PERFORMANCE EVALUATION OF PARAMETRIC BIAS FIELD CORRECTION

Vitali Zagorodnov

Nanyang Technological University

Mads Fogtmann Hansen

Danish Technological University

ABSTRACT

Magnetic Resonance (MR) images often exhibit grayscale nonuniformities, caused by radio frequency (RF) coil design or acquisition sequences. Many algorithms to remove these nonuniformities have been proposed in the past decade. However, only minor attention has been given to the performance evaluation of existing methods. We derive a link between the estimation performance and underlying image structure. For a piecewise constant 1D signal model with equal size regions we demonstrate that the variance in estimation grows as M^2 , where M is the number of regions. In 2D case the growth becomes linear in M.

1. INTRODUCTION

Magnetic Resonance Imaging (MRI), a popular non-invasive 2D - 3D medical visualization technique, often suffers from the presence of grayscale nonuniformities. This artifact reveals itself as a smoothly varying multiplicative field, also called the bias field. Removal of bias field is essential for many automated segmentation and tissue classification algorithms.

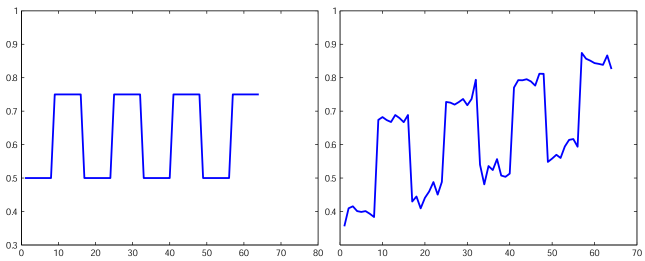
Knowledge of image segmentation greatly simplifies the problem of bias field estimation. To obtain preliminary segmentation early publications suggested using phantom objects [1] or user-defined set of control points [2]. Many automatic segmentation-based methods for removal of bias fields have also been developed [3], [4], [5], [6]. For example, Meyer et al [3] obtains a preliminary segmentation using an inhomogeneity-tolerant segmentation algorithm. Least square estimates of the parameters of a polynomial bias field are then obtained by combining local estimates from the segmented regions. Wells et al [4] uses EM algorithm to iterate between segmentation and estimation of the bias field. There are also solutions that do not use image segmentation. These are based on energy minimization formulation [5], sharpening of the intensity histogram [6], and homomorphic filtering [7].

Even though great effort has gone into developing new approaches, only minor attention has been given to the performance evaluation of existing methods. A recently conducted empirical evaluation of six algorithms [8] concludes that "none of the six chosen algorithms perform ideally under all circumstances". There is also no established link between the image structure and the estimation performance. For example, inhomogeneity-tolerant segmentation algorithm in [3] stops partitioning the image after region sizes become smaller than the number of unknown parameters. However, the effect of such a fine partitioning on the estimation performance is unclear.

The main contribution of this paper is a simple formula that links the variance in estimation with the number of image partitions M. We demonstrate that for a linear bias field and image with a uniform partition, the variance of the bias field slope estimate grows as M^2 for 1D signals and as M for images. For non-linear bias field these values are further multiplied by a constant that depends on the chosen set of basis functions.

2. BIAS FIELD ESTIMATION

Medical images typically consist of a set of uniform in intensity regions corresponding to particular organs and tissues. Hence we model the given data image as a piecewise constant model image that is corrupted by noise and bias field, see Figure 2 for an example of such image. The theory developed can also be applied to 1D signals corrupted by bias field (Figure 1).



(a) (b)

Fig. 1. (a) 1D Model piecewise constant signal. (b) 1D model signal corrupted by additive bias and noise

Taking the logarithm of image intensities converts a multiplicative bias field into an additive one. Following the no-

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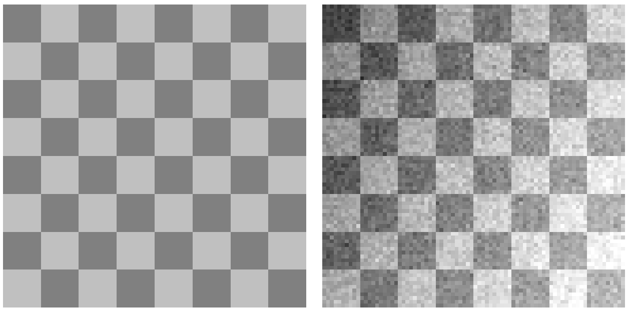

(a) (b)

Fig. 2. (a) Model piecewise constant image. (b) Model image corrupted by additive bias and noise

tation introduced in [3], the log intensity of the data image

$$o(s) = c(s) + b(s) + e(s) \quad (1)$$

Here s is a pixel, $o(s)$ and $c(s)$ are the intensities of the data image and piecewise constant model image respectively, $b(s)$ is the bias field and $e(s)$ is the noise. Smoothness of the bias field is captured by modeling it as a linear combination of P basis functions, $b(s) = \sum_{k=1}^{P} a_k \varphi_k(s)$. We assume that the model image contains M regions, where m-th region contains N_m pixels with intensity c_m. Let S be a set of all pixels. The underlying structures define a partition $\pi = (\pi_1, \pi_2, \cdots, \pi_M)$ on S.

