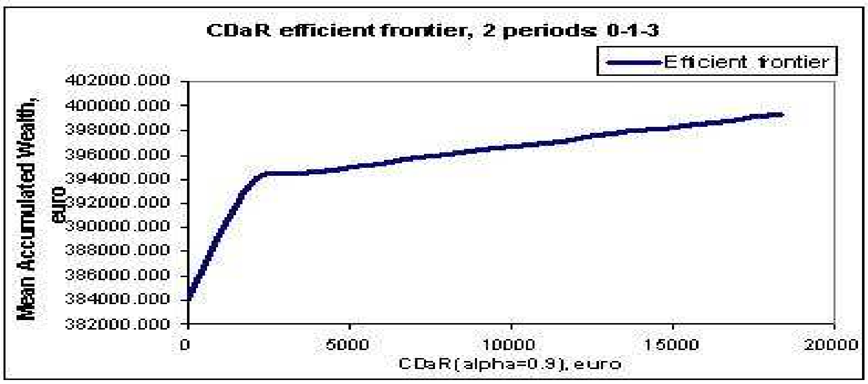
Figure 21 presents the results of solving the model for the sample two-period 0y-1y-3y portfolio. In particular, it illustrates efficient frontiers found by the CVaR formulation in the 0y-1y-3y test case on 21(a) and 0y-1y-2y-3y case - on 21(c). Further, CDaR efficient frontiers in the 0y-1y-3y and 0y-1y-2y-3y test cases are shown on 21(b) and 21(d) correspondingly.

One may observe that the CDaR efficient frontier has a steeper shape if compared to the corresponding CVaR one. This implies that even with high risk aversion an investor is offered larger margins in portfolio wealth provided small relative (to the total interval of risk measure values) change in its risk. Moreover, the CDaR efficient frontier is characterized by more steadiness for the investors with other than high risk aversion, which allows for a presumably effective wealth risk application of this model.

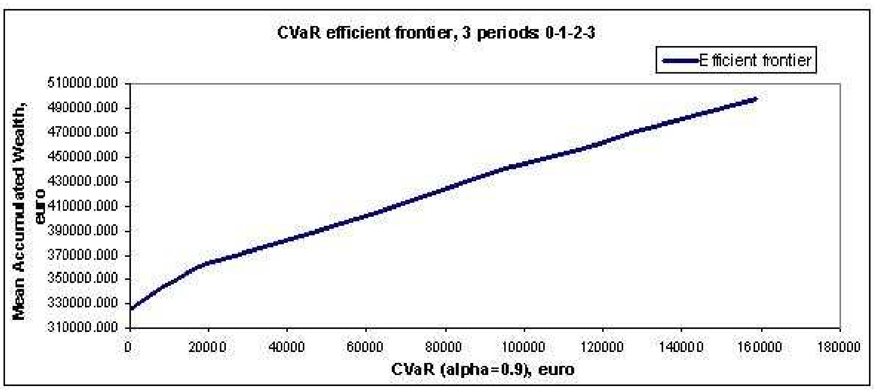
It is noteworthy to mention that the solution time used to optimize the CDaR model formulation is shorter than that used to optimize its corresponding CVaR counterpart.



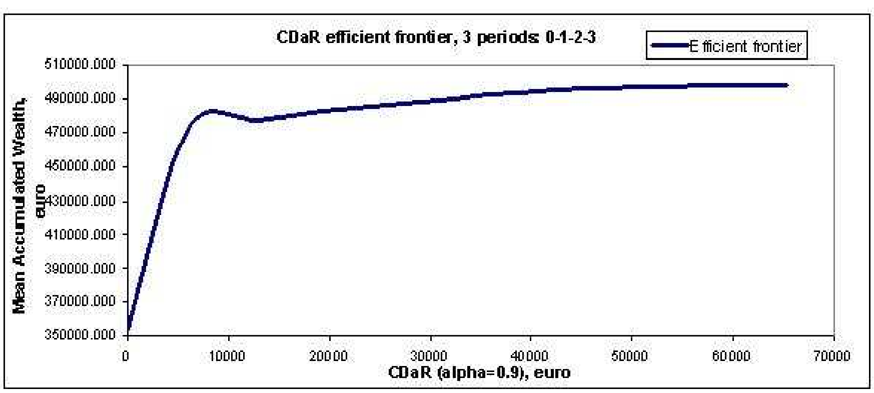
(a) CVaR efficient frontier of the 0y-1y-3y portfolio



(b) CDaR efficient frontier of the 0y-1y-3y portfolio



(c) CVaR efficient frontier of the 0y-1y-2y-3y portfolio



(d) CDaR efficient frontier of the 0y-1y-2y-3y portfolio

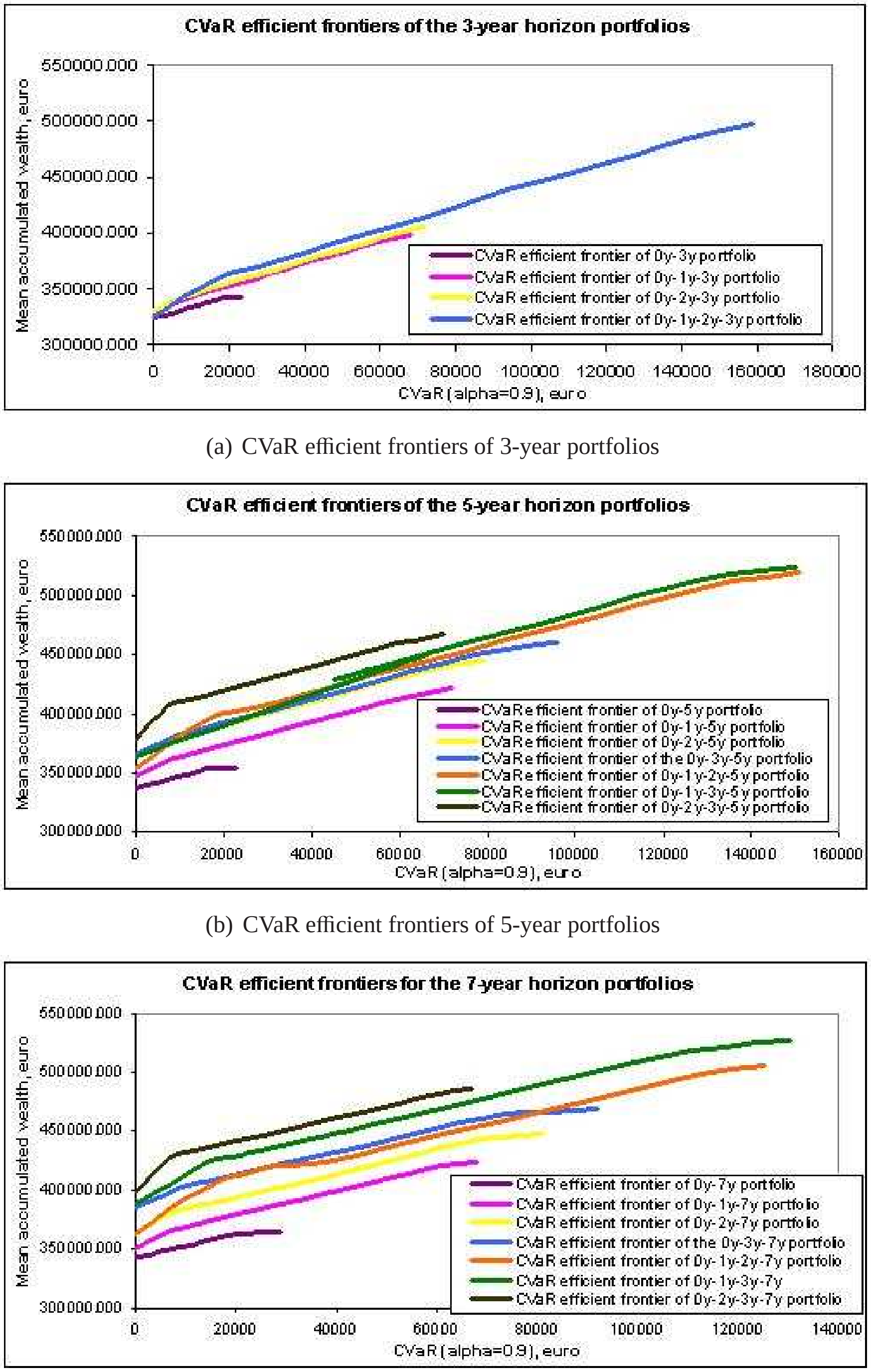
Figure 21: Sample Test: CVaR and CDaR Efficient Frontiers.

7.3.2 Multiperiod Study

Comparing efficient frontiers of the models with the same optimization time horizon but different number of periods, it is noteworthy to mention that multi-stage problems overperform the corresponding single-stage models. Moreover, among the multi-stage models, one may observe that the cases with the higher number of stages are adding significant value to the efficient frontiers. When the solutions of the models with identical number of periods but different time stages are compared, the efficient frontiers of the later-staged solutions are predominantly outperforming the earlier-staged ones correspondingly[21].

Figures 22 and 23 illustrate efficient frontiers of the portfolios with three, five and seven-year optimization horizons with different number of periods and time stages, found by the CVaR and CDaR models correspondingly. As one may observe, the $0-1-2-3$ is the CVaR efficient frontier yields highest returns over the widest range of CVaR values (Figure 22(a)) among other 3-year efficient frontiers (namely, $0-3$, $0-1-3$ and $0-2-3$ cases, which are performing in the ascending order). In the CDaR sense, the $0-1-2-3$ efficient frontier is significantly overperforming compared to other 3-year frontiers (Figure 23(a)) when other than absolutely risk-averse investor is concerned. Similar trends in the 5- (Figures 22(b) and 23(b) in the CVaR and CDaR sense correspondingly) and 7-year cases (Figures 22(c), 23(c)) conform to the conclusion that the three-staged models yield better performing efficient frotniers than one- or two-staged ones. They also validate that among the models with the same number of stages, the later-staged are in general outperforming the earlier-staged ones.

[21]The case X is considered a later-staged if vs. the case Y if the time stages of X are greater than those of Y, when compared numerically. E.g. the case $0-1-3-5$ is later-staged compared to the case $0-1-2-5$.

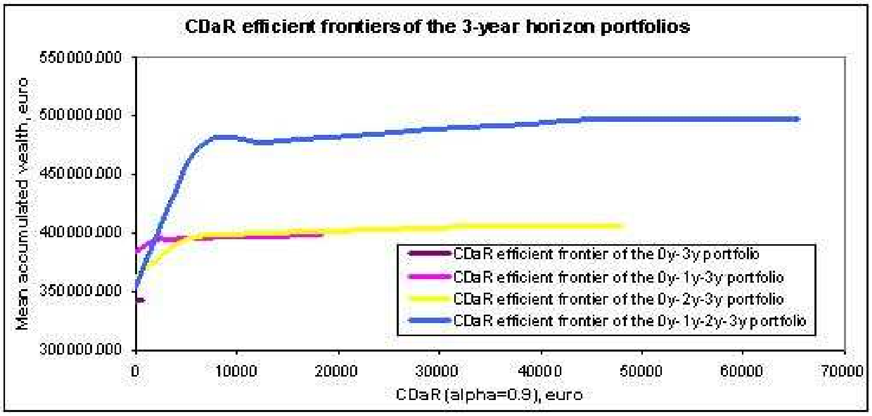


(a) CVaR efficient frontiers of 3-year portfolios

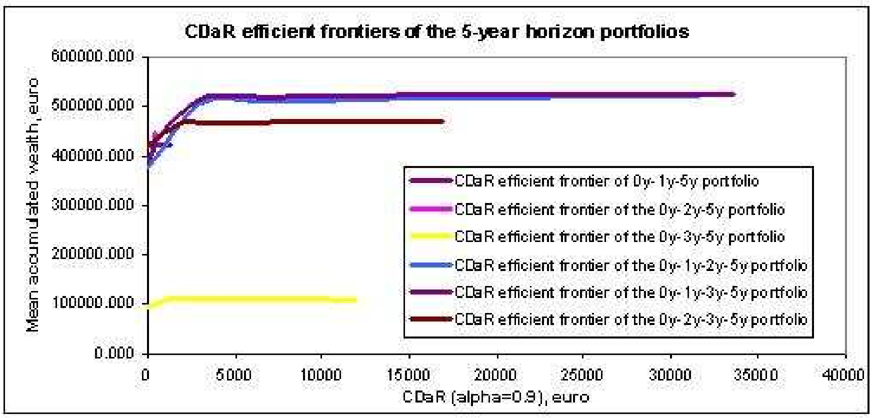
(b) CVaR efficient frontiers of 5-year portfolios

(c) CVaR efficient frontiers of 7-year portfolios

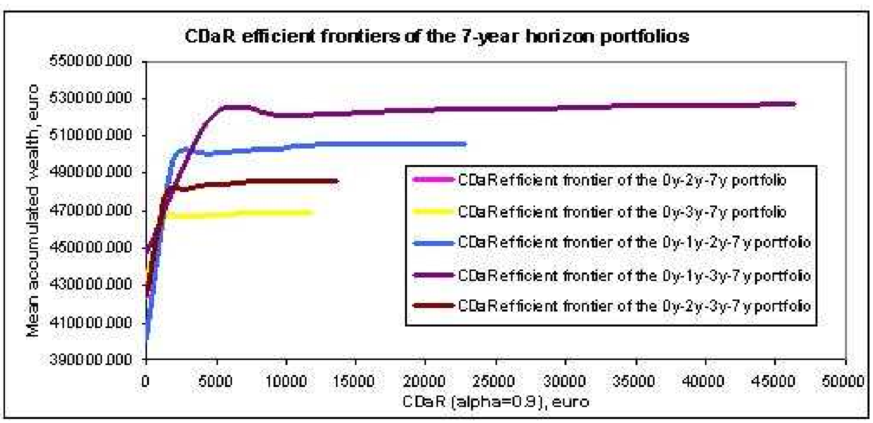
Figure 22: Multiperiod Study: CVaR Efficient Frontiers of the Portfolios with 3-, 5- and 7-year Horizons.



(a) CDaR efficient frontiers of 3-year portfolios



(b) CDaR efficient frontiers of 5-year portfolios



(c) CDaR efficient frontiers of 7-year portfolios

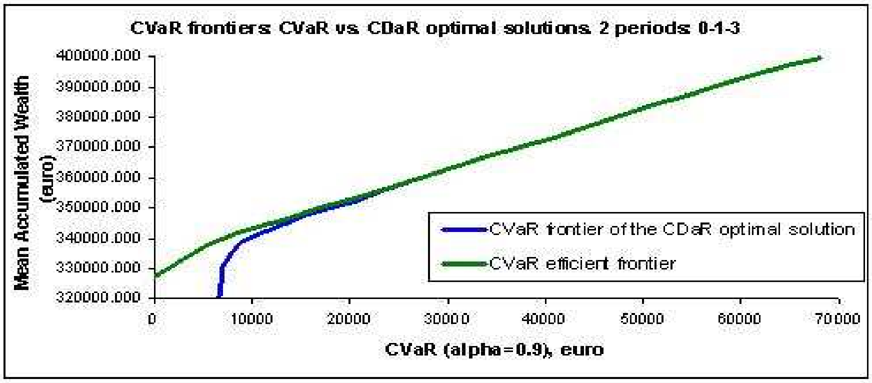
Figure 23: Multiperiod Study: CDaR Efficient Frontiers of the Portfolios with 3-, 5- and 7-year Horizons

7.4 Robustness Analysis

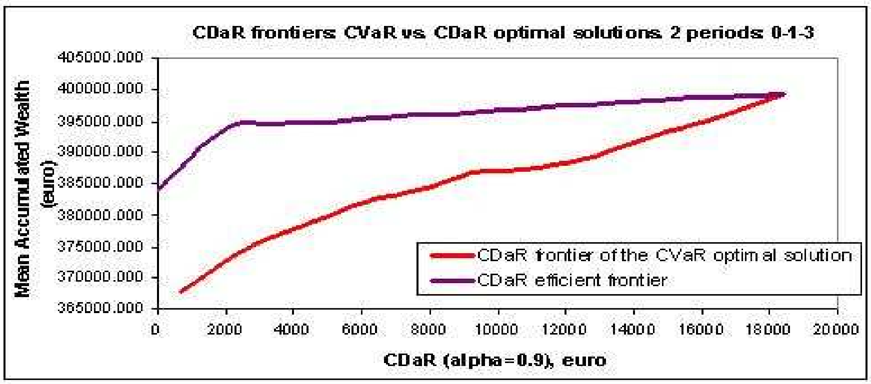
For the purpose of analyzing robustness of the integrated pension and mortgage portfolio management model, the CVaR efficient frontier is compared with the CVaR frontier yielded by the use of CVaR optimal solution in the CDaR model. Similarly, the CDaR efficient frontier is then compared to the CDaR frontier of the CVaR model with the CDaR optimal solution. Figure 24 presents such frontiers for the 0y-1y-3y and 0y-2y-3y cases.

No efficient frontier can be overperformed by other feasible frontiers. Both CVaR and CDaR models conform to this statement. Namely, in the CVaR cases, efficient frontiers are above the ones yielded by corresponding CDaR models with the original CVaR optimal solutions. Further, in the CDaR cases, efficient frontier are also above frontiers yielded by the CVaR models with the CDaR optimal solutions.

