Construction of Timetables Based on Periodic Event Scheduling

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Abstract

This thesis deals with scheduling of periodic timetables for railways. The model is based on the *Periodic Event Scheduling Problem* (PESP) formulated in terms of some cycle basis of the constraint graph.

Important properties of the PESP relevant for train scheduling such as sequencing and matching are examined. Constraints appearing in timetable planning are identified and formulated using the PESP, and integration with other planning phases are achieved, partly by extending the PESP model. In particular, aspects of lineplanning is introduced into the model by matching of predefined linesegments.

Instances are created and solved for the Copenhagen commuter train service S-train. The main objective is to minimise the number of required train units used to operate the service, and thereby the estimated operating costs. Simultaneously, minimising passenger waiting times for certain connections is also attemted for one instance. In general, results show that solutions can easily be obtained for basic instances, where lineplanning is not considered. When lineplanning is allowed, the complexity and solution time increases significantly.

For periodic train networks the method seems to be a relevant alternative to existing manual or ad-hoc timetabling mehtods, since it is based on a widely recognised mathematical model and is able to integrate most aspects of timetabling and some aspects of lineplanning, leading to better solutions in some cases.

Resumé

Dette projekt omhandler konstruktion af periodiske jernbanekøreplaner. Den benyttede model er baseret på *Periodic Event Scheduling Problem* (PESP), formuleret vha. en kredsbasis i grafen svarende til problemets betingelser.

Vigtige egenskaber af PESP i forhold til konstruktion af køreplaner, såsom sekvensering og matching, undersøges. De mest almindelige begrænsninger, der forekommer i planlægning af køreplaner, bliver identificeret og formuleret vha. PESP. Ydermere undersøges integration med andre planlægningsfaser, hvilket tildels kræver en udvidelse af PESP modellen. Specielt undersøges en partiel integration med lineplanlægning ved at betragte kombinationer af foruddefinerede liniesegmenter.

Køreplaner konstrueres for Københavns S-tog. Hovedformålet er at minimere antallet af anvendte togstammer og dermed de forventede operationelle omkostninger. I et enkelt tilfælde vises, at passagerventetiden på givne forbindelser kan minimeres simultant. Generelt viser resultaterne, at løsninger forholdsvis let kan opnås, når linieplanlægning ikke betragtes. Når linieplanlægning integreres stiger kompleksiteten og løsningstiden kraftigt.

Metoden fremstår som et relevant alternativ til eksisterende manuelle eller adhoc metoder, da den bygger på en anerkendt matematisk model og da det er muligt at integrere de fleste aspekter af køreplanlægning samt enkelte aspekter fra linieplanlægning, der kan føre til bedre løsninger.

Preface

This thesis was prepared at Informatics and Mathematical Modelling, Technical University of Denmark in accordance with the requirements for obtaining the M.Sc. degree in engineering.

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Jonas Christoffer Villumsen

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Contents

Chapter 1

Introduction

1.1 Problem Formulation

The purpose of this thesis is to investigate how the Periodic Event Scheduling Problem may be used efficiently to construct *good* railway timetables for periodic train services. A *good* timetable is one that has low operating costs as well as a high level of passenger service. In particular, several timetables will be constructed for the Copenhagen S-train. These timetables should satisfy basic railway requirements wrt. infrastructure, rolling stock, and safety. In addition, special requirements may be explored wrt. for example passenger service. Is it possible to construct such timetables for S-train in reasonable time? And, if so, is it possible to integrate aspects of line planning into the construction of timetables, while still obtaining results in reasonable time?

1.2 Timetabling for Railway Systems

A railway timetable defines for each train the time of departure from and arrival at each station in the network. When creating such a timetable, some constraints must be respected. The time between a departure at one station and the arrival at the next station must be sufficient for the train to travel the distance between the two stations. The time a train stops at a station must be sufficient to let passengers alight and board. There should be some slack in the timetable, allowing trains to catch up with potential delays. At certain stations, there must be sufficient time for train operations, such as turning the train or coupling or decoupling of train cars. Conflicts between trains, e.g. where trains need to use the same track at the same time, must be avoided. It must be possible to assign drivers to the trains in the timetable without violating union rules and allowing for drivers to operate the trains safely. A good timetable has low cost of operation, low travel and waiting time of passengers, and a high degree of robustness, i.e. it is insensitive to delays in the system.

In periodic railway systems, a line defines a set of trains serving the same stations in the same sequence with a fixed time interval. Hence, for each line only one train needs to be scheduled. If all lines have the same frequency corresponding to the time interval T, a timetable for a timeperiod of length T is sufficient to define the entire timetable.

1.3 Scheduling Periodic Events

The Periodic Event Scheduling Problem (PESP) faces the issue of scheduling a number of events in a periodic context, such that for certain ordered pairs of events the difference in time must be in a prespecified interval (the span interval). That the events are periodic means that all events are assumed to recur with a global fixed interval (the timeperiod). Given a timeperiod of length T each event may be scheduled by assigning to it a point in time in the interval $[0, T[$.

The PESP may be formulated as a mixed integer program with one continuos variable for each event (the time of occurrence) and one integer variable for each span interval. Alternatively, a formulation, known as the cycle periodicity formulation, with fewer integer variables can be used.

1.4 Concepts in Railway Systems

Train units are the physical trains used to operate a timetable. A train unit or rolling stock unit may consist of several rail cars and possibly a locomotive if the rail cars are not motorised. At least one driver is needed to operate a train unit. Each train unit in operation make up the main part of the operational cost of the railway network. These costs related to the operation of rolling stock units include driver salary (per hour), electricity or fuel consumption (per km), maintenance of rolling stock, and maintenance of track.