Rewriting equation (1) in a matrix form yields

$$o = Am + e \quad (2)$$

where $o \in \mathbb{R}^N$ is the observation vector of logarithmic pixel intensities, $m = (a_1, \cdots, a_P | c_1, \cdots, c_M)^\top$ is the vector of unknown parameters and $A \in \mathbb{R}^{N \times (P+M)}$ has the following form:

$$A = \begin{bmatrix} \varphi_1(v_1) & \dots & \varphi_P(v_1) & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ \varphi_1(v_{N_1}) & \dots & \varphi_P(v_{N_1}) & 1 & 0 & \dots & 0 \\ \varphi_1(v_{N_1+1}) & \dots & \varphi_P(v_{N_1+1}) & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ \varphi_1(v_{N_1+N_2}) & \dots & \varphi_P(v_{N_1+N_2}) & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ \varphi_1(v_N) & \dots & \varphi_P(v_N) & 0 & 0 & \dots & 1 \end{bmatrix} \quad (3)$$

where v_i is the Cartesian coordinates of pixel s_i. A least squares error (LSE) estimate of m is

$$\hat{m} = (A^T A)^{-1} A^T o \quad (4)$$

3. PROBLEM STATEMENT

We assume that the noise is i.i.d. Gaussian r.v. with zero mean and variance $\sigma^2 = 1$. Then $\hat{m}$ in (4) is a maximum likelihood (ML) estimate with covariance matrix $\Sigma = (A^T A)^{-1}$. The first P diagonal entries of Σ correspond to the variances of $\hat{a}_1, \hat{a}_2, \cdots \hat{a}_P$. Let $\sigma_k^2 = \mathrm{var}(\hat{a}_k)$. Let $\bar{o}(s) = c(s) + n(s)$ be the true bias free image and $\hat{o}(s) = c(s) + b(s) - \hat{b}(s) + n(s)$ its ML estimate.

Choosing an orthonormal set of basis functions yields a simple relationship between the averaged over all pixels estimation error variance and σ_k^2's:

$$\begin{aligned} \nu^2 &= \frac{1}{N} \sum_s E(\hat{o}(s) - \bar{o}(s))^2 \\ &= \frac{1}{N} \sum_s E(\hat{b}(s) - b(s))^2 \\ &= \frac{1}{N} \sum_{k=1}^{P} \sigma_k^2 \end{aligned}$$

Here N is the total number of pixels. Due to popularity of polynomial modeling of the bias field in the literature, we will limit our analysis to normalized Legendre polynomials. In 2D we will use separable basis function set $\phi_{k,l}(x,y) = \phi_k(x)\phi_l(y)$, where $\phi_k(x)$ and $\phi_l(y)$ are 1D Legendre polynomials.

The goal is to find the relationship between ν^2 and the number and relative sizes of the partition elements. To simplify the initial consideration we will assume that in $2D$ case the partition elements are rectangular in shape.

4. PERFORMANCE EVALUATION

For simplicity of notation we will write φ_k instead of $\varphi_k(v_i)$, so that $\sum_{\pi_1} \varphi_1 = \sum_{i=1}^{N_1} \varphi_1(v_i)$. Then

$$A^T A = \begin{bmatrix} 1 & \cdots & 0 & \sum_{\pi_1} \varphi_1 & \cdots & \sum_{\pi_M} \varphi_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \sum_{\pi_1} \varphi_P & \cdots & \sum_{\pi_M} \varphi_P \\ \sum_{\pi_1} \varphi_1 & \cdots & \sum_{\pi_1} \varphi_P & N_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_{\pi_M} \varphi_1 & \cdots & \sum_{\pi_M} \varphi_P & 0 & \cdots & N_M \end{bmatrix}$$

which can be written in a block matrix form as:

$$A^T A = \begin{bmatrix} I & B \\ B^T & N \end{bmatrix} \quad (5)$$

Let $\Sigma = (A^T A)^{-1} = \begin{bmatrix} K & L \\ L^T & M \end{bmatrix}$ where matrices K,L and M have the same sizes as I, B and N respectively. According to the well known result from the matrix theory [9],

$$K = (I - BN^{-1}B^T)^{-1} \quad (6)$$

Expanding (6) yields

$$K = \begin{bmatrix} 1 - \sum_m \frac{(\sum_{\pi_m} \varphi_1)^2}{N_m} & \cdots & -\sum_m \frac{\sum_{\pi_m} \varphi_1 \sum_{\pi_m} \varphi_P}{N_m} \\ \vdots & \ddots & \vdots \\ -\sum_m \frac{\sum_{\pi_m} \varphi_1 \sum_{\pi_m} \varphi_P}{N_m} & \cdots & 1 - \sum_m \frac{(\sum_{\pi_m} \varphi_P)^2}{N_m} \end{bmatrix}^{-1}$$