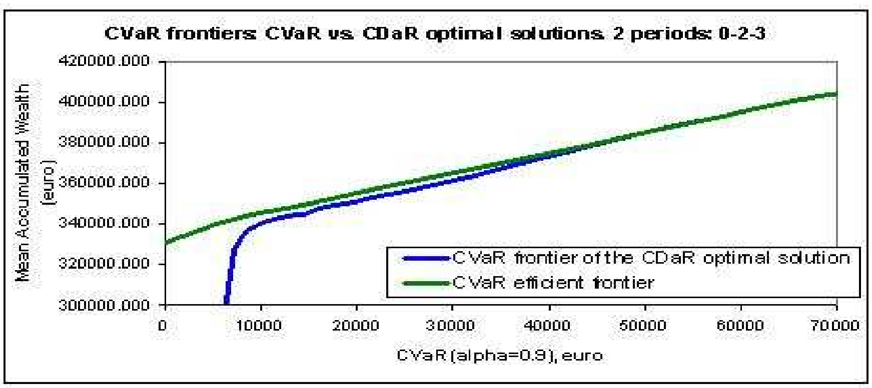
It is noteworthy to observe that CVaR frontier yielded by the CDaR solution is fairly close to the CVaR efficient frontier. This indicates fine robustness of CDaR solution in the CVaR sence. This may have value when the solution time is a concern (CVaR models are longer to optimize). On the other hand, CDaR frontier yielded by the CVaR solution offers less reward for more risk, with convergence in its right-most part (for the risk-neutral attitude). Hence, the CVaR is only partially suitable for use in the CDaR sence in terms of desired similarities in the resulting efficient frontier.



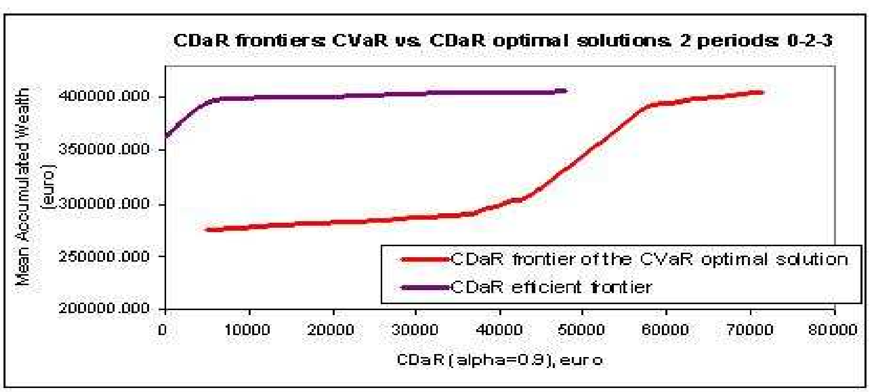
(a) CVaR frontiers in the 0y-1y-3y case



(b) CDaR frontiers in the 0y-1y-3y case



(c) CVaR frontiers in the 0y-2y-3y case



(d) CDaR frontiers in the 0y-2y-3y case

Figure 24: Robustness Analysis: CVaR vs. CDaR Frontiers

7.5 Sensitivity Analysis

To provide confidence in the integrated pension and mortgage portfolio management model, the sensitivity analysis has been carried out. This section briefly evaluates the sensitivity of CVaR and CDaR efficient frontiers to changes in the investment trust returns (the "most uncertain" parameters).

In particular, the statistics of investment trust returns is perturbed by:

- changes in the first and second central moments (i.g. consistent shift in the selected values),
- changes in the correlation matrix (i.g. other than original asset class is the most correlated with the short rates),

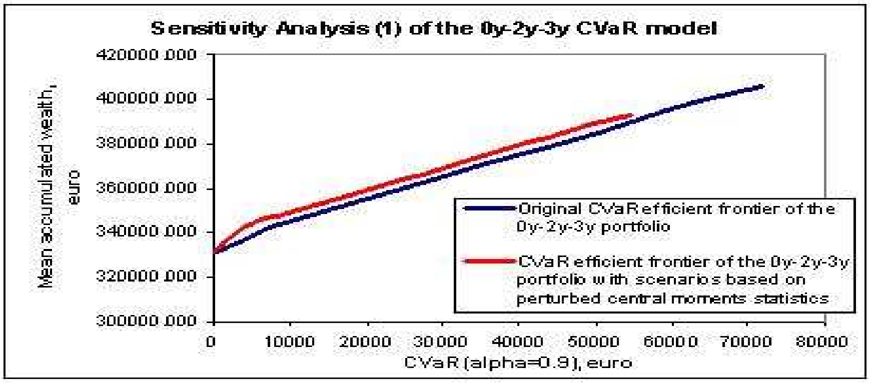
Next, new investment trust returns scenarios are generated (see Figures 32 and 33 in the Appendix B for visualization of the original and perturbed scenario trees accordingly). These are the scenarios over which the sensitivity of CVaR and CDaR efficient frontiers is analyzed.

Figures 25(a) and 25(b) present efficient frontiers yielded by the CVaR and CDaR models with scenarios based on the original and perturbed central moments statistics. Similarly, Figure 25(c) and 25(d) show efficient frontiers yielded by the CVaR and CDaR models with scenarios based on the original and perturbed correlation statistics.

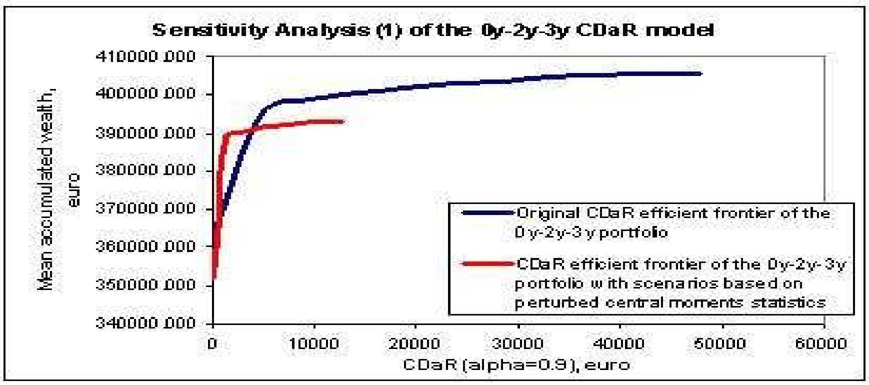
One may observe that the shape of CDaR efficient frontier is more sensitive to the changes in scenarios. Yet, when the absolute distances between the curves are evaluated (see Table 6), the gap between CVaR efficient frontier in the case of original scenarios and the one in the case of perturbed scenarios, may be similar or even larger than that yielded by the CDaR model.

In order to validate the stability of CVaR and CDaR optimal solutions, the corresponding contribution in the asset products is compared among the portfolios based on the original and perturbed scenarios in the two-stage portfolio in the following.

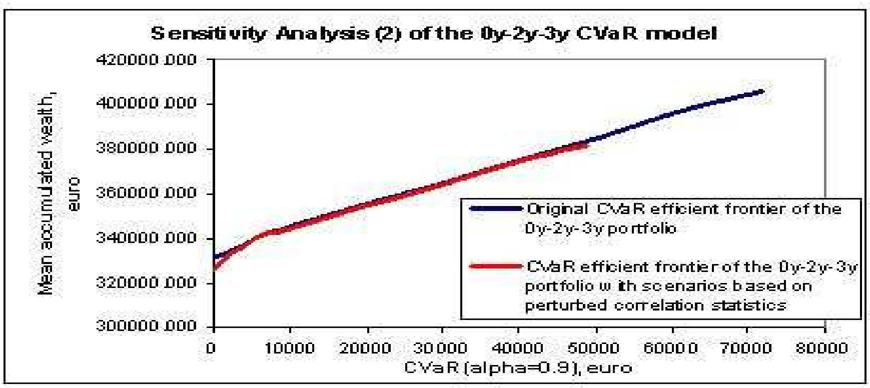
Solution stability analysis with respect to perturbation in central moments statistics



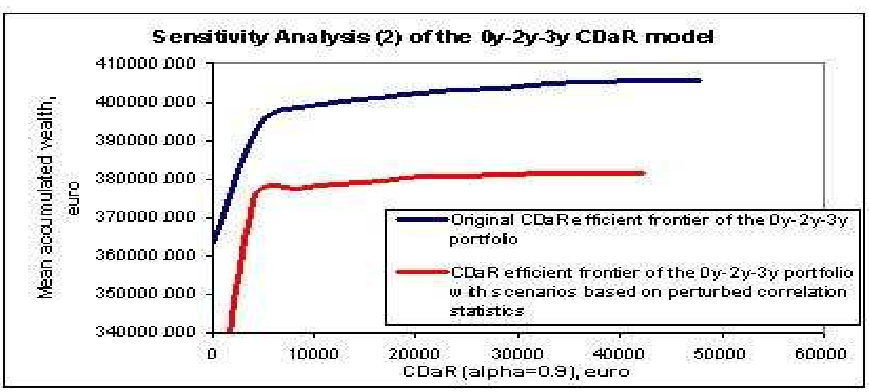
(a) Sensitivity analysis (perturbed central moments statistics) of CVaR 0y-2y-3y model



(b) Sensitivity analysis (perturbed central moments statistics) of CDaR 0y-2y-3y model



(c) Sensitivity analysis (perturbed correlation statistics) of CVaR 0y-2y-3y model



(d) Sensitivity analysis (perturbed correlation statistics) of CDaR 0y-2y-3y model

Figure 25: Sensitivity Analysis of the CVaR and CDaR Models

	CVaR (1)	CDaR (1)	CVaR (2)	CDaR (2)
0-1-3	23369.3474	22836.393	35532.268	30786.745
0-2-3	21305.22002	37483.212	33338.04	50700
0-1-2-3	505600.0646	69855.0828	523151.9282	59705.805

Table 6: Distance between the efficient frontiers in the Sensitivity Analysis. (1) - Sensitivity Analysis of the models with scenarios based on the original and perturbed central moments statistics, and (2) - original and perturbed correlation statistics correspondingly. The distance (in euro) is expressed as the maximum of distances between the efficient frontiers.

CVaR model

The contribution structure (see Figure 26) is almost identical at the starting year in the CVaR solutions with scenarios based on original and perturbed statistics. Further similarities are observed in the contribution dynamics at the second and third years correspondingly. It is noteworthy to mention that the CVaR model with scenarios based on perturbed statistics yields more steady contribution structures if compared to those in the case of original scenarios.

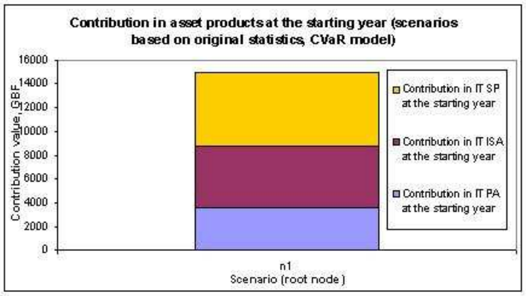
CDaR model

Figure 27 illustrates the contribution element of solution in the case of CDaR model. As in the CVaR case, the CDaR model yields contribution structure which is similar yet more steady in the case of pertubed scenarios rather than in the case of original ones.

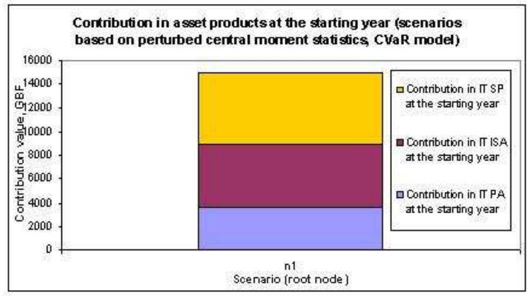
The accumulated wealth yielded by the solutions to the original and perturbed models is analyzed by means of histograms shown on Figure 28. In both CVaR and CDaR cases, the accumulated wealth distribution is sensitive to changes in scenarios, with similarity in effects (observe the shape of distribution in the CVaR and CDaR models correspondingly). This infers the consistency in response to such a change in input parameters in both CVaR and CDaR models.

In conclusion, both CVaR and CDaR efficient frontiers are sensitive to the changes in the investment trust returns statistics. In general, the solution structure is affected in terms of its steadiness quality.

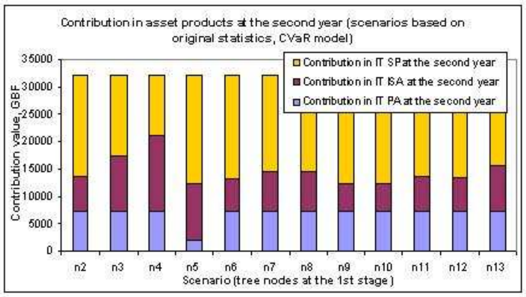
For more details on sensitivity analysis carried out, see *Appendix E.*



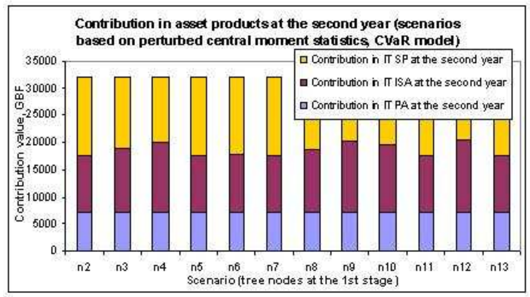
(a) Contribution decisions (original scenarios) at the starting year: CVaR 0y-2y-3y model



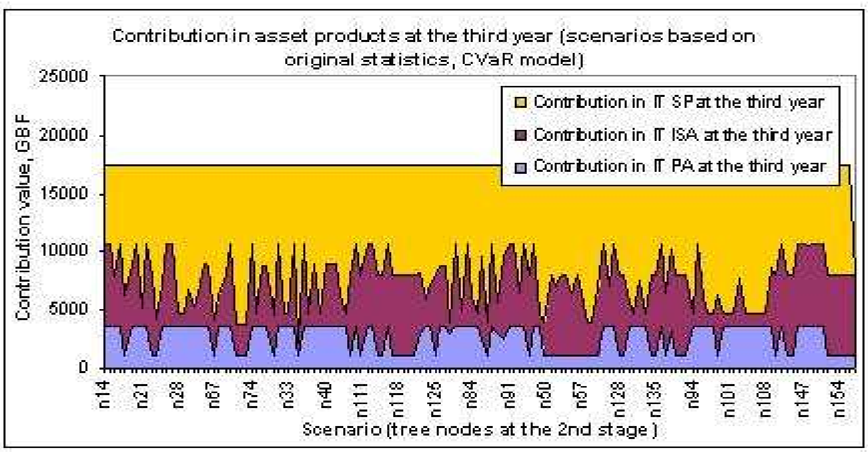
(b) Contribution decisions (perturbed scenarios) at the starting year: CVaR 0y-2y-3y model



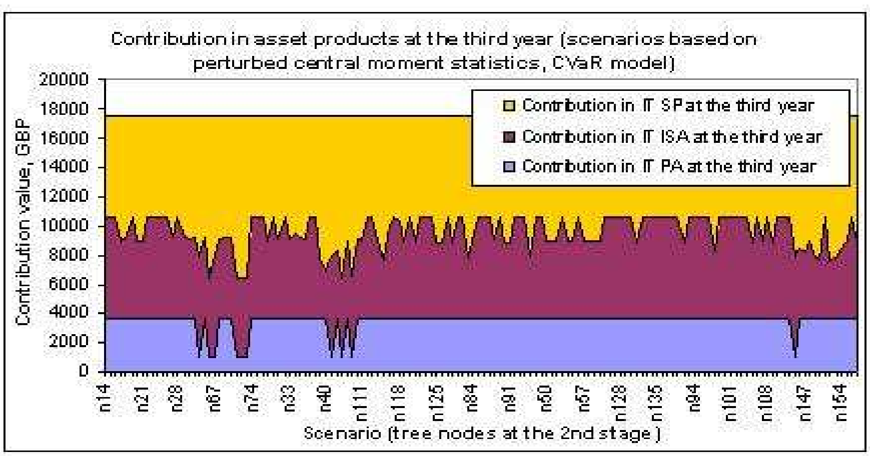
(c) Contribution decisions (original scenarios) at the second year: CVaR 0y-2y-3y model



(d) Contribution decisions (perturbed scenarios) at the second year: CVaR 0y-2y-3y model

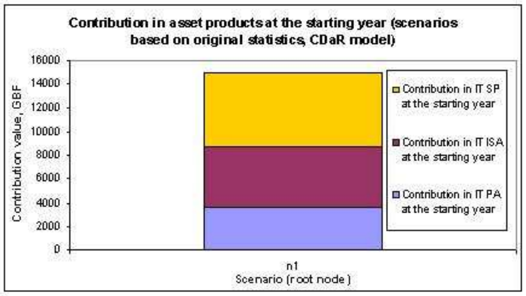


(e) Contribution decisions (original scenarios) at the third year: CVaR 0y-2y-3y model

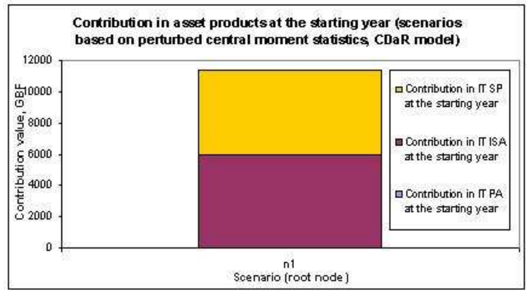


(f) Contribution decisions (perturbed scenarios) at the third year: CVaR 0y-2y-3y model

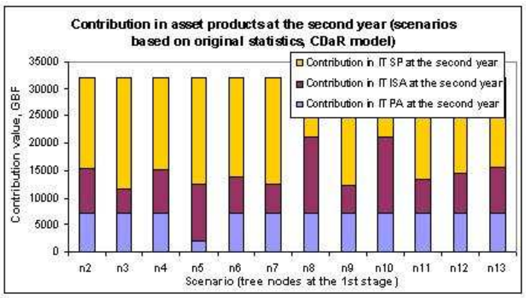
Figure 26: Contribution decisions (original vs. perturbed scenarios) of the CVaR model