A *train line* defines the sequence of stations, that are served by a train in some direction with fixed interval, such that the starting and ending station is the same. For a *symmetric line*, each station is served twice, once in each direction, such that the sequence of stations visited is reversed for the opposite direction, e.g. some line visiting the stations $s_1, s_2, ..., s_k$, could visit the stations in the sequence $s_1, s_2, ..., s_{k-1}, s_k, s_k, s_{k-1}, ..., s_2, s_1$. A *lineplan* is a set of lines, while a *symmetric lineplan* is a lineplan in which all lines are symmetric. A symmetric lineplan is easily communicated to passengers by a graphical representation, for example as shown in figure 4.3.

If the arrival time in some direction and the departure time in the opposite direction for some line at some station sum to 0 modulo the timeperiod holds for all lines and stations, then the timetable is said to be symmetric. In a *symmetric timetable*, the running and dwell times of a line on the same track segment / at the same station is the same for the two opposite directions. Also, transfer time between two lines at some station is the same, whether transferring from one line to the other or vice versa. Hence, passenger travel time between two stations is also independent on the direction of travel.

Headway constraints ensures a minimum safety distance between trains. In operation, the track network is subdivided into blocks, a single track of length from 200 m. to 2 km. Restricting the number of trains in each block to maximum one, and requiring a minimum time between two consecutive trains in the same block, ensures no collissions. However, in strategic timetabling, a simple minimum time difference between arrivals of two consecutive trains at the beginning of a common track segment is often used. One must then be able to show, that no overtakings (collissions) will take place on the common track segment. If running and dwelling times are fixed for the respective lines and track segments, it is clear that no overtakings will take place.

The length and condition of the track as well as rolling stock properties determines the *running time* of trains between stations. That is, the time it takes for a train unit to travel from one station to the subsequent station. Some extra time, slack, may be added to the running time to prevent and to be able to recover from disruptions. The *dwell time* is the time a train unit stops at a station platform. The dwell time needs to be large enough for passengers to alight and board, but should not delay subsequent trains more than necessary.

At terminal stations, trains must be able to turn, i.e. a northbound train change direction to become a southbound train and vice versa. This usually involves, that the driver must go to the other end of the train before continuing in the opposite direction. Such *turnarounds* may take place when the train is parked either at a shunting track or at a platform track adjacent to a passenger platform. I will refer to the latter as platform turnaround. Platform turnarounds may be done faster, since no driving to and from the shunting yard is needed, but take up scarce ressources at the passenger station.

At certain stations, where many passengers are expected to transfer from one line to another, good connections may be ensured by restricting the *transfer time* between arrival of one line and departure of the other line. The transfer time consists of a minimum transfer time and *passenger waiting time*. The minimum transfer time must reflect f.ex. the walking distance between the two platforms plus some buffer time, whereas the maximum transfer time is the maximum waiting time plus minimum transfer time.

Passenger travel time is the time a passenger uses in the system from he departs at the origin station until arrival at the destination station. It includes time spent in trains (train running and dwell time) as well as potential transfer time between lines. Waiting time at origin and destination is not included.

Merging of lines at a terminal station (the merging station) allows for rolling stock units to be used on more than one line. This is done by allowing vehicles on some line entering its terminal station to leave the station as a different line. Merging of lines requires that at least two lines have a common terminal station. Merging of lines at a non-terminal station can cause the lineplan to be altered. This can also be understood as matching of some linesegments incident to the merging station. Allowing *matching of linesegments* is a powerful way to expand the solution space.

1.5 Litterature Review

In the article *A Mathematical Model for Periodic Scheduling Problems* [23] from 1989 by Serafini and Ukovich the Periodic Event Scheduling Problem (PESP) is defined and a solution method is proposed, which iteratively fixes the integer variables and checks for feasibility. Backtracking is used to recover feasibility, when infeasibility is detected. Serafini and Ukovich also propose an extended model with multiple time periods for ressource scheduling.

Odijk uses in *Railway Timetable Generation* [20] the PESP to randomly generate a set of railway timetables. These timetables are then used for evaluating some proposed infrastructure changes in the railway network, in particular changes to the capacity of stations. He also shows that the PESP is NP-complete by reduction from the node colouring problem.

The Ph.D. thesis *Train Schedule Optimization in Public Rail Transport* [18] by Lindner conciders the PESP and a minimum cost scheduling model. The polyhedral structure of the PESP is investigated and a new class of cutting planes are developed. Several algorithms are presented and evaluated on instances from Germany (Deutsche Bahn) and the Netherlands (Nederlandse Spoorwegen).

In *On Cyclic Timetabling and Cycles in Graphs* [15], Liebchen and Peeters discusses cycle bases of graphs in the cycle periodicity formulation of the PESP in relation to periodic railway timetabling, and what characterises a good cycle basis. They propose to look at integral cycle bases and not only strictly fundamental cycle bases when formulating the PESP.

In *Finding Short Integral Cycle Bases for Cyclic Timetabling* [10], Liebchen evaluates the effect of using generalised fundamental cycle bases instead of strictly fundamental cycle bases. In particular, an algorithm by Berger to construct generalised fundamental cycle bases is evaluated on two timetabling instances from Deutsche Bahn AG and the Berlin Underground. Significantly faster solution times are reported when using Bergers algorithm.