Matrix K^{-1} is square; the number of rows and columns is equal to the number of the unknown parameters. This makes the size of K^{-1} much smaller than the size of $A^T A$, making it easier to compute the inverse. Before considering a general case in Section 7 we solve the problem for a simpler linear bias case.

5. LINEAR BIAS FIELD IN 1D

For linear bias field in 1D case $b(x) = a_1 \varphi_1(x)$. Then K^{-1} is a scalar and the variance of $\hat{a}_1$ is

$$\sigma_1^2 = \left(1 - \sum_{m=1}^{M} \frac{(\sum_{\pi_m} \varphi_1)^2}{N_m}\right)^{-1}$$

For an unpartitioned image, $\sigma_1^2 = 1$, which means that the variance of $\hat{a}_1$ is equal to the variance of the noise. Partitioning the image leads to larger variance. When the number of regions is M, we have

$$\begin{aligned} \sigma_1^2 &= \left(1 - \sum_{m=1}^{M} \frac{(\sum_{\pi_m} \varphi_1)^2}{N_m}\right)^{-1} \\ &= \left(\sum_{m=1}^{M} (\sum_{\pi_m} \varphi_1^2 - \frac{(\sum_{\pi_m} \varphi_1)^2}{N_m})\right)^{-1} \end{aligned}$$

Note that $\sum_{\pi_m} \varphi_1^2 - \frac{(\sum_{\pi_m} \varphi_1)^2}{N_m}$ is the variance of φ_1 over pixels in π_m. Since φ_1 is linear, for sufficiently large N_m this variance is proportional to N_m^3. Hence

$$\sum_{\pi_m} \varphi_1^2 - \frac{(\sum_{\pi_m} \varphi_1)^2}{N_1} \approx \frac{N_m^3}{N^3} \sum_{i=1}^{N} \varphi_1^2 = \frac{N_m^3}{N^3}$$

and

$$\sigma_1^2 = \frac{N^3}{N_1^3 + N_2^3 + \cdots + N_M^3} \geq 1 \qquad (7)$$

In particular, when all partitions have the same size $N_m = \frac{N}{M}$, the estimation variance grows as the second degree of the number of partition $\sigma_1^2 \approx M^2$, yielding $\nu^2 = \frac{M^2}{N}$

6. LINEAR BIAS FIELD IN 2D

In 2D case the bias field is a weighted sum of two linear functions in x and y directions:

$$b(x, y) = a_1 \varphi_1(x) \varphi_0(y) + a_2 \varphi_0(x) \varphi_1(y)$$

Here φ_0 and φ_1 are zero and first degree Legendre polynomials. Let N_x and M_x be the image dimension and number of partition elements in the horizontal direction, N_y and M_y in the vertical direction. Clearly, $M = M_x M_y$ and $N = N_x N_y$.

Because of the assumed rectangular shape of partition elements the 2D problem splits into two 1D problems, where N and M are substituted for N_x, M_x or N_y, M_y:

$$K = \begin{bmatrix} \frac{N_x^3}{N_{1,x}^3 + \cdots + N_{M_x,x}^3} & 0 \\ 0 & \frac{N_y^3}{N_{1,y}^3 + \cdots + N_{M_y,y}^3} \end{bmatrix}$$

In particular, when all partition elements have the same size and $M_x = M_y = \sqrt{M}$, the diagonal entries of K are $K_{11} = K_{22} = M$ and

$$\nu^2 = \frac{2M}{N} \qquad (8)$$

Hence, in the 2D case the parameter variances grow linearly with the number of partitions. This is a remarkable result, indicating a much slower decline of the estimation performance in 2D, compared with the 1D case. This result agrees well with the experiment, as shown in Figure 3.

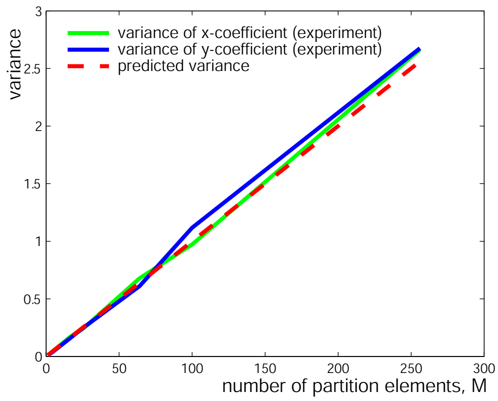

Fig. 3. Variances of parameter estimates: predicted vs. experimental. Noise variance is $\sigma^2 = 0.01$.