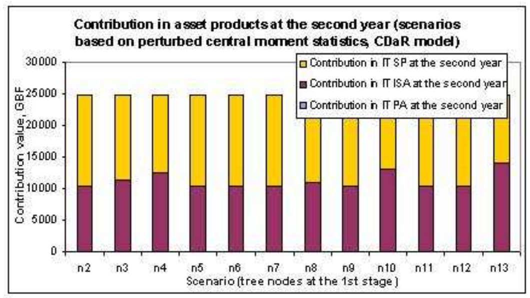
(a) Contribution decisions (original scenarios) at the starting year: CDaR 0y-2y-3y model



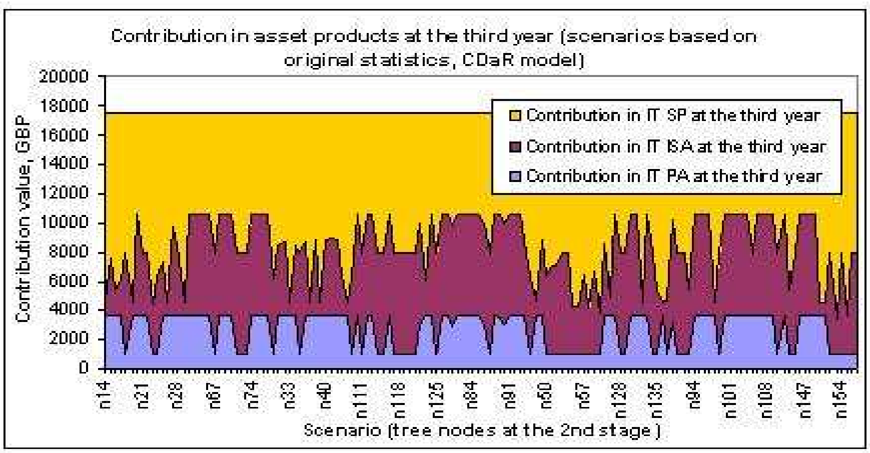
(b) Contribution decisions (perturbed scenarios) at the starting year: CDaR 0y-2y-3y model



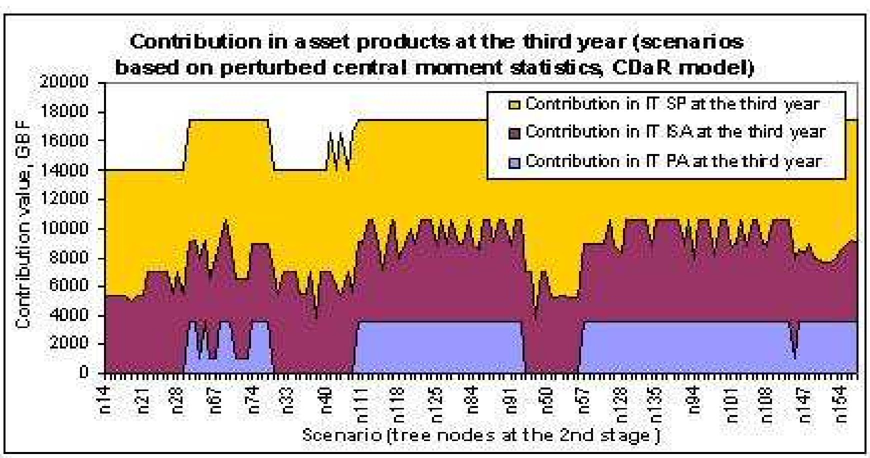
(c) Contribution decisions (original scenarios) at the second year: CDaR 0y-2y-3y model



(d) Contribution decisions (perturbed scenarios) at the second year: CDaR 0y-2y-3y model

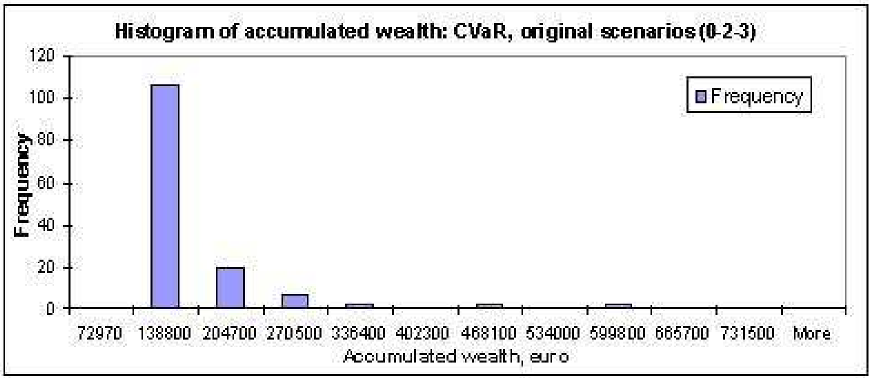


(e) Contribution decisions (original scenarios) at the third year: CDaR 0y-2y-3y model

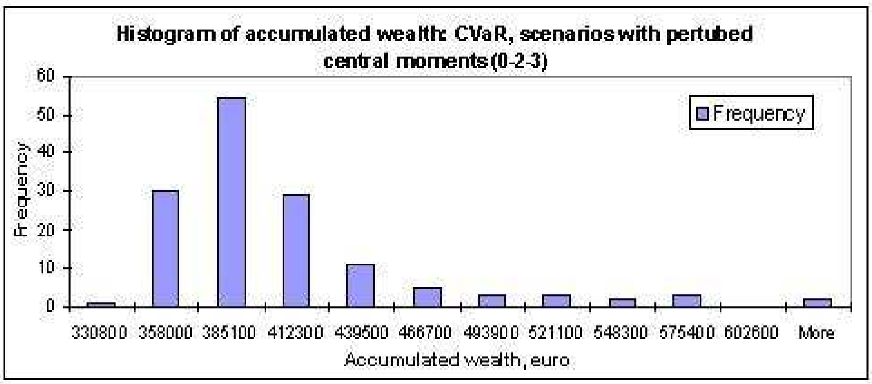


(f) Contribution decisions (perturbed scenarios) at the third year: CDaR 0y-2y-3y model

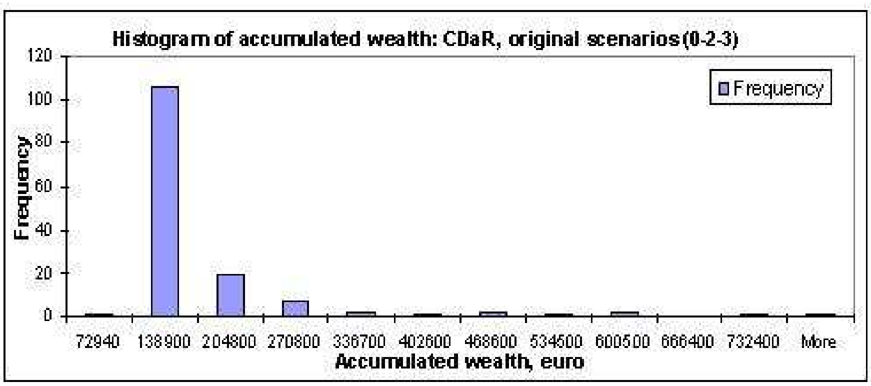
Figure 27: Sensitivity of contribution decisions (original vs. perturbed scenarios) of the CDaR model



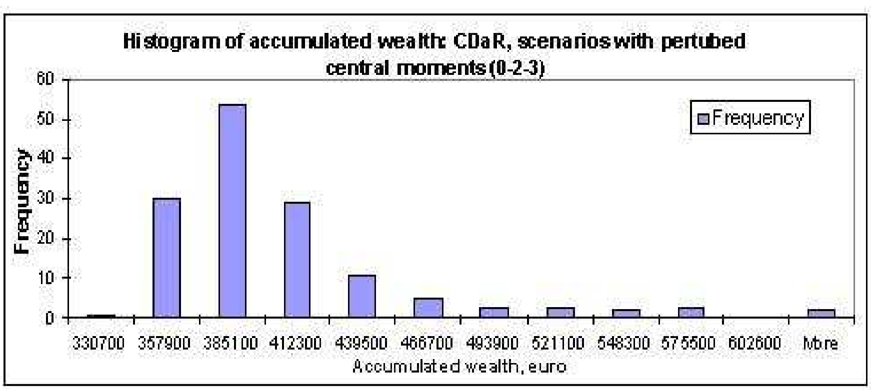
(a) 0y-2y-3y CVaR model based on original scenarios



(b) 0y-2y-3y CVaR model based on scenarios with perturbed central moments statistics



(c) 0y-2y-3y CDaR model based on original scenarios



(d) 0y-2y-3y CDaR model based on scenarios with perturbed central moments statistics

Figure 28: Sensitivity Analysis of CVaR and CDaR models: Accumulated Wealth

8 Conclusions

8.1 Summary

The hereby presented Master Thesis proposes an integrated approach to the pension and mortgage portfolio management problems that are traditionally solved separately. These are viewed from the perspective of a household with needs to capitalize on its financial wealth towards preset goals. Such goals may be long-term, such as accumulation of retirement funds or shorter-term, e.g. school fees to pay on the future education of their children. At the same time, the household may be facing mortgage obligations on their existing or planned property. Both such goals and obligations can be met by setting up and actively managing a financial portfolio of specialized products. These products usually differ in their structure and require deep understanding of benefits and risks associated with their use. Thus, an essential challenge is to study and model their policies and dynamics to a reasonable level of detail.

In the scope of this work, the Pension Account, Investment Savings Account and Share Plan products offered to a UK investor are participating at the asset side of the portfolio. The underlying links of these products are shares of investment trusts which are inherently stochastic in terms of their market prices, returns, dividends, and etc. Danish mortgage products are chosen for participation on the liability side. These are based on the Fixed interest Rate Mortgage (FRM) and Adjustable interest Rate Mortgage (ARM) loans having unique properties and being advancely researched in the financial optimization sense which makes them attractive to include in the study. Interest rate and market risks are the main determinants of volatility in the prices and rates of these mortgage loans. Hence, the problem translates into the integration of the UK pension and investment products with the Danish mortgage loans in anticipation of specific risks effecting returns and rates of their underlying links.

To capture uncertainties in the model, the stochastic programming approach is used. At its cornerstone is the stochastic event tree which reflects the finite number of scenarios over which the change in every uncertain parameter may happen. Such a stochastic event tree has one or more stages over which decisions can be made. These include contribution into product accounts, prepayment of mortgage installments, investment dealing operations, refinancing certain loans, and etc. At the portfolio horizon the mortgage principal prepayment is made and the profit from the market price of property is evaluated. This added to the reward of the portfolio at the asset side, comprises the accumulated wealth of the integrated portfolio. However, not only it is important to maximize final wealth of the portfolio, but also to manage the risks it incurs. The popular risk measures used in the context of financial optimization are Conditional Value-at-Risk (CVaR) and Conditional Drawdown-at-Risk (CDaR), both evaluated at a certain confidence level α. Furthermore, the investor's risk attitude defines the weights put on the goal of maximizing the accumulated wealth and on minimizing the portfolio risk measure. Combining these aspects, the bilinear utility function is defined in the objective of the problem. Having established the connection between the asset and liability portfolios and projected their influence on the portfolio goals, the integrated multistage stochastic programming model is formulated.

In order to investigate the solution structure and ascertain correctness of the model, an illustrative case followed by a number of tests has been carried out. These have shown the consistency of efficient frontiers yielded by both CVaR and CDaR formulations of the model. The correlation between the number of periods in the model and its solution is studied. Further, the robustness of the CVaR vs. CDaR models is analyzed. Finalizing the work, sensitivity analysis is carried out, aiming to assess an impact uncertain parameters such as investment trust returns have on the solution structure.

8.2 Research Contribution

The research accomplished in this thesis has a number of values. Firstly, integrated portfolio management has evolved as the well-shaped domain of financial industry from the corporate perspective, but little has been studied at the household side, resulting in a gap on the market space. A modern private investor is increasingly demanding asset and wealth management products and services, which implies a need for innovation in financial applications that can effectively fulfill such needs. On the practical side of setting up and managing such applications is the complexity of products they incorporate and multiple risk exposure they should anticipate. Traditional modeling techniques often experience performance problems in terms of resources needed to give viable answers to financial problems with uncertainties. The multistage stochastic programming approach is one of the recent instruments allowing for the relatively simple formulation of a traditional optimization model that captures uncertainties including equity, fixed income, interest rate and background risks. Having undertaken this approach, the integration of participating products and risks has resulted in the consistent and robust portfolio management solutions.

A number of research ideas are noteworthy to distinguish. The moment matching approach is used to generate input scenarios. This involves integration of the interest rates and investment trust returns into the same stochastic event tree. Such a problem is challenging due to the different uncertainty structures that describe investment trust return and interest rates correspondingly and solving it is important to the resiliency in scenario generation. Modeling of the CVaR and CDaR risk measures in the arena of an integrated portfolio problem is another task that requires careful consideration of risk positions both at the product and portfolio levels. Formulation of the objective function that balances household utility versus their risk profile and imposes a CVaR or CDaR managed control of the integrated portfolio strategies as different product scenarios unfold has been accomplished.

8.3 Future Work

One of the future research aspirations of the author is to conduct sequential historical testing of the models, which means to study the consistency and trends of the model solution with variable starting time. It is also desired to adjust the model to a changing risk profile of the household as well as to account for property, labour capital and currency risks in the stochastic sense. Analyzing the models on how unexpected circumstances influence the integrated portfolio strategies is another interesting line of further study. Such circumstances span from positively affecting investment opportunities of a household, e.g. large increase in their income or significant lump-sum premium, to negative, e.g. household bancruptcy, other than mortgage liability commitment in the future, and etc. Other products than scoped in this study are worth integrating in order to establish understanding of the model flexibility and to increase its modularity. Finally, it is of great interest to tailor the modelling approach proposed to the financial product design purposes. This may require experiments with variable product parameters, qualitative and scalability analysis, inferring large-scale tests over the wide spectrum of investor profiles.

9 Bibliography

References

[1] Bodie, Zvi , *Thoughts on the future: Life-cycle investing in theory and practice*, Financial Analysts Journal, January-February 2003.

[2] Bodie, Zvi, Robert C. Merton and William Samuelson, *Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model*, Journal of Economic Dynamics and Control, issue 16, 1991.

[3] Campbell, John T. , *Household Finance*, January 2, 2006.

[4] Carino, David R. and Willian T. Ziemba, *Formulation of the Russel-Yasuda Kasai Financial Planning Model*, Operations Research, Vol. 46, No. 4, July-August 1998.

[5] Chekhlov, A., S. Uryasev and M. Zabarankin, *Portfolio Optimization with Drawdown Constraints*, January 13, 2003.

[6] Consiglio, Andrea, Flavio Cocco and Stavros Zenios, *Scenario Optimization Asset and Liability Modeling for Individual Investors*, Working Paper 02-10, HERMES Center of Excellence on Computational Finance and Economics, January 25, 2005.

[7] Consiglio, Andrea, Flavio Cocco and Stavros Zenios, *The Value of Integrative Risk Management for Insurance Products with Guarantees*, Financial Institutions Center, The Wharton School, University of Pennsylvania, July 2001.

[8] Drijver, Sibrand *Asset Liability Management for Pension Funds using Multistage Mixed-integer Stochastic Programming*, Rijksuniversiteit Groningen, The Netherlands, July 2005

[9] Hoyland Kjetil and Stein Wallace, *Generating Scenario Trees for Multistage Decision Problems*, Management Science Vol. 47, No. 2, February 2001, INFORMS.

[10] Hull, John, *Options, Futures and other Derivatives*, 5th edition, 2003.

[11] Klaassen, Pieter, *Comment on "Generating Scenario Trees for Multistage Decision Problems"*, Management Science Vol. 48, No. 11, November 2002, INFORMS.

[12] Nielsen, Søren S. and Rolf Poulsen, *A Two-Factor Stochastic Programming Model of Danish Mortgage-Backed Securities*, Journal of Economic Dynamics and Control, 28, 2004.

[13] Rasmussen, Kourosh M. and Jens Clausen, *Mortgage Loan Portfolio Optimization using Multi Stage Stochastic Programming*, November 9, 2005.

[14] Rasmussen, Kourosh M. and Stavros S. Zenios, *Well ARMed and FiRM: Diversification of mortgage loans for homeowners*, May 2006, Working Paper 06-02 HERMES Center on Computational Finance, University of Cyprus, Nicosia.

[15] Rockafellar, R.Tyrrell and Stanislav Uryasev, *Optimization of Conditional Value-at-Risk*, Journal of Risk 2 (2000), 21-41.

[16] Rockafellar, R.Tyrrell and Stanislav Uryasev, *Conditional Value-at-Risk for General Loss Distributions*, Research report 2001-5, (2001)

[17] Thijssen, Jacco, *A Stochastic Discount Factor Approach to Investment under Uncertainty*, January 2006, University of Dublin

[18] Viceira, Luis M., *Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income*, Journal of Finance, Vol. 56 Issue 2, April 2001

[19] United Nations, *World Population Prospects. Highlights*, the 2004 Revision, United Nations New York, 2005

[20] Zenios, Stavros A., *Practical Financial Optimization*, Draft of June 16, 2005.

[21] Ziemba, William T. and Nomura, *The Stochastic Programming Approach to Asset, Liability and Wealth Management*, AIMR Publisher, May 30, 2003.

[22] Bank of England, *http://www.bankofengland.co.uk*

[23] Danmarks Nationalbank, *http://www.nationalbanken.dk*

[24] Thomson Financial Datastream Advance, *http://www.datastream.com*

A Complete Model Formulation

In the following the integrated pension and mortgage portfolio management model is formulated in CVaR and CDaR versions:

- CVaR model: all Asset Portfolio Constraints, Mortgage Portfolio Constraints, complemented by the integrated portfolio constraints (18), (19) and (20). Objective function (22).