The article *The Modelling Power of the Periodic Event Scheduling Problem: Railway Timetables - and Beyond* [13] by Liebchen and Möhring explores the use of PESP constraints in modelling features in periodic railway timetabling. In addition to simple running time, dwell time, and train separation constraints, more sophisticated features are modelled. These include bundling of lines, train sharing, variable trip times as well as modelling aspects of vehicle scheduling, line planning, and infrastructure planning.

In *A Case Study in Periodic Timetabling* [12], timetables are constructed using the PESP framework for the Berlin Underground and evaluated wrt. passenger waiting time and number of required vehicles. The model was modified iteratively to accomodate expectations of practitioners. The final timetable reduced passenger waiting time, while using the same number of train units compared to the timetable in operation.

Peeters Ph.D. thesis *Cyclic Railway Timetable Optimization* [22] is concerned with the construction of periodic timetables using the cyclic periodicity formulation of the PESP. A number of timetable instances are created for intercity and interregional trains in the Netherlands. For the larger timetabling instance consisting of both intercity and interregional trains, no solution could be found within reasonably computation time. Therefore, a heuristic solution procedure was developed, in which first a solution for the intercity network, only, was found, whereafter the obtained sequence of intercity trains was fixed when solving for the entire network. He also shows how merging of two lines at a terminal station can be allowed within the framework of the PESP model, and how the objective function minimising the number of required train units can be obtained when merging is allowed.

In *Symmetry of Periodic Railway Timetables* [11], Liebchen discusses symmetry for periodic railway timetables. Constraints ensuring symmetric timetables does not fall within the PESP framework and are therefore added explicitly to the MIP formulation. Liebchen reports faster solution times for timetabling instances with symmetry constraints.

Liebchen and Peeters considers in *Some Practical Aspects of Periodic Timetabling* [16] various practical aspects regarding scheduling of periodic public transit systems. First they show, that if vehicles are not allowed to change lines, and running and dwell times are fixed, the number of vehicles required differs by at most the number of lines. Secondly, they consider a heuristic for allowing vehicles to change line at a terminal. They report that for the tram network of Halle, timetables that minimise passenger waiting time uses more rolling stock than timetables that minimise the number of vehicles needed. Also, minimising rolling stock results in timetables that have considerably larger passenger waiting times. Finally, they discuss sequencing of rail lines and introduce cuts limiting the solution space of the PESP based on the fact that the lines must be sequenced. They also show that the linear ordering problem is polynomially reducible to the PESP.

In *Infrastructure Update According to Schedule?* [14], Liebchen, Möhring, and Wagner explores integrated fixed-interval timetables (IFIT), which are timetables in which for some (hub) station all lines depart at the same time. IFIT's are attractive from a passenger view, but very unflexible in a planning context. The authors question previous practices of upgrading the existing infrastructure to accomodate a desired IFIT, and propose to use mathematical optimisation, specifically PESP, to construct attractive timetables fitting the existing infrastructure.

Liebchen, Proksch, and Wagner investigates in *Performance of Algorithms for Periodic Timetable Optimization* [17], various solution methods for the PESP. These include solving the MIP formulation using CPLEX, local search procedures, and constraint programming. The solution methods are evaluated on three PESP instances, optimising periodic timetables. The three instances are for the Berlin U-bahn, a small intercity rail network, and a larger intercity network, all in Germany. In particular, for experiments with CPLEX they report that for the larger instances, changing the CPLEX parameters from their default values may improve solution time or best integer solution considerably.

1.6 Report Overview

In chapter 2, the Periodic Event Scheduling Problem is presented and important aspects of the model are investigated. In particular, the cycle periodicity formulation of the PESP is derived. Also, one class of cutting planes are defined and investigated. Finally, the possibility of *matching* periodic events is introduced.

Chapter 3 deals with the problem of constructing feasible and optimal timetables for periodic railway services. The basic constraints of timetabling are introduced as well as the possibility of integrating aspects of lineplanning and vehicle scheduling.

A case study of constructing timetables for a real size railway network is presented in chapter 4. First, the current structure of the S-train network and lineplan is introduced. Next, timetabling instances are created for two scenarios, one with 20 minute lineplans and eleven lines and an alternative scenario with 10 minute lineplans and six lines. Solutions for the timetabling instances are obtained minimising the number of train units used. For the latter scenario, the number of train units used can be further reduced when integrating aspects of lineplanning.

Finally, the thesis is summarised and suggestions for further research are proposed in chapter 5.

CHAPTER 2

The Periodic Event Scheduling Problem

2.1 Definition and Notation

The Periodic Event Scheduling Problem (PESP) is the problem of scheduling a number of recurring events, such that each pair of events fulfills certain constraints on the time between them. More specifically, given n events to be scheduled and a timeperiod of length T, a particular point in time u (the potential) within the timeperiod must be determined for each of the events, satisfying a number of constraints. Each constraint a defines a lower bound d_a^- and an upper bound d_a^+ on the time difference modulo T between two events.

Let each event be represented by a node and each constraint by an edge in a directed multigraph $G = (N, A)$, and let u_ε denote the potential of node $\varepsilon \in N$. Also, let ε_a^- and ε_a^+ denote the initial and final node of edge a, respectively. Furthermore, let $v_a = u_{\varepsilon_a^+} - u_{\varepsilon_a^-}$ define the tension of an edge $a \in A$. For each edge $a \in A$ a span $[d_a^-, d_a^+]$ is defined by $d_a^-, d_a^+ \in \mathbb{R}$ with $d_a^- \leq d_a^+$.