7. NON-LINEAR BIAS FIELD IN 2D

In the non-linear 2D case we model the bias field as

$$b(x, y) = \sum_{k=0}^{P_x - 1} \sum_{l=0}^{P_y - 1} a_{k,l} \varphi_k(x) \varphi_l(y)$$

The following conjecture yields an upperbound on the variances of the unknown parameters in the asymptotic case as $M \to N$. Here $\overline{f(x)}$ denotes the average value of $f(x)$ on $[0, 1]$.

Conjecture 1 *Let* $\sigma^2_{k,l} = \mathrm{var}(\hat{a}_{k,l})$*. Then if all partition elements have the same size and* $M_x = M_y = \sqrt{M}$*, the variance of* $\hat{a}_{k,l}$ *satisfies*

$$\sigma^2_{k,l} < C\Phi_{k,l}M \tag{9}$$

where $\Phi_{k,l} = \frac{1}{\left[\frac{\varphi_k'(x)\varphi_l(y)}{\varphi_1'(x)\varphi_0(y)}\right]^2 + \left[\frac{\varphi_k(x)\varphi_l'(y)}{\varphi_0(x)\varphi_1'(y)}\right]^2}$

Proof outline: The non-diagonal entries of K^{-1} are not zero, but small. Hence we will assume that their influence on the diagonal entries of K can be limited by a constant C. The diagonal entry of K^{-1} corresponding to $a_{k,l}$ is the sum of variances of $\varphi_k(x)\varphi_l(y)$ on all partition elements. If the partition elements are sufficiently small, we can use a piecewise linear approximation of $\varphi_k(x)\varphi_l(y)$. Then on π_m we have

$$\varphi_k(x)\varphi_l(y) \approx r_{x,m}\varphi_1(x)\varphi_0(y) + r_{y,m}\varphi_0(x)\varphi_1(y) + D$$

where $r_{x,m} = \frac{\varphi_k'(x^*_m)\varphi_l(y^*_m)}{\varphi_1'(x)\varphi_0(y)}$, $r_{y,m} = \frac{\varphi_k(x^*_m)\varphi_l'(y^*_m)}{\varphi_0(x)\varphi_1'(y)}$ and (x^*_m, y^*_m) are coordinates of some pixel in π_m. Using the linear case result and the fact that M is large to convert the sum into integral yields (9). ■

This result agrees well with the experiment, as shown in Figure 4. Here we plotted variances $\hat{a}_{1,0}$, $\hat{a}_{1,1}$ and $\hat{a}_{3,0}$ as functions of M. Straightforward calculations yield $\Phi_{1,0} = 1$, $\Phi_{1,1} = \frac{1}{2}$, and $\Phi_{3,0} = \frac{1}{14}$. Hence, assuming that constant C is the same for all estimates, we should have $\sigma^2_{1,0} : \sigma^2_{1,1} = \sigma^2_{3,0} = 1 : 2 : 14$. The actual ratio is $1 : 2.67 : 14.1$, which is close to the predicted ratio. The value of constant C is approximately equal to 1.35.

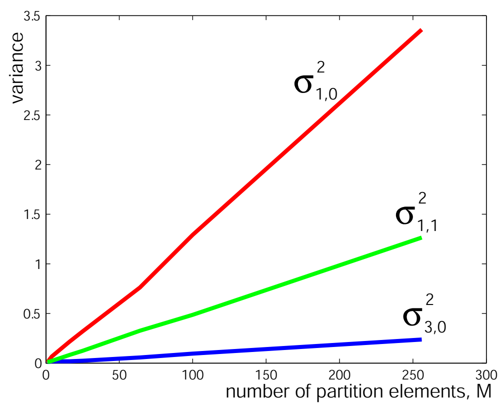

Fig. 4. Variances of estimates vs. number of partition elements. Maximum polynomial degree is $P_x = P_y = 3$. Noise variance is $\sigma^2 = 0.01$

8. CONCLUSION AND FUTURE WORK

We demonstrated a link between the bias field estimation performance and the image structure. When the bias field is modelled as a linear combination of orthonormal basis functions, the variance of estimated coefficients grows as M^2 in 1D case and linearly with M in 2D case, where M is the number of partition elements. This relationship is precise for linear bias field. However, there is still work to be done on proving the derived upper bound for the non-linear bias field. Nevertheless, its validity is confirmed empirically.

9. REFERENCES

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