- CDaR model: all Asset Portfolio Constraints, Mortgage Portfolio Constraints, complemented by the integrated portfolio constraints (18), (19) and (21). Objective function (23).

The notation for the model parameters and variables is consistent with that in *Section 4.1.6 Investment Products Dynamics and Policy Constraints* and *Section 4.2.3 Mortgage Products Dynamics and Policy Constraints.*

Asset Portfolio Constraints

Contribution and accounting constraints

$$
\begin{aligned}
C^l_{kt} + \textstyle\sum_{k\in\mathcal{K}} X^{-l}_{ikt} &= \textstyle\sum_{k\in\mathcal{K}}(X^{+l}_{ikt} + TER_i \cdot Z^l_{ikt}) + W^l_{kt}, \quad \forall k \in \mathcal{K}, \forall t \in \{t_1, \dots, t_T\}, \forall l \in \Omega \\
C^l_{kt_0} &= \textstyle\sum_{i\in\mathcal{I}}(X^{+l}_{ikt_0} + TER_i \cdot Z^l_{ikt_0}), \quad \forall k \in \mathcal{K}, \quad \forall l \in \Omega \\
C_{(Min)k} &\le C^l_{kt} \le C_{(Max)k}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_0, \dots, t_T\}, \quad \forall l \in \Omega \\
Z^l_{ikt} &= PM^l_{it} \cdot z^l_{ikt}
\end{aligned}
$$

Cash account dynamics

$$
ACI_t \ \ge \ \textstyle\sum_{k\in\mathcal{K}}(C^l_{kt} + ATP_k), \quad \forall t \in \{t_0, \dots, t_T\}, \quad \forall l \in \Omega
$$

Investment dealing

$$
\begin{aligned}
x^{+l}_{ikt} + z^{anc(l)}_{ikt-1}(1 + r^{anc(l)}_{(Inv)it-1}) &= z^l_{ikt} + x^{-l}_{ikt}, \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \forall t \in \{t_1, \dots, t_T\}, \forall l \in \Omega \\
x^{+l}_{ikt_0} &= z^l_{ikt_0}, \quad \forall i \in \mathcal{I}, \forall k \in \mathcal{K}, \quad \forall l \in \Omega
\end{aligned}
$$

Variable cost structure (on investment dealing)

$$
\begin{aligned}
X_{ikt}^{+l} &= PO_{it}^{l} \cdot x_{ikt}^{+l}(1 + GovStamp_k) + pf_{ikt}^{l} \\
&\quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_0, \ldots, t_T\}, \quad \forall l \in \Omega \\
\Delta pf_{ikt}^{l} &= PFCap_k - PFR_k \cdot PO_{it}^{l} x_{ikt}^{+l} \\
&\quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_0, \ldots, t_T\}, \quad \forall l \in \Omega \\
pf_{ikt}^{l} &= PFCap_k - \Delta pf_{ikt}^{l} \\
&\quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_0, \ldots, t_T\}, \quad \forall l \in \Omega \\
X_{ikt}^{-l} &= PB_{it}^{l} \cdot x_{ikt}^{-l} + sf_{ikt}^{l} \\
&\quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega \\
\Delta sf_{ikt}^{l} &= SFCap_k - SFR_k \cdot PB_{it}^{l} x_{ikt}^{-l} \\
&\quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega \\
sf_{ikt}^{l} &= SFCap_k - \Delta sf_{ikt}^{l} \\
&\quad \forall i \in \mathcal{I}, \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega
\end{aligned}
$$

Withdrawal constraints

$$
\begin{aligned}
W_{ktl} &\leq W_{(Max)k} \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega \\
W_{ktl} &\leq \textstyle\sum_{i \in \mathcal{I}} Z_{ikt}^{l} - W_{(Rem)k} \quad \forall k \in \mathcal{K}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega
\end{aligned}
$$

Mortgage Portfolio Constraints

Cash account dynamics

$$
\begin{aligned}
\textstyle\sum_{j \in \mathcal{J}} K_j^{l} \cdot y_{jt_0}^{+l} &\geq IA, \quad \forall l \in \Omega \\
\textstyle\sum_{j \in \mathcal{J}} K_j^{l} \cdot y_{jt}^{+l} &= \textstyle\sum_{j \in \mathcal{J}} CallK_j^{l} \cdot y_{jt}^{-l} \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega.
\end{aligned}
$$

Liability flow

$$
\begin{aligned}
RG_{jt}^{l} &= RG_{jt-1}^{anc(l)} + y_{jt}^{+l} - y_{jt}^{-l} - Ajt^{l}, \quad \forall j \in \mathcal{J}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega \\
RG_{jt_0}^{l} &= y_{jt_0}^{+l}, \quad \forall j \in \mathcal{J}, \quad \forall l \in \Omega \\
y_{jt_T}^{+l} &= 0, \quad \forall j \in \mathcal{J}, \quad \forall l \in \Omega.
\end{aligned}
$$

Principle prepayment and other payments (incl. costs)

$$
\begin{aligned}
A^l_{jt} &= RG^{anc(l)}_{jt-1}\Big(\frac{r^l_{(M)j}}{1-(1+r^{anc(l)}_{(M)j})^{-T+t-1}} - r^{anc(l)}_{(M)j}\Big), \quad \forall j \in \mathcal{J}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega \\
B^l_{jt} &= A^l_{jt} + r^{anc(l)}_{(M)j}(1-\gamma)RG^{anc(l)}_{jt-1} + b(1-\beta)RG^{anc(l)}_{jt-1} + \eta(y^{+l}_{jt} + y^{-l}_{jt}), \\
&\quad \forall j \in \mathcal{J}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega \\
B^l_{jt_0} &= \eta(y^{+l}_{jt_0}) + \varrho, \quad \forall j \in \mathcal{J}, \quad \forall l \in \Omega \\
B^l_t &= \textstyle\sum_{j\in\mathcal{J}} B^l_{jt}, \quad \forall t \in \{t_0, \ldots, t_T\}, \quad \forall l \in \Omega
\end{aligned}
$$

Total prepayment

$$PP^l_T = \sum_{j\in\mathcal{J}} CallK^l_j \cdot y^{-l}_{jT}, \quad \forall j \in \mathcal{J}, \quad \forall l \in \Omega$$

Final value of the mortgage portfolio

$$
\begin{aligned}
Profit^l_{(T)} &= HP^l_T - PP^l_T, \quad \forall l \in \Omega \\
PP^l_T &\leq HP^l_T, \quad \forall l \in \Omega
\end{aligned}
$$

Integrated Portfolio Constraints

Linking the asset and liability portfolios

$$B^l_t = \textstyle\sum_{k\in\mathcal{K}} W^l_{kt}, \quad \forall t \in \{t_1, \ldots, t_T\}, \quad \forall l \in \Omega \tag{18}$$

Accumulated wealth of the portfolio

$$AW^l = \textstyle\sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{I}} Z^l_{ikt_T} + Profit^l_{(T)}, \quad \forall l \in \mathcal{N}_{\mathcal{T}} \tag{19}$$

Conditional Value at Risk (CVaR) of the portfolio

Defining the parameter α as the confidence level and the following variables:

$CVaR$ = CVaR of the portfolio,

VaR = VaR of the portfolio,

$VaRDev^l$ = Difference from the Value of Risk of the portfolio in the scenario l, CVaR of the portfolio is formulated in the following:

$$\begin{aligned} VaRDev^l &\geq \textstyle\sum_{ll\in\Omega}(p(ll)\cdot AW(ll) - AW(l)) - VaR, \quad \forall l \in \Omega \\ CVaR &= VaR + \frac{\sum_{l\in\Omega}(p(l)\cdot VaRDev(l))}{(1-\alpha)} \end{aligned} \tag{20}$$

Conditional Drawdown at Risk (CDaR) of the portfolio

Defining the following variables:

$CDaR$ = CaR of the portfolio,

THR = Threshold such as (1-α)100% drawdowns dont exceed it,

D_t^l = Portfolio Drawdown,

CDaR of the portfolio is formulated in the following:

$$\begin{aligned} CDaR &= THR + \frac{1}{T(1-alpha)} \textstyle\sum_{l\in\Omega}\sum_t(p(l)\cdot(D_t^l - THR) \\ D_t^l &= Max(\frac{\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} Z_{k,i,t}^l}{\sum_{\tau\leq t} ACI_\tau}) - \frac{\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} Z_{k,i,t}^l}{\sum_{\tau\leq t} ACI_\tau} \end{aligned} \tag{21}$$

Objective Functions

Denoting by λ - risk averseness of the investor, the objective functions are formulated in the following:

CVaR objective function

$$Maximize \quad (1-\lambda)\sum_{l\in\Omega} p^l AW^l - \lambda(CVaR_\alpha) \tag{22}$$

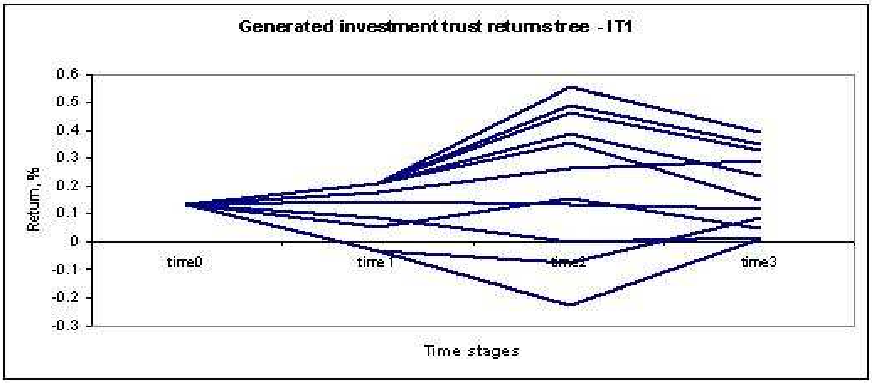
CDaR objective function

$$Maximize \quad (1-\lambda)\sum_{l\in\Omega} p^l AW^l - \lambda(CDaR_\alpha) \tag{23}$$

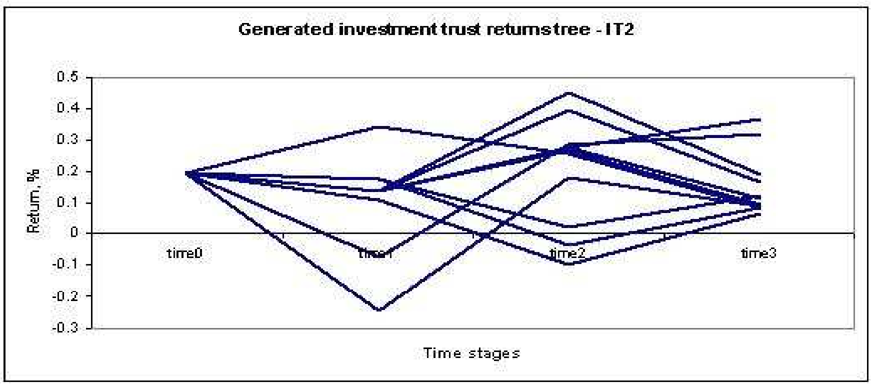
B Scenario Trees of Investment Trust Returns

To model the uncertainty of the investment trust returns the scenario trees are generated as discussed in the *Section 6.2 Interest Rate and Investment Trust Returns Tree Scenario Generation*. To convince in the appropriateness of the scenario structure, Figures 29, 30, and 31 present more scenario trees used in the test and analysis of the models.

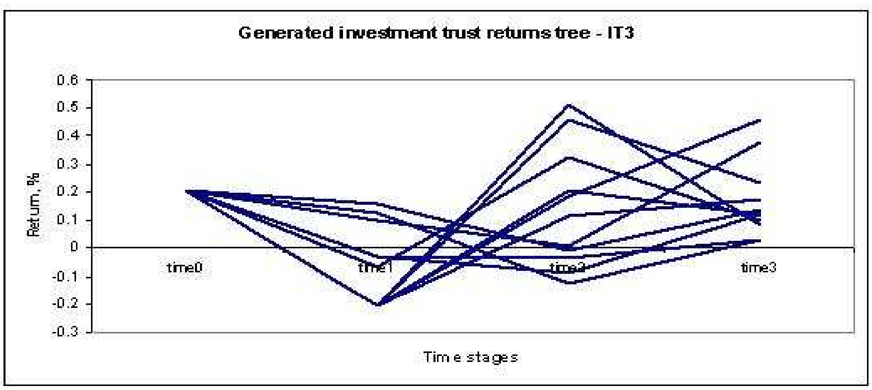
Further, Figures 32 and 33 illustrate scenario trees used in the sensitivity analysis. These are generated based on the perturbed central moments and correlaton statistics accordingly.



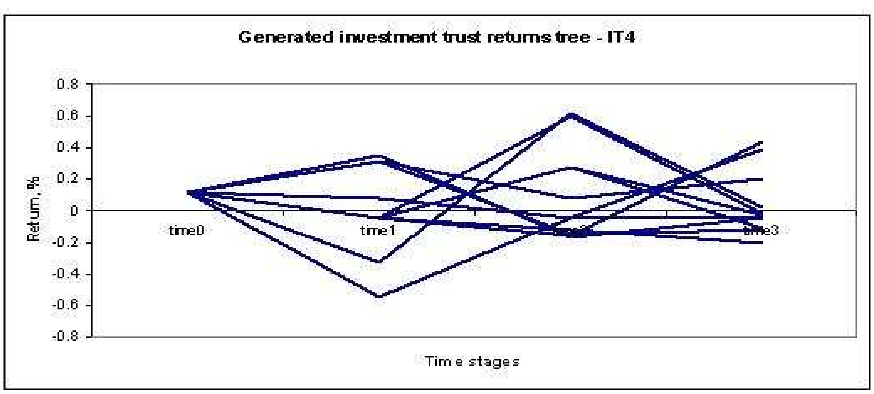
(a) The 0y-1y-2y-3y tree of IT1 returns



(b) The 0y-1y-2y-3y tree of IT2 returns

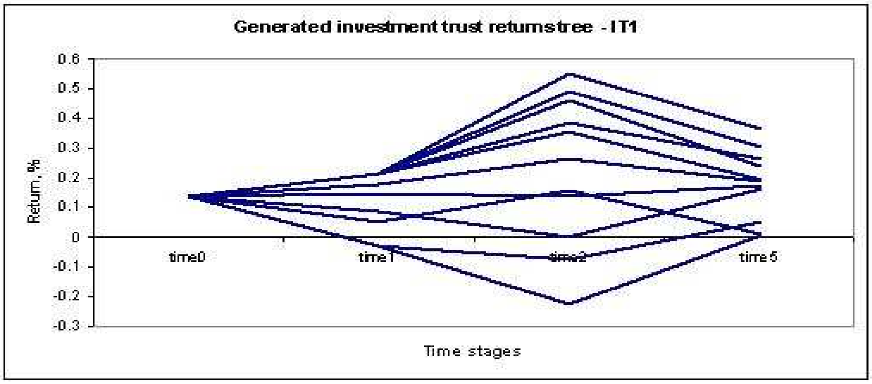


(c) The 0y-1y-2y-3y tree of IT3 returns

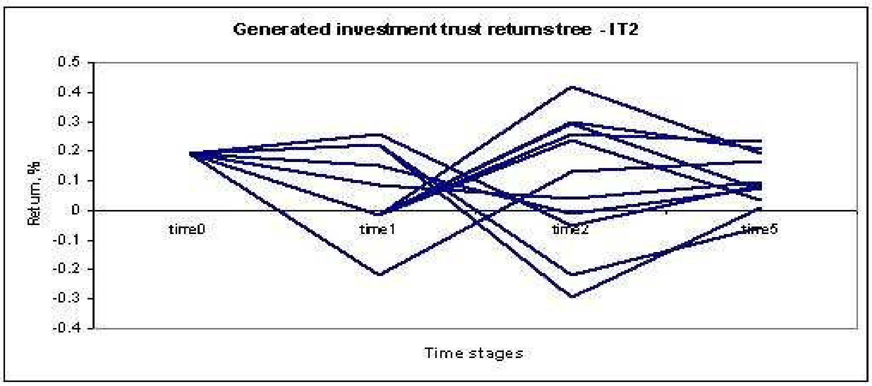


(d) The 0y-1y-2y-3y tree of IT4 returns

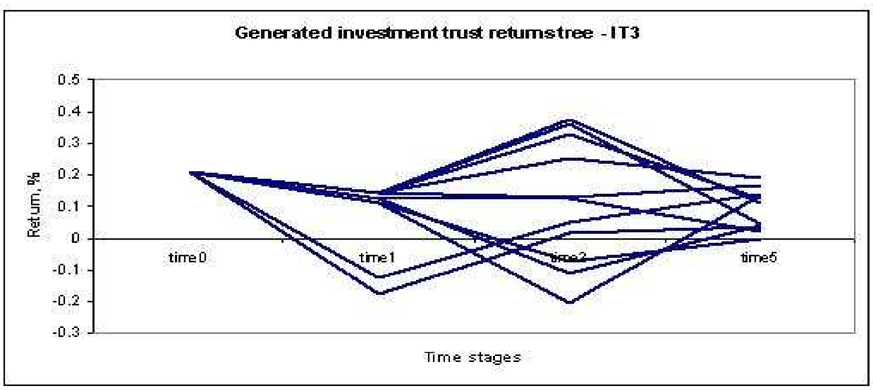
Figure 29: Stochastic event trees of investment trust returns in the 0-1-2-3 case