The PESP may be defined as the problem of finding a set of feasible node potentials u for G satisfying the set of constraints

$$u_{\varepsilon_a^+} - u_{\varepsilon_a^-} - z_a T \in [d_a^-, d_a^+], \forall a \in A \tag{2.1}$$

for some integers z_a. In the following, I will sometimes refer to a PESP instance by referring to its constraint graph $G = (N, A, d^-, d^+)$.

Below is a small example of a PESP instance with three events and four constraints and timeperiod $T = 10$. Figure 2.1 shows the constraint graph for the PESP instance. Note, that the last constraint is equivalent to $2 \leq u_0 - u_2 - 10z_3' \leq 5$.

Find u, z such that

$$\begin{array}{rcl} 1 \leq & u_1 - u_0 - 10z_0 & \leq 3 \\ -1 \leq & u_1 - u_2 - 10z_1 & \leq 1 \\ 0 \leq & u_2 - u_0 - 10z_2 & \leq 4 \\ -8 \leq & u_0 - u_2 - 10z_3 & \leq -5 \end{array}$$

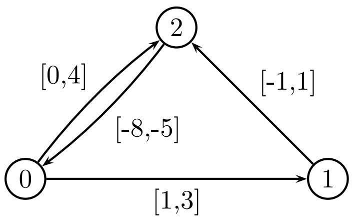


Figure 2.1: Constraint graph for small PESP instance with three events and four constraints.

In the following, it is assumed that $0 \leq d_a^+ < T$, and that the width of the span $d_a^+ - d_a^- < T$ for all a in A. These assumptions do not affect the feasibility of the instance, since the span $[d_a^-, d_a^+]$ can always be shifted such that $d_a^+ \in [0, T[$ by changing the value of z_a and a span with width greater than or equal to the period length T does not impose any restrictions to the instance and is therefore redundant.

Let $\pi_\varepsilon = u_\varepsilon \bmod T$ define the periodic potential. Now, the constraints (2.1) may be rewritten in terms of the periodic potentials,

$$\pi_{\varepsilon_a^+} - \pi_{\varepsilon_a^-} - z_a T \in [d_a^-, d_a^+], \forall a \in A \tag{2.2}$$

which for all edges $a \in A$ limits the integers z_a to take values from the set $\{-1, 0\}$ only. For short, we write (2.2) as

$$\pi_{\varepsilon_a^+} - \pi_{\varepsilon_a^-} \in [d_a^-, d_a^+]_T, \forall a \in A \tag{2.3}$$

Define $\vartheta_a = \pi_{\varepsilon_a^+} - \pi_{\varepsilon_a^-} - z_a T$, such that for a feasible solution $\vartheta_a \in [d_a^-, d_a^+]$.

When scheduling periodic events, an edge a may be thought of as representing a set of activities initiated by the events ε_a^- and finalised by the events ε_a^+. One occurence of the activity will then have starting time $\pi_{\varepsilon_a^-}$ and ending time $\pi_{\varepsilon_a^+}$ in the same or in one of the subsequent time periods. The periodic tension ϑ_a mod T then represents the time between the starting time of the activity and the first subsequent ending time.

2.2 Properties of PESP

Now, some basic properties are discussed. Firstly, it is shown that parallel edges in the constraint graph may result in disjunctive span constraints. Secondly, the integrality property of the PESP is shown.

2.2.1 Parallel Edges

Parallel edges are allowed in the PESP constraint graph, as they simply model several span constraints between the same pair of events, e.g. the two edges between node 0 and 2 in figure 2.1. Note that an edge in the constraint graph may be reversed by multiplying the respective constraint by -1. Thus, the edge from 2 to 0 with span interval $[-8, -5]$ may be replaced by an oppositely directed edge with span interval $[5, 8]$.

When parallel edges exist for a pair of events all the respective constraints must be fulfilled, i.e. the potential difference between the two events must belong to the intersection of the span intervals. In figure 2.2, the principle of intersecting periodic intervals is shown for two constraints. If the resulting span interval is non-disjoint (figure 2.2 (1, 2, 4)), the respective span constraints may be replaced by one span constraint (unless the resulting interval is equivalent to $[0, T]$, in which case no constraint is required), thereby reducing the number of constraints.

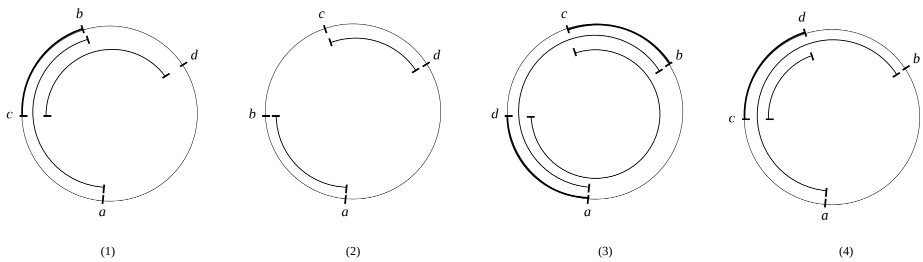


Figure 2.2: Intersecting two periodic intervals $[a, b]$ and $[c, d]$

A union of two disjoint intervals may be modelled as an intersection of two periodic intervals as shown in figure 2.2 (3). k constraints is necessary and sufficient to model a union of k disjoint intervals.

The two parallel edges in figure 2.1 are seen to yield an empty interval, and the PESP instance is thus infeasible.