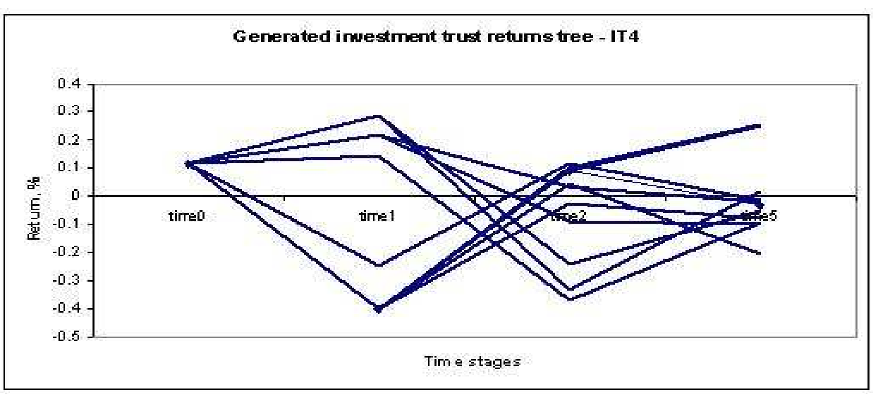
(a) The 0y-1y-2y-5y tree of IT1 returns



(b) The 0y-1y-2y-5y tree of IT2 returns

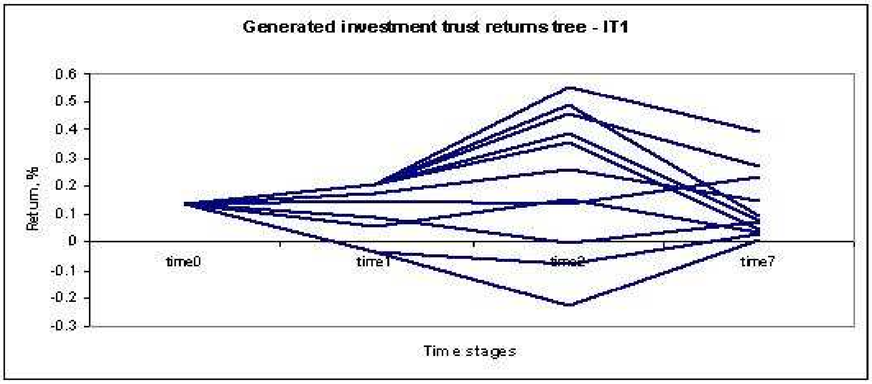


(c) The 0y-1y-2y-5y tree of IT3 returns

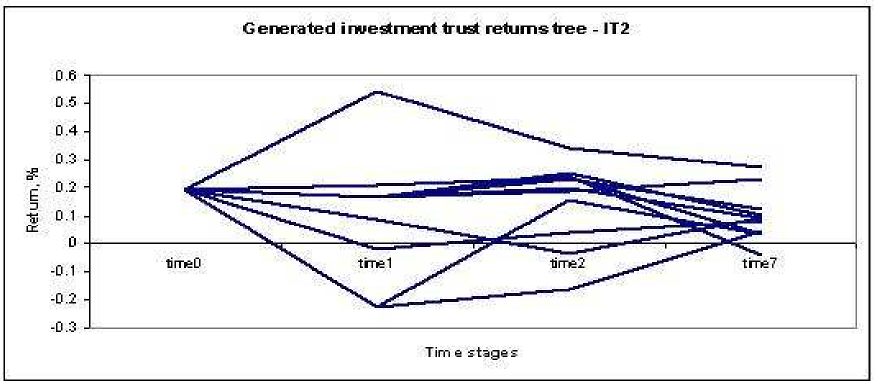


(d) The 0y-1y-2y-5y tree of IT4 returns

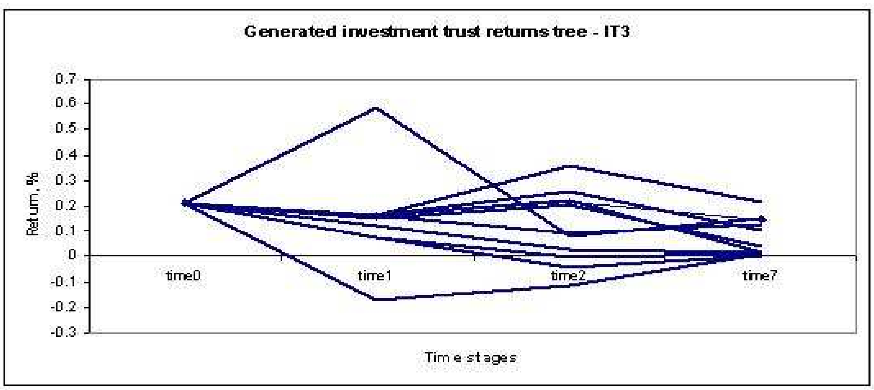
Figure 30: Stochastic event trees of investment trust returns in the 0-1-2-5 case



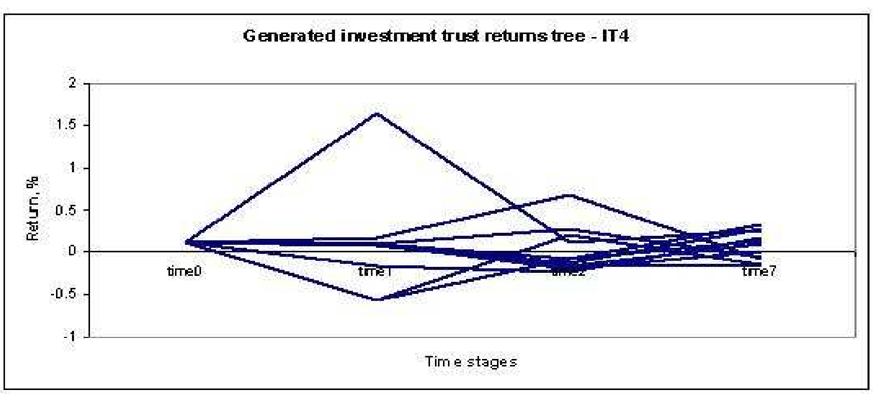
(a) The 0y-1y-2y-7y tree of IT1 returns



(b) The 0y-1y-2y-7y tree of the IT2 returns

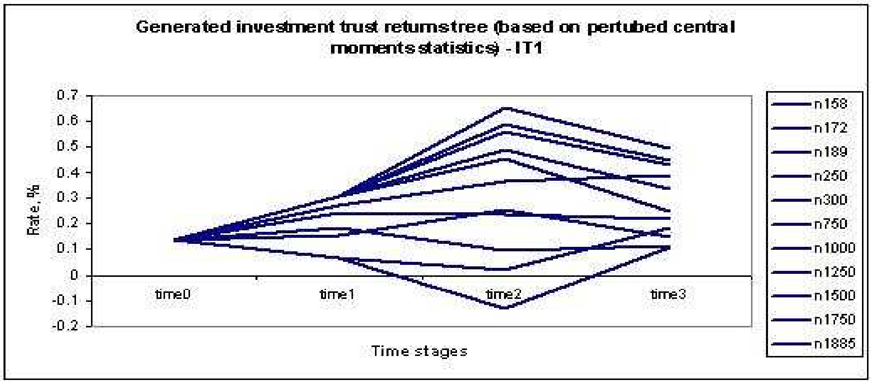


(c) The 0y-1y-2y-7y tree of the IT3 returns

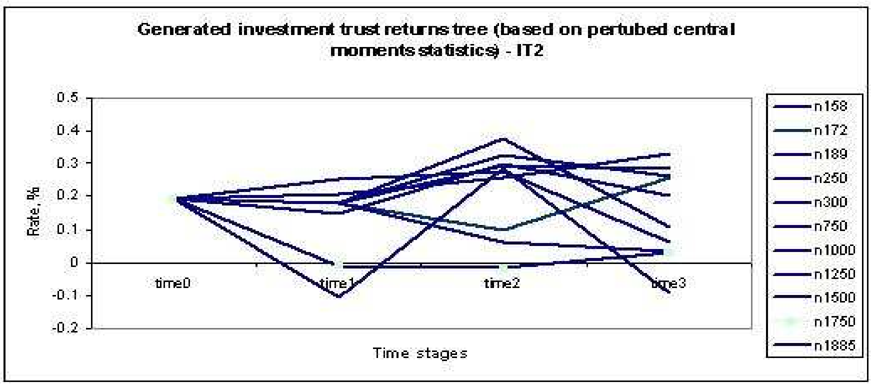


(d) The 0y-1y-2y-7y tree of the IT4 returns

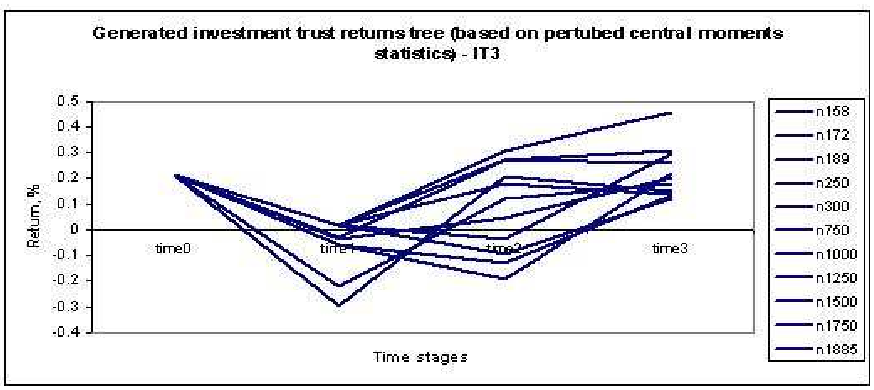
Figure 31: Stochastic event trees of investment trust returns in the 0-1-2-7 case



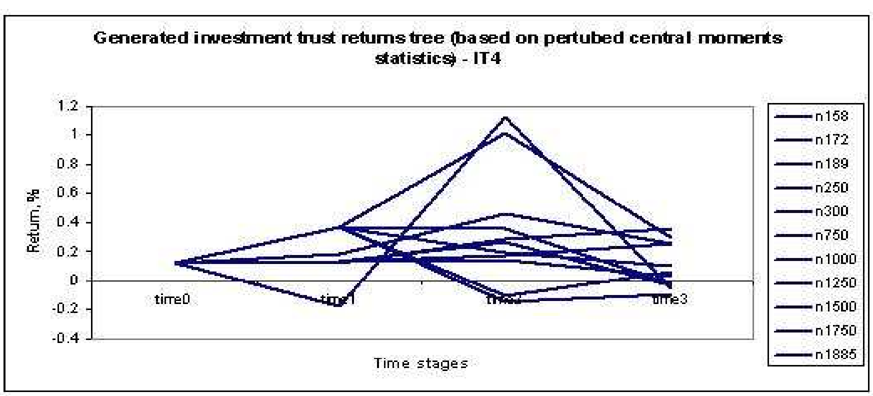
(a) The 0y-1y-2y-3y scenario tree of IT1 returns based on perturbed central moments statistics



(b) The 0y-1y-2y-3y scenario tree of IT2 returns based on perturbed central moments statistics

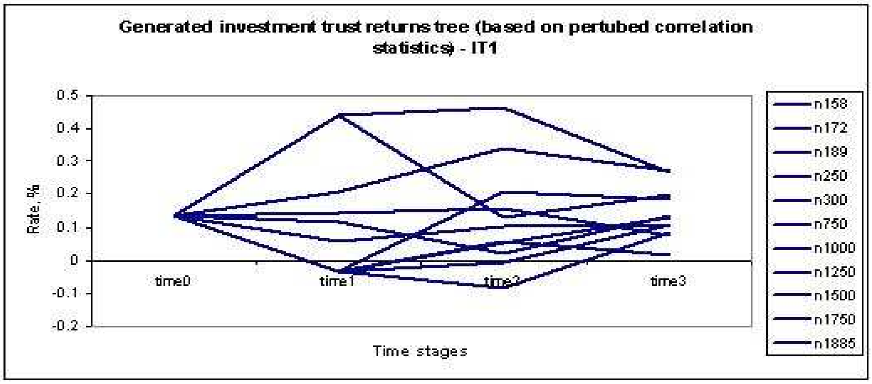


(c) The 0y-1y-2y-3y scenario tree of the IT3 returns based on perturbed central moments statistics

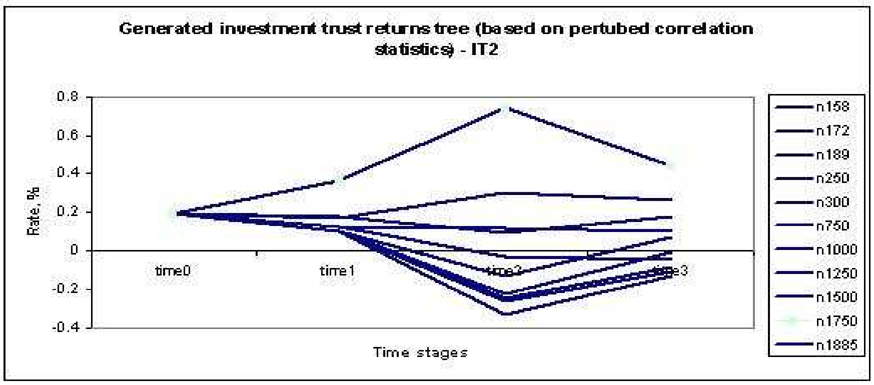


(d) The 0y-1y-2y-3y scenario tree of the IT4 returns based on perturbed central moments statistics

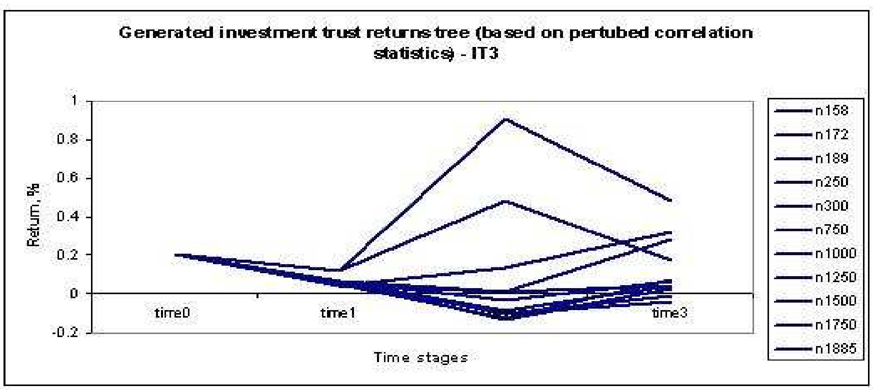
Figure 32: Stochastic event trees of investment trust returns based on perturbed central moments statistics



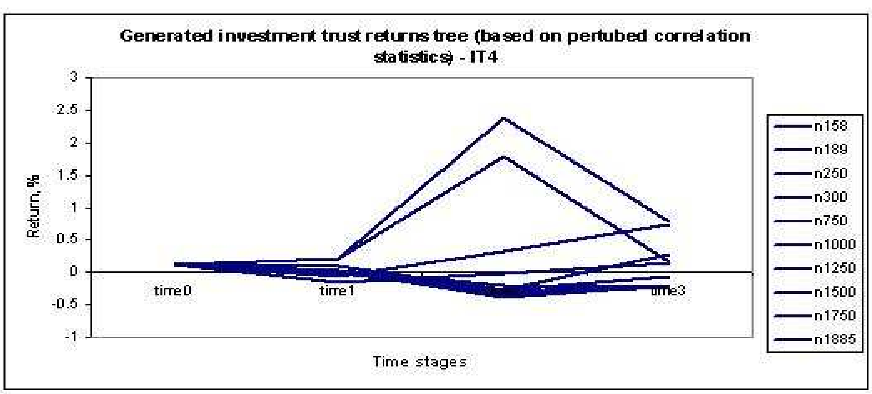
(a) The 0y-1y-2y-3y scenario tree of IT1 returns based on perturbed correlation statistics



(b) The 0y-1y-2y-3y scenario tree of IT2 returns based on perturbed correlation statistics



(c) The 0y-1y-2y-3y scenario tree of IT3 returns based on perturbed correlation statistics



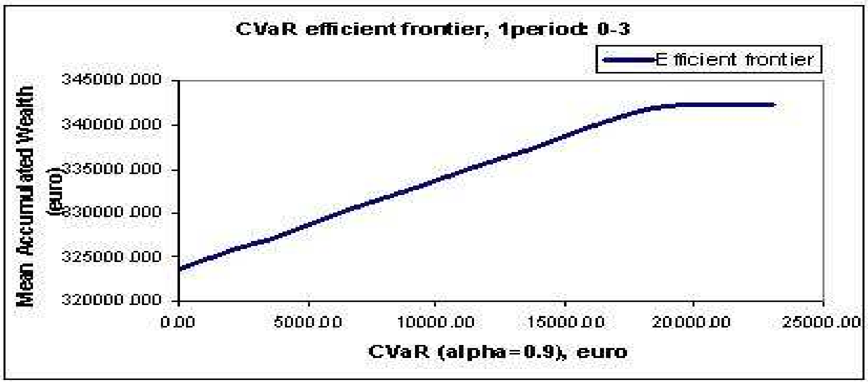
(d) The 0y-1y-2y-3y scenario tree of IT4 returns based on perturbed correlation statistics

Figure 33: Stochastic event trees of investment trust returns based on perturbed correlation statistics

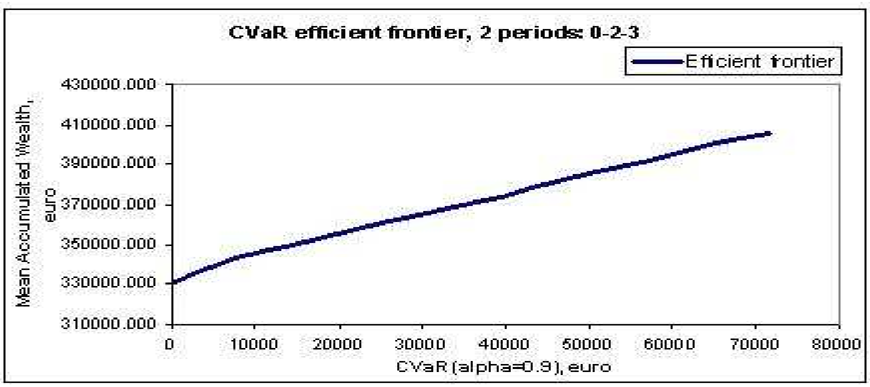
C Details on the Major Test Findings

CVaR and CDaR efficient frontiers

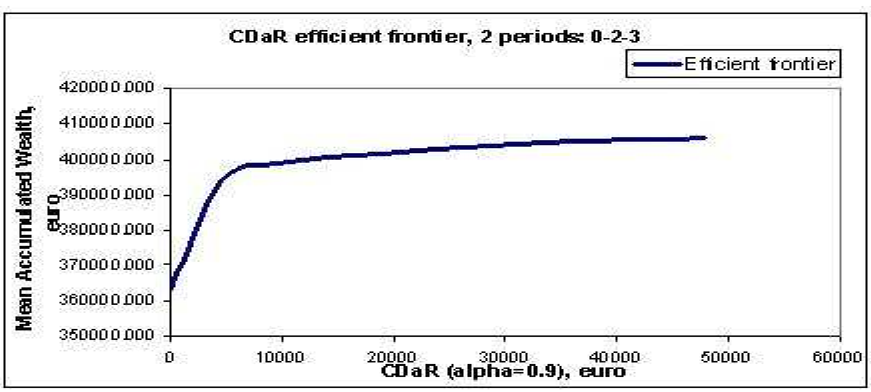
Figures 34 show efficient frontiers generated by the CVaR and CDaR models for all test cases included in the Sample Test (see *Section 7.3.1 Sample Test*).



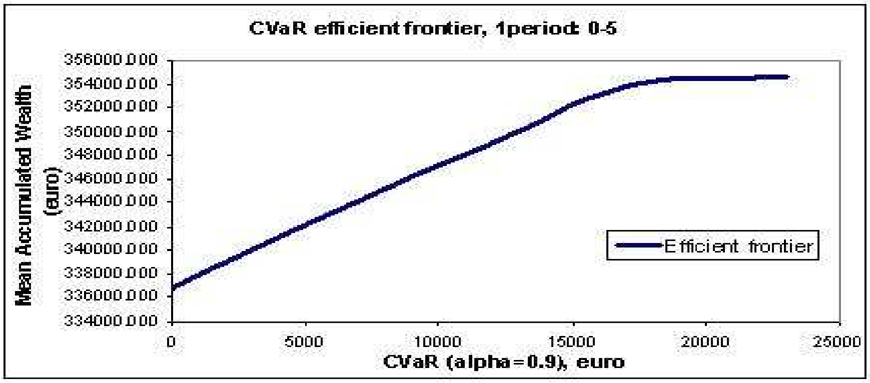
(a) CVaR efficient frontier of the 0y-3y portfolio



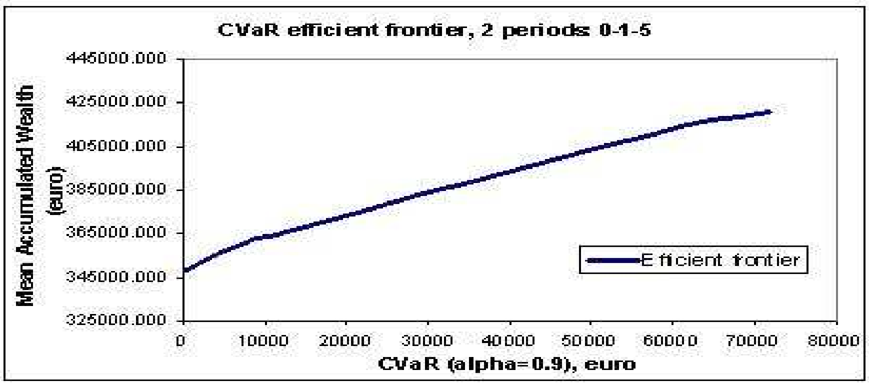
(b) CVaR efficient frontier of the 0y-2y-3y portfolio



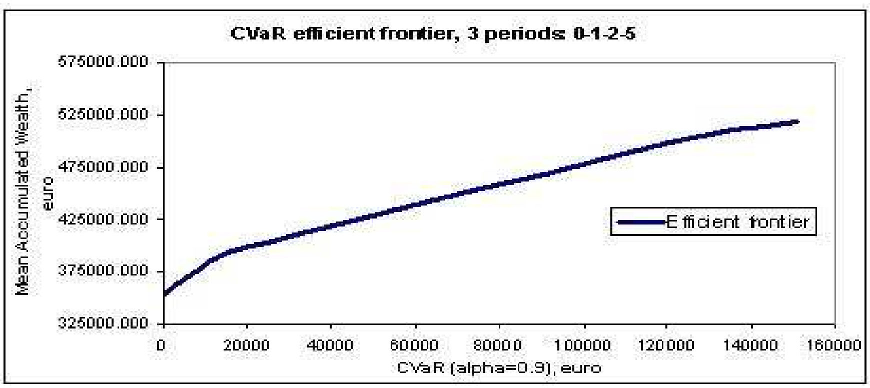
(c) CDaR efficient frontier of the 0y-2y-3y portfolio



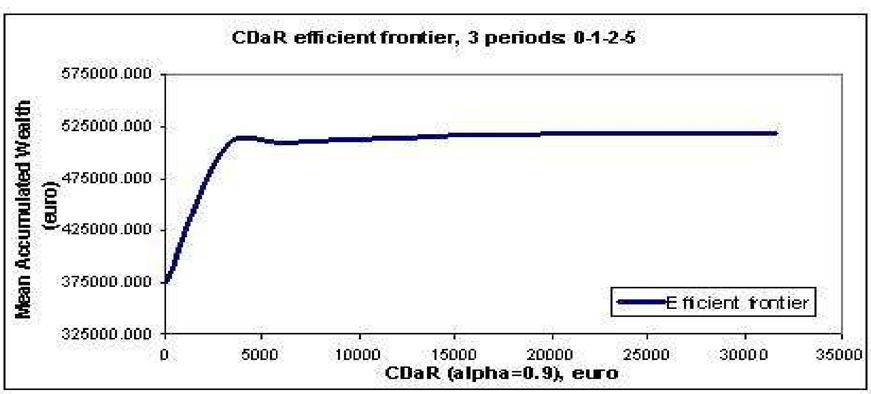
(d) CVaR efficient frontier of the 0y-5y portfolio



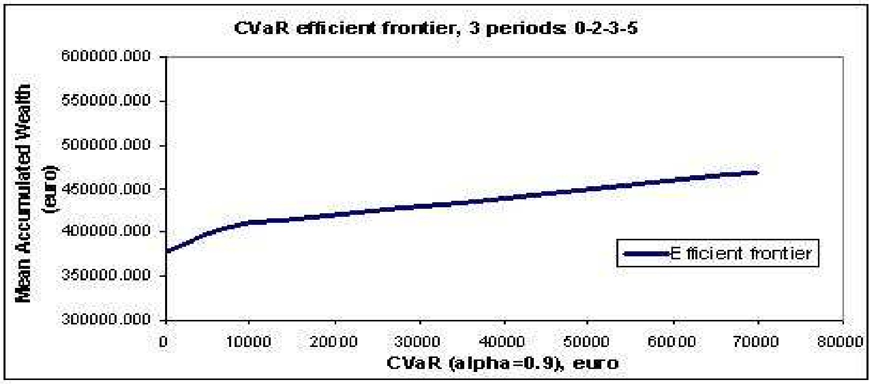
(e) CVaR efficient frontier of the 0y-1y-5y portfolio



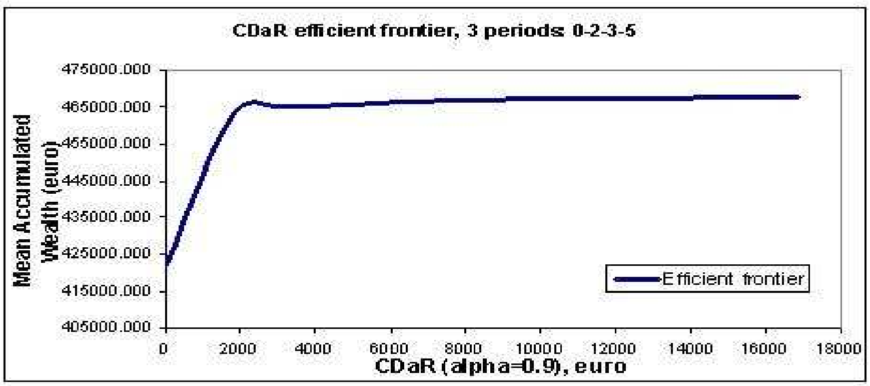
(f) CVaR efficient frontier of the 0y-1y-2y-5y portfolio



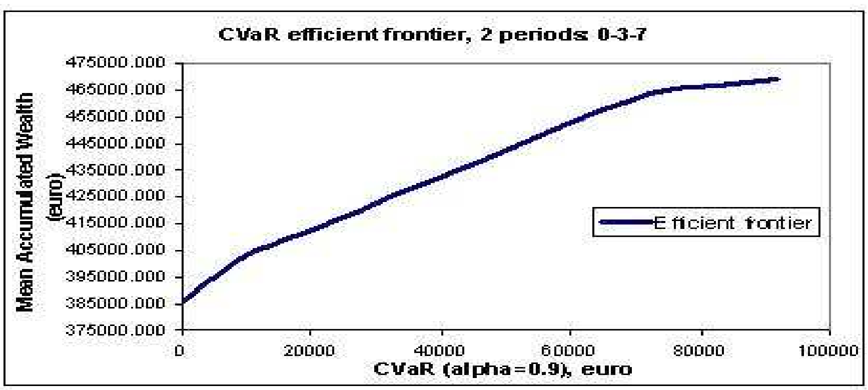
(g) CDaR efficient frontier of the 0y-1y-2y-5y portfolio



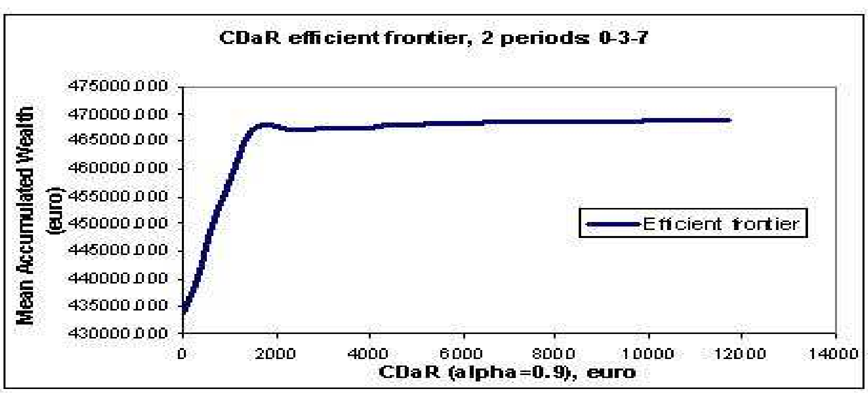
(h) CVaR efficient frontier of the 0y-2y-3y-5y portfolio



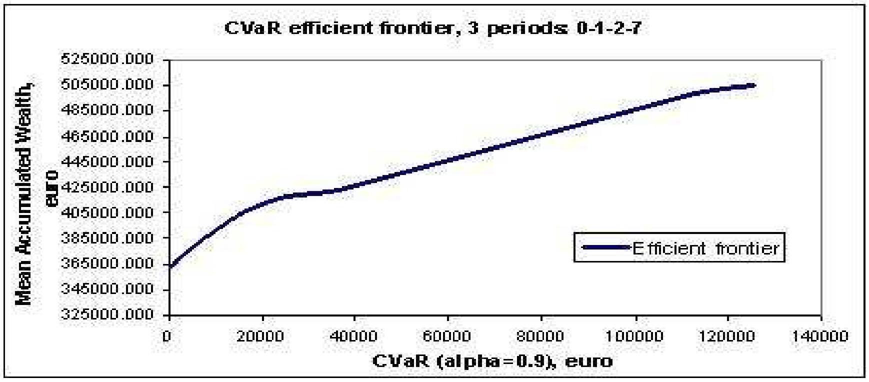
(i) CDaR efficient frontier of the 0y-2y-3y-5y portfolio



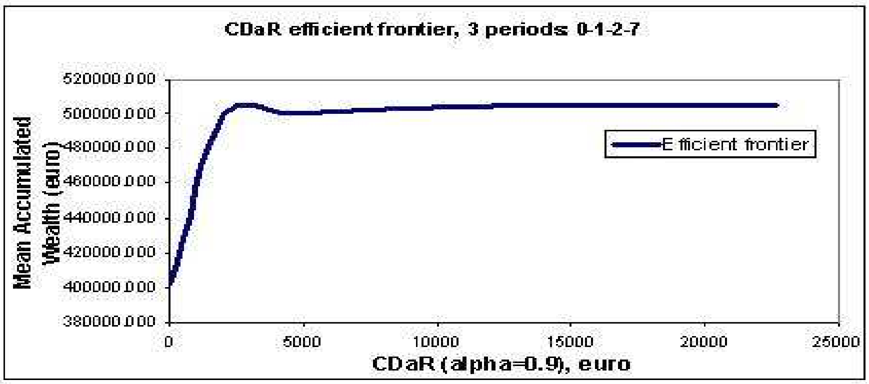
(j) CVaR efficient frontier of the 0y-1y-7y portfolio



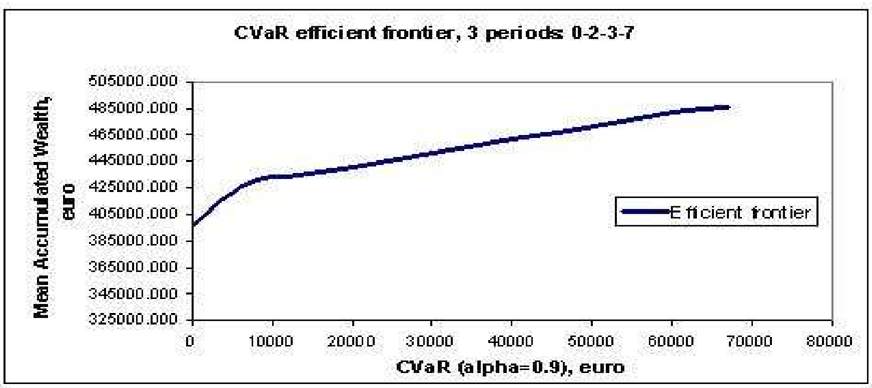
(k) CDaR efficient frontier of the 0y-3y-7y portfolio



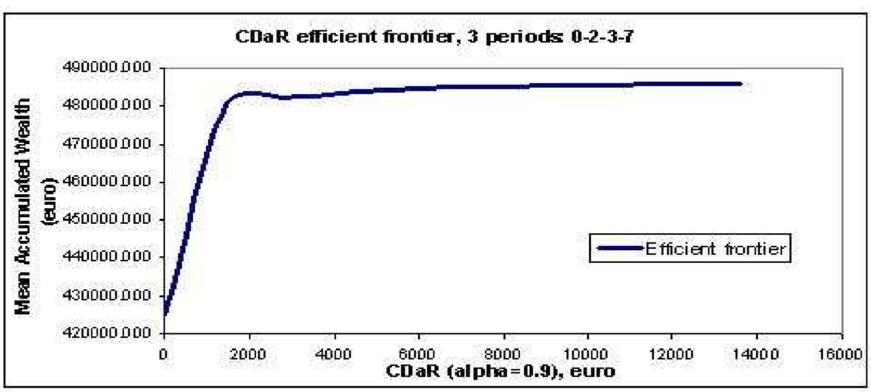
(l) CVaR efficient frontier of the 0y-1y-2y-7y portfolio



(m) CDaR efficient frontier of the 0y-1y-2y-7y portfolio



(n) CVaR efficient frontier of the 0y-2y-3y-7y portfolio

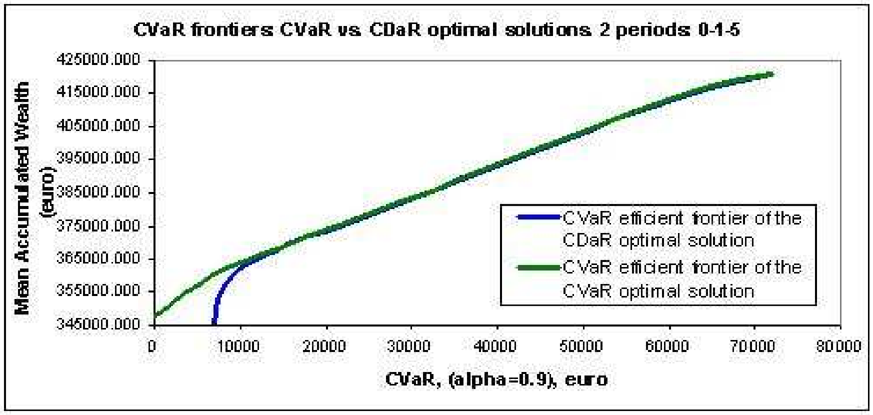


(o) CDaR efficient frontier of the 0y-2y-3y-7y portfolio

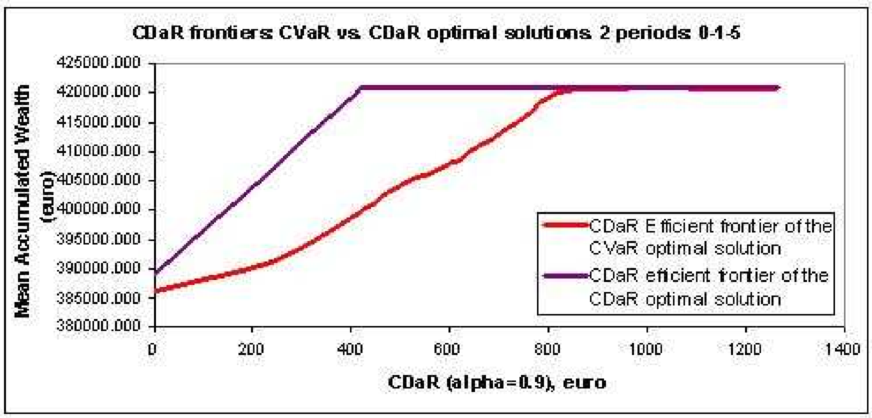
Figure 34: CVaR and CDaR Efficient Frontiers in the Sample Test

D Robustness Analysis Details

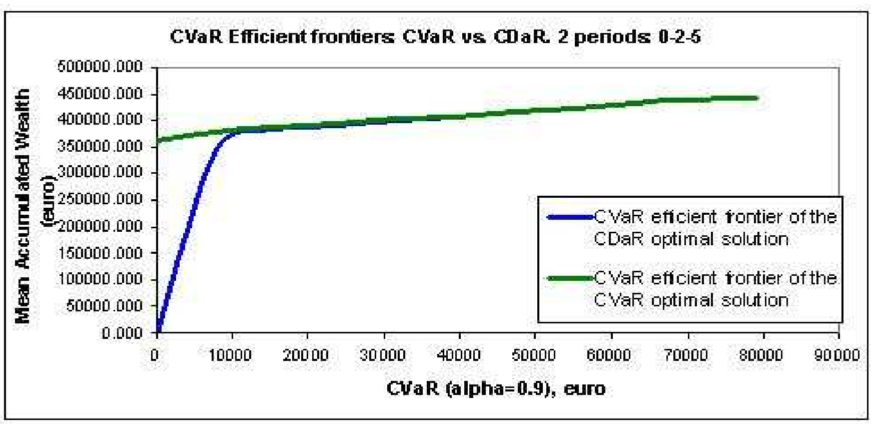
Figures 35 and 36 presents different 5-year cases of robustness analysis conducted.