2.2.2 Integrality Property of PESP

The integrality property states that a feasible solution to a PESP instance with integral upper and lower bounds, i.e. $d^-, d^+ \in \mathbb{Z}$, always has integral tensions. This is used in the reduction from the node colouring problem.

If (v, z) is a feasible solution to a PESP instance with constraint graph $G = (N, A, d^-, d^+)$ and integral upper and lower bounds, then the tension v is integral. This is shown below:

Suppose a feasible solution (v, z) to the PESP instance given by G exists, and let d^- and d^+ be the vectors of lower and upper bounds respectively, then we have

$$
\begin{gathered}
d^- \leq v - zT \leq d^+ \\
\Updownarrow \\
v \leq d^+ + zT \\
-v \leq -(d^- + zT)
\end{gathered}
$$

This is equivalent to

$$\begin{bmatrix} I \\ -I \end{bmatrix} v \leq \begin{matrix} d^+ + zT \\ -d^- - zT \end{matrix}$$

where I is the $m \times m$ identity matrix. Since, $\begin{bmatrix} I \\ -I \end{bmatrix}$ is totally unimodular ([24], prop. 3.1 and 3.2) and the right hand side is integral, it follows that v is integral ([24], prop. 3.3).

2.3 Complexity

Now, the complexity of the PESP is investigated. PESP is shown to be NP-complete by reduction from the Node Colouring Problem, as well as, from the Hamiltonian Cycle Probelm.

2.3.1 Node Colouring Problem

The node k-colouring problem (NCP), may be formulated as the problem of assigning k integer values in the interval $[0, k-1]$ to nodes in a graph, such that no two adjacent nodes are assigned the same value. Let the undirected graph $G = (N, E)$ represent such a k-coloring instance P^{NCP}. Also, let a directed complete graph $K = (N, A)$ represent a PESP instance P^{PESP} with period k and span constraints

$$v_a - z_a k \in [1, k-1], \quad \forall a \in A' \tag{2.4}$$

where $A' = \{(i, j) \in A | (i, j) \in E\}$.

The problem of finding a feasible node colouring on the graph G is now equivalent to the problem of finding a feasible node potential for the PESP represented by K. We will show this in the following:

First note that, an integer solution u to the problem P^{NCP} with values in $[0, k-1]$ is feasible if and only if it satisfies, that the absolute distance between two adjacent nodes in G is greater or equal to 1, i.e.

$$\begin{array}{c} 1 \leq u_a^+ - u_a^-, \quad \forall a \in A' : u_a^- < u_a^+ \\ u_a^+ - u_a^- \leq -1, \quad \forall a \in A' : u_a^+ < u_a^- \\ 0 \leq u_\varepsilon \leq k-1, \quad \forall \varepsilon \in N \end{array} \tag{2.5}$$

If P^{NCP} has a feasible solution then P^{PESP} has a feasible solution:

Let u be a feasible node colouring, and consider an edge a in A'. If $u_a^+ > u_a^-$, then since $u_a^+ \leq k$ and $u_a^- \geq 1$, we have $u_a^+ - u_a^- \leq k-1$. Otherwise $u_a^+ < u_a^-$, and since $u_a^- \leq k$ and $u_a^+ \geq 1$, we have $1-k \leq u_a^+ - u_a^-$.

Let $z_a = 0$, when $u_a^+ \geq u_a^-$ and $z_a = -1$ otherwise. Then we have, $1 \leq u_a^+ - u_a^- - z_a k \leq k-1$ for all $a \in A'$, which corresponds to the span constraints (2.4) in the P^{PESP}. The colouring u is therefore a feasible solution to P^{PESP}.

Conversely, if P^{PESP} has a feasible solution then P^{NCP} has a feasible solution:

A solution u to P^{PESP} is feasible if and only if it satisfies (2.4).

Let $z'_a = z_a - \left[\frac{u_a^+ - u_a^-}{k}\right]$, where $[r]$ denotes rounding towards 0, such that $|r| - |[r]| \in [0,1[$. Now, $z'_a = 0$ when $u_a^+ \geq u_a^-$ and $z'_a = -1$ otherwise. Define a periodic potential $\pi = u \bmod k - \epsilon$, for some $\epsilon \in [0,T[$, such that for some $i \in \{1,...,n\}$, $\pi_i = 0$. Then we get $\pi_a^+ - \pi_a^- = u_a^+ - u_a^- - k\left[\frac{u_a^+ - u_a^-}{k}\right] \in]-k,k[\ \forall a \in A$ and (2.4) becomes

$$1 \leq \pi_a^+ - \pi_a^- - z'_a k \leq k-1, \quad \forall a \in A$$

If $\pi_a^+ \geq \pi_a^-$, we satisfy $1 \leq \pi_a^+ - \pi_a^-$, otherwise $\pi_a^+ - \pi_a^- \leq -1$.

Since the upper and lower bounds in (2.4) are integral, any feasible solution is integral (see section 2.2.2). Since the tensions are in the range $[0, k-1]$ and there exists a potential $\pi_i = 0$, it follows that π is integral and in the range $[0, k-1]$. Hence π is a feasible node colouring of G.

Since, P^{NCP} is polynomially reducible to P^{PESP} and the NCP is NP-complete, it follows that PESP is NP-complete.

2.3.2 Hamiltonian Cycle Problem

Consider the problem P^{HC} of finding a hamiltonian cycle in an undirected graph $G = (N, E)$ with n nodes. A PESP instance $P^{PESP}$with period n may now be constructed, which is equivalent to P^{HC}.