(a) CVaR frontiers in the 0y-1y-5y case

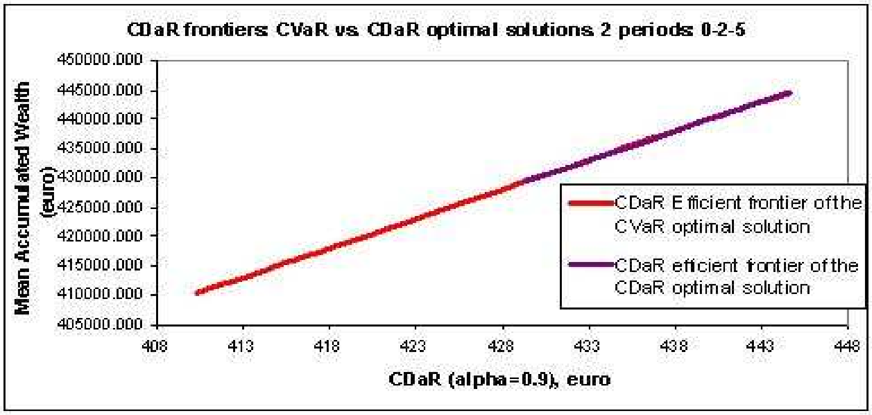


(b) CDaR frontiers in the 0y-1y-5y case

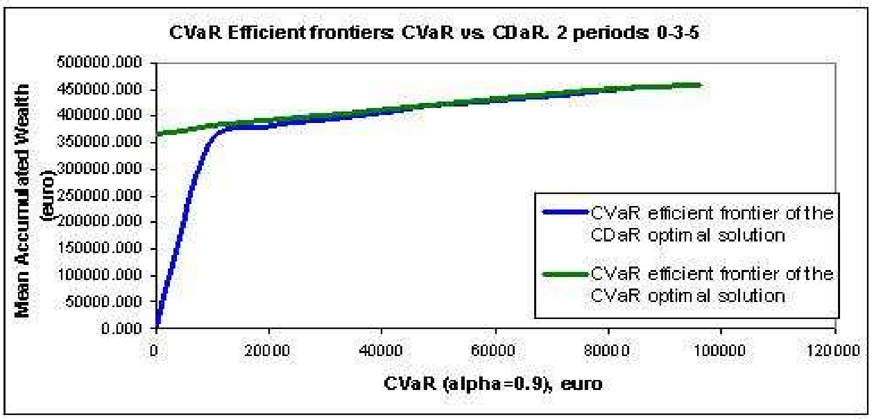


(c) CVaR frontiers in the 0y-2y-5y case

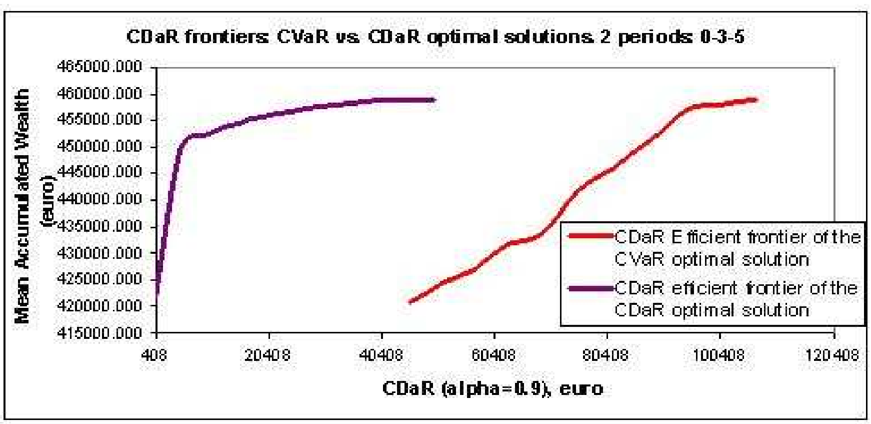
Figure 35: Robustness Analysis of the CVaR and CDaR Models. Cases 0y-1y-5y and 0y-2y-5y (CVaR).



(a) CDaR frontiers in the 0y-2y-5y case



(b) CVaR frontiers in the 0y-3y-5y case

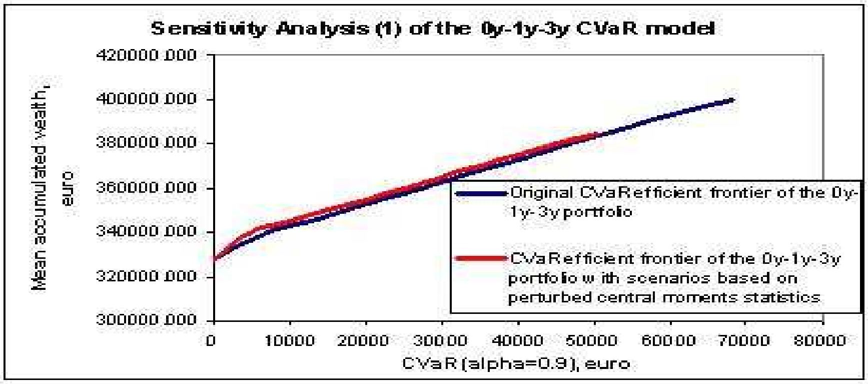


(c) CDaR frontiers in the 0y-3y-5y case

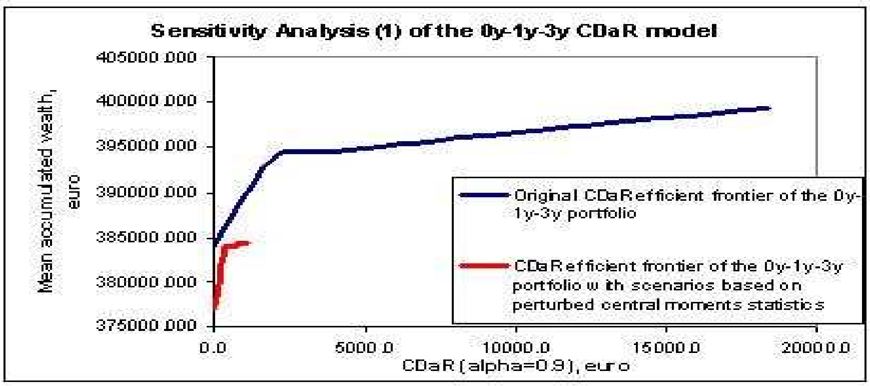
Figure 36: Robustness Analysis of the CVaR and CDaR Models. Cases 0y-2y-5y (CDaR) and 0y-3y-5y.

E Sensitivity Test Details

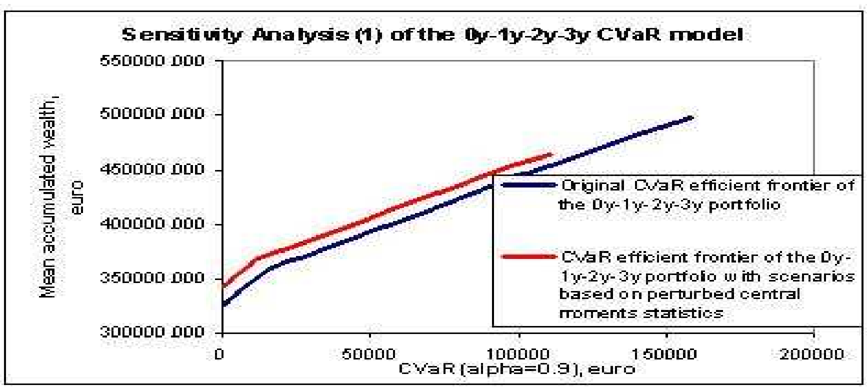
Figure 37 shows different cases of sensitivity analysis conducted for the CVaR and CDaR models with scenarios based on the original and perturbed central moments statistics. Figure 38 - scenarios based on the original and perturbed correlation statistics, correspondingly.



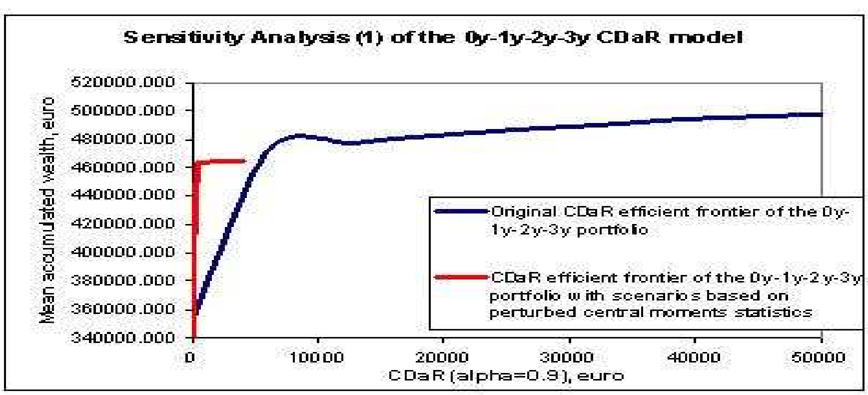
(a) Sensitivity analysis (perturbed central moments statistics) of CVaR 0y-1y-3y model



(b) Sensitivity analysis (perturbed central moments statistics) of CDaR 0y-1y-3y model

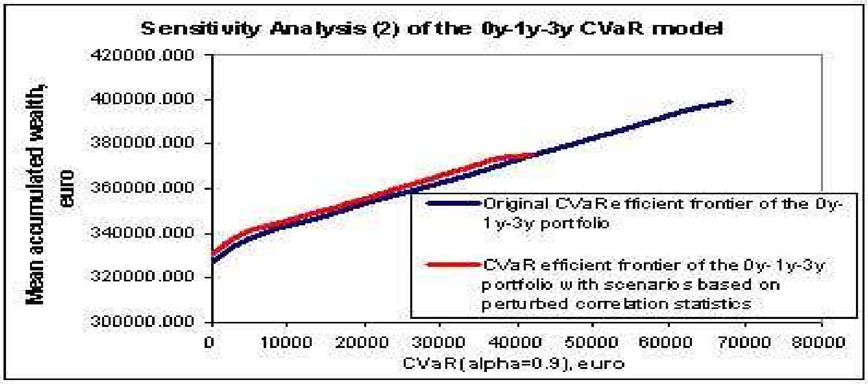


(c) Sensitivity analysis (perturbed central moments statistics) of CVaR 0y-1y-2y-3y model

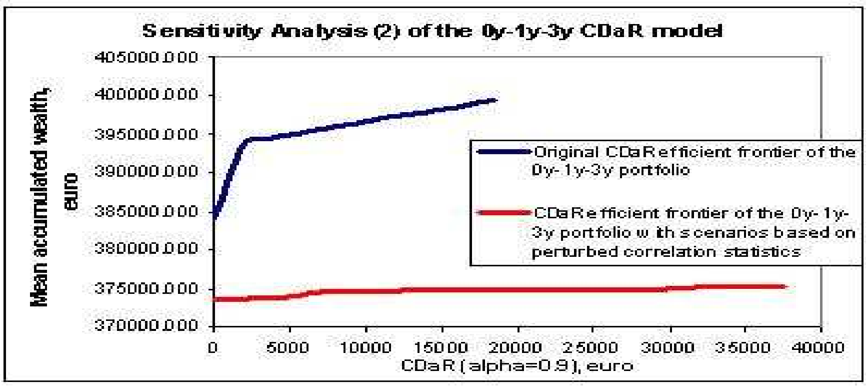


(d) Sensitivity analysis (perturbed central moments statistics) of CDaR 0y-1y-2y-3y model

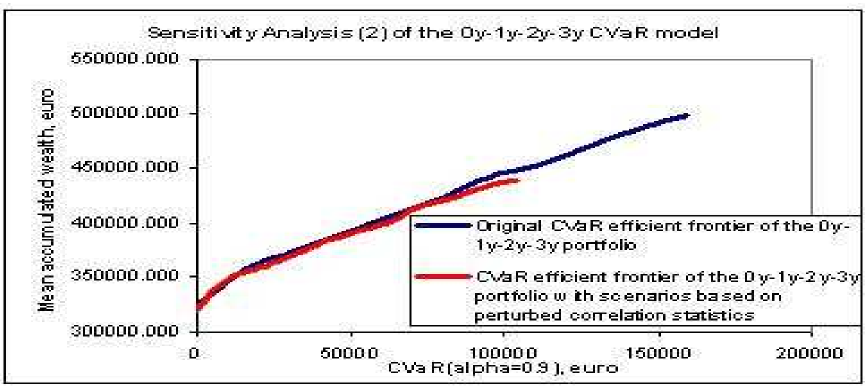
Figure 37: Sensitivity analysis (perturbed central moments statistics) of the CVaR and CDaR models. Cases 0-1-3 and 0-1-2-3.



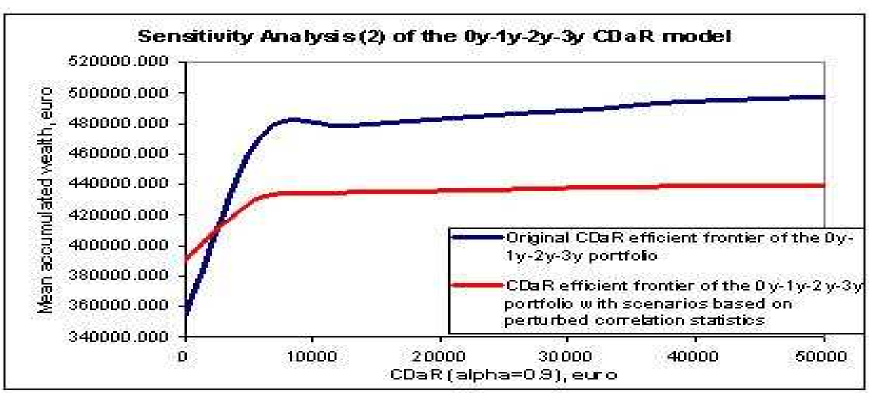
(a) Sensitivity analysis (perturbed correlation statistics) of CVaR 0y-1y-3y model



(b) Sensitivity analysis (perturbed correlation statistics) of CDaR 0y-1y-3y model



(c) Sensitivity analysis (perturbed correlation statistics) of CVaR 0y-1y-2y-3y model



(d) Sensitivity analysis (perturbed correlation statistics) of CDaR 0y-1y-2y-3y model

Figure 38: Sensitivity analysis (perturbed correlation statistics) of the CVaR and CDaR models. Cases 0-1-3 and 0-1-2-3.