Let P^{PESP} be given by the complete directed graph $K = (N, A)$ with n nodes and arbitrarily oriented edges. Consider the set of edges $A' = \{(i,j) \in A | (i,j) \in E\}$, and introduce into P^{PESP} the following span constraints

$$v_a - z_a k \in [1, n-1], \quad \forall a \in A'$$
$$v_a - z_a k \in [2, n-2], \quad \forall a \in A \setminus A'$$

Then, the problem of finding a hamiltionian cycle on the graph G is equivalent to the problem of finding a feasible node potential in the PESP instance represented by K. This is shown below:

First note that, since all span intervals are symmetric of the form $[a, n-a]$, the direction of any edge in A may be reversed, yielding the equivalent span interval $[a-n, -a] = [a, n-a]$. Therefore, the edges of A may be oriented arbitrarily.

If P^{HC} has a feasible solution, then P^{PESP} has a feasible solution:

A hamiltonian cycle C may be defined by an assignment of distinct integer potentials to nodes, such that the difference modulo n between two adjacent nodes in the cycle is either 1 or $n-1$.

Then P^{HC} obviously satisfies $v_a \in [1, n-1]$ for all edges $a \in C$ in the cycle and $v_a \in [2, n-2]$ for all $a \in A \setminus C$. Since, $[2, n-2]$ is completely contained in $[1, n-1]$, the span constraints for the edges in $A' \setminus C$ are satisfied, and a solution to P^{PESP} is obtained.

If P^{PESP} has a feasible solution, then P^{HC} has a feasible solution:

Given a feasible solution (u, z) to P^{PESP}. Since K is the complete graph, for every node $\varepsilon \in N$ there must exist two nodes $\varepsilon^-, \varepsilon^+$, such that

$$(u_\varepsilon - u_{\varepsilon^-}) \mod n = 1$$
$$(u_\varepsilon - u_{\varepsilon^+}) \mod n = n-1$$

Hence, the potentials u define a hamiltonian cycle in G.

Since, P^{HC} is polynomially reducible to P^{PESP} and the hamiltonian cycle problem is NP-complete, it follows that the PESP is NP-complete.

2.4 Cycle Periodicity Formulation

Given the formulation (2.1) of the PESP, a formulation with fewer integer variables can be obtained, namely one in which, the integer variables are associated with each cycle in some integral cycle basis of the constraint graph G.

Let the periodic tension be defined by $\vartheta_a = v_a - z_a T$ and let for some oriented cycle c in G, c^+ and c^- denote the set of edges in c oriented along, respectively against, the direction of c. Since, for any cycle c in G, the sum of the aperiodic tensions v along c wrt. its orientation is 0, we have for all cycles in G,

$$\sum_{a \in c^+} (v_a - z_a T) - \sum_{a \in c^-} (v_a - z_a T) = T \left(- \sum_{a \in c^+} z_a + \sum_{a \in c^-} z_a \right), \ \forall c \in G$$

where $z_a \in \mathbb{Z}$.

Now, define for each cycle in G a new integer variable $y_c = -\sum_{a \in c^+} z_a + \sum_{a \in c^-} z_a$. Inserting, we have

$$\sum_{a \in c^+} \vartheta_a - \sum_{a \in c^-} \vartheta_a = y_c T, \ \forall c \in G \tag{2.6}$$

where $y_c \in \mathbb{Z}$.

Conversely, given (2.6), define for all edges a, $v_a = \vartheta_a + z_a T$, where z_a is any integer. Let H be any spanning tree in G and let C be the strictly fundamental cycle basis defined by H. From equation 2.6, we get

$$\sum_{a \in c^+} v_a - \sum_{a \in c^-} v_a - \sum_{a \in c^+} z_a T + \sum_{a \in c^-} z_a T = y_c T, \ \forall c \in C$$

Now, set $z_a = 0$ for all $a \in H$. Since C is strictly fundamental, there is exactly one edge from $A \setminus H$ in each cycle $c \in C$ in the cycle basis. Therefore it suffices to set $z_a = -y_c$ if $a \in c^+$ and $z_a = y_c$ if $a \in c^-$ for all $a \in A \setminus H$. We immediately obtain, $\sum_{a \in c^+} v_a - \sum_{a \in c^-} v_a = 0$, for all cycles in the cycle basis C. Furthermore, for the linear combination $\lambda_1, ..., \lambda_{m-n+1}$, that defines a given non-basic cycle b, we have

$$\sum_{c \in C} \lambda_c \left[\sum_{a \in c^+} v_a - \sum_{a \in c^-} v_a + T \left(- \sum_{a \in c^+} z_a + \sum_{a \in c^-} z_a \right) \right] = \sum_{c \in C} \lambda_c y_c T$$

Since $-\sum_{a \in c^+} z_a + \sum_{a \in c^+} z_a = y_c$,

$$\sum_{c \in C} \lambda_c \left[\sum_{a \in c^+} v_a - \sum_{a \in c^-} v_a \right] = 0$$

Also, b is defined by the linear combination λ_c of cycles in C. Therefore, we have

$$\sum_{a \in b^+} v_a - \sum_{a \in b^-} v_a = 0$$

for any cycle b in G.

Hence, the PESP instance with constraint graph G has a periodic tension ϑ if and only if the cycle periodicity property (2.6) holds for all cycles in G.

If in addition to the cycle periodicity property the periodic tension ϑ respects the span constraints $[d^-, d^+]$, i.e. $d_a^- \leq \vartheta_a \leq d_a^+$ for all a in A, it defines a feasible solution to the PESP. Therefore, the PESP can be formulated using the cycle periodicity property.

2.4.1 Valid cycle bases for CPF

We showed above that it is sufficient to require that the cycle periodicity property (2.6) holds for the edges in a strictly fundamental cycle basis in order to obtain a periodic tension for the PESP. In fact, it is enough to enforce the cycle periodicity property for all edges in an integral cycle basis. We show this below.

A cycle basis C in G is integral if and only if any non-basic cycle b with incidence vector γ_b can be expressed as an integer linear combination λ^b of the cycles in C with incidence vectors $\gamma_1, ..., \gamma_{m-n+1}$.That is, $\gamma_b = \sum_{c \in C} \lambda_c^b \gamma_c$ such that $\lambda_c^b \in \mathbb{Z}$ for all cycles b in G [15].

Let C be an integral cycle basis for which the cycle periodicity property holds. Consider the non-basic cycle b defined by the linear combination λ of C. Then we have,

$$\sum_{a \in b^+} \vartheta_a - \sum_{a \in b^-} \vartheta_a = \sum_{c \in C} \lambda_c \left(\sum_{a \in c^+} \vartheta_a - \sum_{a \in c^-} \vartheta_a \right) = T \sum_{c \in C} \lambda_c y_c$$

Since $y_c, \lambda_c \in \mathbb{Z}$, we get that the sum of the periodic tension along b wrt. the orientation of the edges is an integer multiple of the period length, T. Therefore, the cycle periodicity property holds for all cycles in G.

2.4.2 Bounds on the cycle integer variables

Bounds on the cycle integer variables y can be obtained by summing the bounds along each cycle in the cycle basis.

For each cycle c in G, an arbitrary orientation is chosen. Let c^+ and c^- be the sets of edges oriented along, respectively against, the direction of c. Summing along the cycle wrt. the orientation of the edges yields,

$$\sum_{a \in c^+} d_a^- - \sum_{a \in c^-} d_a^+ \leq \sum_{a \in c^+} \vartheta_a - \sum_{a \in c^-} \vartheta_a \leq \sum_{a \in c^+} d_a^+ - \sum_{a \in c^-} d_a^-$$

Let $d_c^- = \sum_{a \in c^+} d_a^- - \sum_{a \in c^-} d_a^+$ and $d_c^+ = \sum_{a \in c^+} d_a^+ - \sum_{a \in c^-} d_a^-$.

Since the cycle periodicity property holds for all cycles in G, we get

$$d_c^- \leq y_c T \leq d_c^+$$

As y_c is integral, the following bounds on y_c are obtained

$$\left\lceil \frac{d_c^-}{T} \right\rceil \leq y_c \leq \left\lfloor \frac{d_c^+}{T} \right\rfloor \tag{2.7}$$

for all cycles $c \in C$.

2.4.3 Good cycle bases for CPF

Solving the PESP using the cycle periodicity formulation with the cuts (2.7), the solution time depends on the number of possible values the cycle integer variables can take. Therefore, it is desirable to have as tight bounds on the cycle integer variables as possible. This can be achieved by choosing the cycle basis carefully.

Each cycle integer variable y_c can take $\lfloor d_c^+/T \rfloor - \lceil d_c^-/T \rceil + 1$ possible values. Define the width of a cycle basis C [10],

$$W_C = \prod_{c \in C} \left(\left\lfloor \frac{d_c^+}{T} \right\rfloor - \left\lceil \frac{d_c^-}{T} \right\rceil + 1 \right)$$

Choosing the cycle basis such that the width W_C is minimal will therefore result in a formulation of the PESP, where the cycle integer variables y can take the smallest possible number of values.

Since strictly fundamental cycle bases are also integral cycle bases (see [15] for a further discussion of a classification of cycle bases) and efficient algorithms to find such cycle bases exists, one might want to consider only strictly fundamental cycle bases, however by considering the entire set of integral cycle bases, a tighter formulation for the PESP may be obtained. The trade-off is longer computation time for finding the cycle basis.

According to [15], a cycle basis C is fundamental if there exists an ordering of the cycles in C, such that each cycle in C contains at least one edge, which is not part of any of its predecessors in that ordering.

In Algorithm 1, an algorithm proposed by Berger (described in [10]) to find a generalised fundamental cycle basis is outlined. Even though the entire set of integral cycle bases is not considered, using generalised fundamental cycle bases

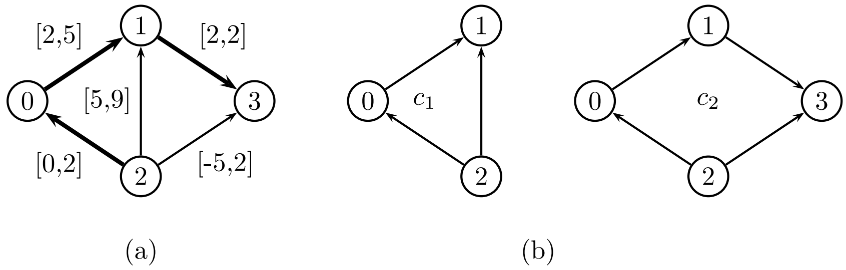


Figure 2.3: (a) Constraint graph for small PESP instance. Bold arcs make up the minimum spanning tree wrt. span widths. (b) A minimal width strictly fundamental cycle basis constructed from the spanning tree in (a).

is a significant improvement compared to using only strictly fundamental cycle bases.