F GAMS Implementation of the Integrated Pension and Mortgage Portfolio Problem

```
** Integrated pension and mortgage portfolio problem, 3 periods:
** Case 0-1-2-3. CVaR version

$eolcom //

option
iterlim=999999999,reslim=25000,optcr=1,solprint=ON,limrow=5,limcol=0,decimals=8;

set
t        'time periods'                              /time0,time1,time2,time3/
tau(t)   'time periods in the scenario tree'         /time0,time1,time2,time3/
nn        'nodes in the target tree'                 /n1*n1885/
n(nn)     'nodes in the target tree'                 /n1*n1885/
k         'asset products'                           /IT_PA,IT_ISA,IT_SP/
i         'investment trusts'                        /IT1*IT4/
m         'mortgage loan products'                   /Loan1*Loan21/
j(m)      'mortgage loan products considered'        /Loan1*Loan21/
q        'states'                                    /p1*p12/
q1(q)    'states in the 1st stage of target tree'    /p1*p12/
q3(q)    'states in the 2nd stage of target tree'    /p1*p12/
q4(q)    'states in the 3rd stage of target tree'    /p1*p12/;

parameter q2(tau,q) 'states at each step in the target tree';
q2(tau,q1)$(ord(tau) le 2) = yes;
q2(tau,q3)$(ord(tau) eq 3) = yes;
q2(tau,q4)$(ord(tau) eq 4) = yes;

// Map the time stages to the actual years (integers)
parameter ActualTimes(t) /time0 0, time1 1, time2 2, time3 3/;
```

```
parameter TauTimes(tau);

TauTimes(tau) = ActualTimes(tau);

alias (n,parent,child);
alias (n,n3,n4);

alias (tau,tau0,tau1);

parameter nq(tau) 'number of states at each time tau';

loop(tau,
  nq(tau) = sum(q,1$q2(tau,q));
);

set preptime(tau) 'prepayment time';
preptime(tau)$(card(tau) eq ord(tau)) =YES;


set
root(n)           'root node' /n1/
tn(tau,n)         'map nodes to time periods in the scenario tree'
anc(child,parent) 'ancestor mapping'
np(n,q)           'map nodes to states'
leaf(n)           'leaf nodes for the scenario tree'
path(n,n3)        'all paths of the target tree';


**** Contruction of scenario trees is provided in the folder with
**** complete GAMS implementation: "Integrated PM Portfolio Mgt"
******************************************************************
```

```
// Confidence level
SCALAR alpha;

// Risk aversion
SCALAR lambda;

// parameters used to construct the efficient frontier
scalar CVaR0, CVaR1, delta_lambda, delta_CVaR;

VARIABLES
  // Asset portfolio variables
  C(k,t,n)           'Contribution in the product k value at the time t, scenario l'
  U(t,n)             'Cash hold (portfolio level) at the time t, scenario l'
  V(k,t,n)           'Cash hold (product level) at the time t, scenario l'
  Z(k,i,t,n)         'Investment holding capital (number of shares by the midmarket price)'
  z_nos(k,i,t,n)     'Inventory hold value (number of shares)'
  X_plus(k,i,t,n)    'Purchase value'
  X_plus_nos(k,i,t,n) 'Purchase amount'
  PF_delta(k,i,t,n)  'Purchase fee positive difference from the Purchase Cap'
  PF(k,i,t,n)        'Purchase dealing fee'
  X_minus(k,i,t,n)   'Sales value'
  X_minus_nos(k,i,t,n)'Sales amount'
  SF_delta(k,i,t,n)  'Sales fee positive difference from the Sales Cap'
  SF(k,i,t,n)        'Sales dealing fee'
  W(k,t,n)           'Withdrawal value (product level)'


  // Liability portfolio variables
  RG(j,t,n)          'Outstanding debt'
  Y_plus(j,t,n)      'Sale variable'
  Y_minus(j,t,n)     'Purchase variable'
  A(j,t,n)           'Principle payment of bond'
  B(t,n)             'Annual payment'
```

```
  PP(n)               'Prepayment amount'
  SB(n)               'Total payment of the path (until the leaf)'
  SA(n)               'Total principle payment of the path'
  Profit(n)           'profit value at the portfolio time horizon'
  mrtgProfit(n)       'Mortgage profit calculation'

  // Integrated pension and mortgage portfolio variables
  AW(n)               'Accumulated wealth'
  VaRDev(n)           'Amount of accumulated wealth exceeding the VaR level'
  VaR                 'VaR at the alpha confidence level'
  CVaR                'CVaR at the alpha confidence level'
  OBJ1                'Objective function value'
  OBJ2                'Objective function value'
;

POSITIVE VARIABLES
  U, V, Z, z_plus, X_plus, x_plus_nos, PF, PF_delta, X_minus, x_minus_nos,
  SF, SF_delta, C, W, AW, VaRDev, VaR,
  RG, Y_plus, Y_minus, A, B, PP, SB, SA, Profit;

EQUATIONS

  // Pension portfolio constraints (IT PA, IT ISA, IT SP products with IT1-IT4)
  CashFlow_init(t,n)                                   'Initialize cash flow equilibrum'
  CashFlow(t,n)                                                   'Cash flow equilibrum'
  ProductCashFlow_init(k,t,n)             'Initialize product level cash flow equilibrum'
  ProductCashFlow(k,t,n)                                'Product level cash flow equilibrum'
  InvFlow_init(k,i,t,n)         'Initialize investment flow equilibrum (IT ISA and IT SP)'
  InvFlow(k,i,t,n)                    'Investment flow equilibrum (IT ISA and IT SP)'
  InvFlow_final(k,t,n)                                 'No purchase at the time horizon'
  xPlus_PA(t,n)                                   'IT PA can purchase only IT1 and IT2'
  xMinus_PA(t,n)                                      'IT PA can sell only IT1 and IT2'
  z_PA(t,n)                                           'IT PA can hold only IT1 and IT2'
```

```
Purchase_calculate(k,i,t,n)                                  'Purchase calculation '
PF_value(k,i,t,n)                                              'Purchase Fee value'
PurchCharge_calculate(k,i,t,n)                        'Purchase charge calculation '
Sell_calculate(k,i,t,n)                                          'Sales calculation '
SF_value(k,i,t,n)                                                   'Sales Fee value'
SellCharge_calculate(k,i,t,n)                             'Sales charge calculation'
PostSalesPA(k,t,n)                 'Sales in must not decrease the face value on the
                                                                   account below minimum'
LumpsumContribution_Min(k,t,n)                 'Lump sum contribution minimum amount'
AnnContribution_Max(k,tau,n)                    'Annual contribution maximum amount'
Withdrawal_max1(k,t,n)                                          'Maximum withdrawal value
                                     (should not exceed the minimum capital required)'
Withdrawal_max2(k,t,n)                                          'Maximum withdrawal value
                                   (should not exceed the policy bounds on withdrawal)'
Capital_Calculation(k,i,t,n)           'Capital asset value of the product x, year t,
                                                                  investment trust/fund i'

// Mortgage portfolio constraints (FRM and ARM loans)
EQ1                     'The initial amount needed must be financed by the mortgage loans'
EQ2(j)                    'The amount of sold bonds at node 1 = initial outstanding debt'
EQ3(j,n)                                          'Sales are not allowed at prepayment time'
EQ4(j,t,n)                                                          'Outstanding debt dynamics'
EQ5(t,n)                                                                      'Cashflow balance'
EQ6(j,t,n)                                                   'Definition of principal payments'
EQ7(t,n)                                                    'Definition of total node payments'
EQ7_0                                     'Initial payment (includes fixed cost of refinancing)'
DefPP(n)                                                     'Calculation of prepayment amount'
BerSB(n)                                    'Calculation of total path payments (short term)'
BerSA(n)                                       'Calculation of total path principal payments'
PP_HousePrice_relation(t,n)      'Relationship between prepayment amount and house price'

// Integrated pension and mortgage portfolio constraints
objective_fn1                        'Objective function (corners of the efficient frontier)'
```

```
  objective_fn2                        'Objective function (inside the efficient frontier)'

  AssetLiabilityBalance(t,n)                         'Asset Liability Balance formulation'
  ProfitDef(t,n)                              'Calculate the profite on the liability side
                                                                  (at the prepayment time)'
  AccumulativeWealth(n)                   ccumulative wealth definition (short term tree)'
  PortfolioVaRDev(n)                               'Calculate the VaRDev at the short term'
  PortfolioCVaR                'Calculate CVaR at the alpha confidence level (short term)';


********* Asset portfolio modeling *********************************************************

* Cash flow equilibrium
CashFlow(tn(tau,n))..           sum(k,C(k,tau,n) + APC(k)*(1+(TauTimes(tau)-TauTimes(tau-1)
    -1)$(ord(tau) gt 1))) =l= ACI(tau)*(1+(TauTimes(tau)-TauTimes(tau-1)-1)$(ord(tau) gt 1));

* Bounds on the contribution
LumpsumContribution_Min(k,tn(tau,n))..     C(k,tau,n) - LumpsumContrMin(k)*(1+(TauTimes(tau)
                                                 -TauTimes(tau-1)-1)$(ord(tau) gt 1)) =g= 0;

AnnContribution_Max(k,tn(tau,n))..           C(k,tau,n) =l= AnnContrMax(k)*(1+(TauTimes(tau)
                                                       -TauTimes(tau-1)-1)$(ord(tau) gt 1));

* Product cash flow equilibrium
ProductCashFlow_init(k,tn(tau,n))$(ord(tau) eq 1)..   C(k,tau,n) =e= sum(i,X_plus(k,i,tau,n)
                                                                    +TER(i)*Z(k,i,tau,n));
ProductCashFlow(k,tn(tau,n))$(ord(tau) gt 1)..       sum(anc(n,n3), C(k,tau,n)
                                       + sum(i,X_minus(k,i,tau,n)) - sum(i,X_plus(k,i,tau,n)
                   +TER(i)*Z(k,i,tau,n)*(TauTimes(tau)-TauTimes(tau-1))) - W(k,tau,n)) =e= 0;

// ITPA can invest only into the IT1 and IT2
InvFlow_init(k,i,tn(tau,n))$(ord(tau) eq 1)..     x_plus_nos(k,i,tau,n) =e= z_nos(k,i,tau,n);
```

```
* Inventory flow equilibrium
InvFlow(k,i,tn(tau,n))$(ord(tau) gt 1)..          sum(anc(n,n3), z_nos(k,i,tau-1,n3)*(1
                                               + itRates_mm(i,n3)) + x_plus_nos(k,i,tau,n)
                                              - z_nos(k,i,tau,n) - x_minus_nos(k,i,tau,n))
                                                                                   =e= 0;

* IT PA invests only in IT1 and IT2
xPlus_PA(tn(tau,n))..                          sum((k,i)$(ord(k) eq 1 and ord(i) gt 2),
                                                          x_plus_nos(k,i,tau,n)) =e= 0;
xMinus_PA(tn(tau,n))..                         sum((k,i)$(ord(k) eq 1 and ord(i) gt 2),
                                                         x_minus_nos(k,i,tau,n)) =e= 0;
z_PA(tn(tau,n))..                              sum((k,i)$(ord(k) eq 1 and ord(i) gt 2),
                                                               z_nos(k,i,tau,n)) =e= 0;

* No purchase at the portfolio horizon
InvFlow_final(k,tn(tau,n))$(ord(tau) eq card(tau))..      sum(i,X_plus(k,i,tau,n)) =e= 0;

* Purchase face value calculation (incl. fees and government stamp)
Purchase_calculate(k,i,tn(tau,n))..            X_plus(k,i,tau,n) =e= x_plus_nos(k,i,tau,n)
                                    *itPrice_offer(i,n)*(1 + GovStamp(k)) + PF(k,i,tau,n);

PF_value(k,i,tn(tau,n))..                        PF_delta(k,i,tau,n) =e= PFCap(k) - PFR(k)
                                             *x_plus_nos(k,i,tau,n)*itPrice_offer(i,n);

PurchCharge_calculate(k,i,tn(tau,n))..      PF(k,i,tau,n) =e= PFCap(k) - PF_delta(k,i,tau,n);

* Sales face value calculation (incl. fees and government stamp)
Sell_calculate(k,i,tn(tau,n))$(ord(tau) gt 1..  X_minus(k,i,tau,n) =e= x_minus_nos(k,i,tau,n)
                                                     *itPrice_bid(i,n) + SF(k,i,tau,n);
SF_value(k,i,tn(tau,n))$(ord(tau) gt 1)..        SF_delta(k,i,tau,n) =e= SFCap(k) - SFR(k)
                                              *x_minus_nos(k,i,tau,n)*itPrice_bid(i,n);
SellCharge_calculate(k,i,tn(tau,n))$(ord(tau) gt 1)..            SF(k,i,tau,n) =e= SFCap(k)
                                                                 - SF_delta(k,i,tau,n);
```

```
* Sales from the IT PA must not decrease it below the MinRemaining level
PostSalesPA(k,tn(tau,n))..        sum(i,X_minus(k,i,tau,n)-Z(k,i,tau,n)) =l= MinRemaining(k);

* Withdrawal restriction
Withdrawal_max1(k,tn(tau,n))..           W(k,tau,n) =l= sum(i,Z(k,i,tau,n))-MinRemaining(k);
Withdrawal_max2(k,tn(tau,n))..                                W(k,tau,n) =l= WithdrawalMax(k);

* Calculation of the investment trust capital
Capital_Calculation(k,i,tn(tau,n))..       Z(k,i,tau,n) =e= z_nos(k,i,tau,n)*itPrice_mm(i,n);



********* Liability portfolio modeling *******************************************************



// The initial amount must be financed by the mortgage loans
EQ1    ..                            SUM(j, loanPrice('n1',j)*Y_plus(j,'time0','n1')) =G= IA;

// The amount of sold bonds at node 1 = initial outstanding debt
EQ2(j) ..                                  RG(j,'time0','n1') - Y_plus(j,'time0','n1') =E= 0;

// Sales are not allowed at prepayment time
EQ3(j,leaf) ..                                 sum(preptime, Y_plus(j,preptime,leaf) ) =E= 0;

// Outstanding debt dynamics
EQ4(j,tn(tau,n)) ..                               sum(anc(n,n3), RG(j,tau-1,n3) - A(j,tau,n)
                                    - Y_minus(j,tau,n) + Y_plus(j,tau,n) - RG(j,tau,n)) =E= 0;

// Cashflow balance
EQ5(tn(tau,n))$(ord(tau)>1)..                          sum(j,loanPrice(n,j)*Y_plus(j,tau,n)) -
                                             sum(j,loanCallPrice(n,j)*Y_minus(j,tau,n)) =E= 0;

// Definition of principal payments
```

```
EQ6(j,tn(tau,n))  ..              sum(anc(n,n3), A(j,tau,n) - RG(j,tau-1,n3)*(((R(n3,j))/(1
                        - (1+(R(n3,j)))**(-NumPer+(TauTimes(tau)-1))) - (R(n3,j))))) =E= 0;


// Definition of total node payments
EQ7(tn(tau,n))$(ord(tau) gt 1) ..                 B(tau,n) =E= sum((j,anc(n,n3)), A(j,tau,n)
                         + (1-gamma)*RG(j,tau-1,n3)*(R(n3,j))*(TauTimes(tau)-TauTimes(tau-1))
                        + (1-betta)*RG(j,tau-1,n3)*(admcost)*(TauTimes(tau)-TauTimes(tau-1)))
                                         + sum(j,transFee*(Y_plus(j,tau,n)+Y_minus(j,tau,n)));


// Initial payment (includes the fixed cost on refinancing)
EQ7_0..                  B('time0','n1') =E= sum(j,transFee*Y_plus(j,'time0','n1'))+FixedCost;


// Calculation of prepayment amount (long term)
DefPP(leaf)..    PP(leaf) =E= sum((j,preptime), RG(j, preptime, leaf) *loanCallPrice(leaf,j));


// Calculation of total path payments (short term)
BerSB(leaf)..                   SB(leaf) =E= sum((tau,n)$(tn(tau,n)*path(n,leaf)), B(tau,n));


// Calculation of total path principal payments
BerSA(leaf)..          SA(leaf) =E= sum((tau,n)$(tn(tau,n)*path(n,leaf)), sum(j, A(j,tau,n)));


// Profit on the mortgage side (long term horizon)
ProfitDef(preptime,leaf)..             mrtgProfit(leaf) - HP(preptime,leaf) + PP(leaf) =l= 0;


// Final mortgage prepayment must not exceed the house price
PP_HousePrice_relation(preptime,leaf)..                   HP(preptime,leaf) - PP(leaf) =g= 0;



*******************Integrated pension and mortgage portfolio modeling*************************


// Objective function
objective_fn1..              OBJ1 =e= lambda*CVaR - (1-lambda)*sum(leaf,prob(leaf)*AW(leaf));
```

```
// Calculate the accumulated wealth
AccumulativeWealth(leaf)..                AW(leaf) =e= sum((tau,k,i)$(ord(tau) eq card(tau)),
                          Z(k,i,tau,leaf))*Spot_PoundToEuro + mrtgProfit(leaf)*Spot_DkkToEuro;

// Asset-liability matching in EURO
AssetLiabilityBalance(tn(tau,n))$(ord(tau)>1)..                      B(tau,n)*Spot_DkkToEuro
                                            - sum(k,W(k,tau,n))*Spot_PoundToEuro =e= 0;

// CVaR of the portfolio
PortfolioCVaR..                    CVaR =e= VaR + sum(leaf,prob(leaf)*VaRDev(leaf))/(1-alpha);

// Define the deviation from VAR in terms of loss on the leaf
PortfolioVaRDev(leaf)..  VaRDev(leaf) =g= (sum(leaf1,prob(leaf1)*AW(leaf1)) - AW(leaf)) - VaR;


MODEL CVaR_1 /objective_fn1,
              AccumulativeWealth,
              CashFlow,
              LumpsumContribution_Min,
              AnnContribution_Max,
              ProductCashFlow_init,
              ProductCashFlow,
              InvFlow_init, InvFlow,
              xPlus_PA, xMinus_PA, z_PA,
              Purchase_calculate,
              PF_value, PurchCharge_calculate,
              Sell_calculate,
              SF_value, SellCharge_calculate,
              PostSalesPA
              Withdrawal_max1,
              Withdrawal_max2,
              Capital_Calculation,
```

```
               EQ1, EQ2, EQ3, EQ4, EQ5, EQ6, EQ7, EQ7_0,
               DefPP, BerSB, BerSA,
               AssetLiabilityBalance, ProfitDef,
               PP_HousePrice_relation,
               PortfolioCVaR, PortfolioVaRDev

/;

OPTION lp = CPLEX;

solve CVaR_1 minimizing OBJ1 using LP;



**** Contruction of efficient frontier is provided in the folder
**** with complete GAMS implementation: "Integrated PM Portfolio Mgt"
******************************************************************
```