Bergers algorithm iteratively adds cycles to the cycle basis, expanding the set of potential edges in each iteration. Starting with a minimum spanning tree, each of the non-tree edges are considered in order of increasing width. In each iteration, the cycle added to the cycle basis is obtained from the current non-tree edge and the shortest path between its initial and final vertices in the undirected graph obtained from the spanning tree and the non-tree edges considered in previous iterations. The algorithm is shown for a small instance in figure 2.4. In the first iteration (a), the shortest path in the undirected minimum spanning tree (solid edges) between node 1 and 2 is considered, yielding the cycle c'_1 (b). The edge from 2 to 1 is added to the spanning tree and the shortest path between node 2 and 3 in the undirected version of the obtained graph (c) yields cycle c'_2 (d).

2.5 Chain Cutting Planes

In this section, a special type of chain cutting planes, introduced by Lindner in [18], is developed for the cycle periodicity formulation. First a class of valid inequlities for the PESP, used to define a class of cutting planes, namely chain cutting planes, is introduced. Secondly, these valid inequlities are used to develop a special class of chain cutting planes suitable for the cycle periodicity formulation.

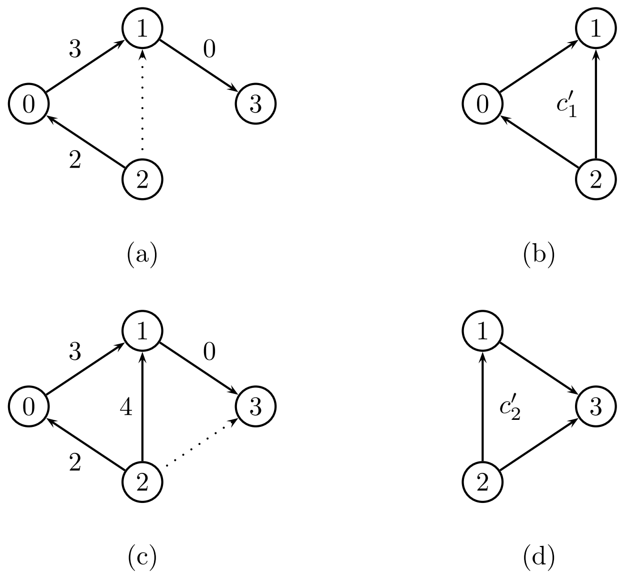


Figure 2.4: The two iterations of Bergers algorithm on the PESP instance in figure 2.3 (a). Edge labels in (a) and (c) denote span widths. The resulting cycle basis consists of the cycles c'_1 (b) and c'_2 (d).

Algorithm 1 Bergers algorithm for finding a minimum width fundamental cycle basis. $S_{i,j}$ denotes the set of edges on the shortest path from i to j without considering the direction of the edges.

find minimum spanning tree H wrt. width of edge spans
let $B \leftarrow H$
while $B \neq A$ **do**
  find edge $a \in A \setminus B$ with min. width of edge span
  let $c_i \leftarrow S_{\varepsilon_a^+, \varepsilon_a^-} \cup \{a\}$
  let $B \leftarrow B \cup \{a\}$
  $i \leftarrow i + 1$
  $C \leftarrow C \cup \{c_i\}$
end while

Consider a set $\{1, 2, ..., k\}$ of k edges with the same initial vertex ε_r and same final vertex ε_s. In the following, let the tension v be defined by $\pi_s - \pi_r$. Also, define by $\hat{d_a^-} = d_a^-$ mod T and let p_a be the integer such that $\hat{d_a^-} = d_a^- + p_a T$ for $a = 1, ..., k$. Furthermore, assume that the edges are ordered, such that

$$0 \leq \hat{d_1^-} \leq \hat{d_2^-} \leq ... \leq \hat{d_a^-} \leq ... \leq \hat{d_k^-} < T \tag{2.8}$$

(otherwise, reassign indices such that the ordering is true). Let $\alpha_a = \hat{d_a^-} - \hat{d_{a-1}^-}$ for $a = 2, ..., k$ and $\alpha_1 = \hat{d_1^-} + T - \hat{d_k^-}$. Then the following inequality is valid (Lindner),

$$\hat{d_k^-} + \sum_{a=1}^{k} \alpha_a \left(z_a - p_a\right) \leq \pi_s - \pi_r \tag{2.9}$$

The validity of this statement is shown below following a proof by Lindner:

First realize that $\sum_{a=1}^{k} \alpha_a = T$ and $\sum_{a=i+1}^{k} \alpha_a = \hat{d_k^-} - \hat{d_i^-}$. Let $\hat{v} = (\pi_s - \pi_r) \bmod T = v + qT$. Then a particular potential π uniquely determines a partition i of the edges, such that

$$0 \leq \hat{d_1^-} \leq \hat{d_2^-} \leq ... \leq \hat{d_i^-} \leq \hat{v} < \hat{d_{i+1}^-} ... \leq \hat{d_k^-} < T$$

where $i = 0$ imply $\hat{v} < \hat{d_1^-}$. (It is necessary to restrict the value of $\hat{d_1^-}$ to be strictly positive whenever $i = 0$).

Since the span constraints are valid for all edges, we have $d_a^- \leq v - z_a T \Leftrightarrow \hat{d_a^-} \leq \hat{v} + (p_a - q - z_a)T$.

Then, for any $a \in \{1, 2, ..., i\}$, $\hat{d_a^-} \leq \hat{v}$ implies that $0 \leq p_a - q - z_a \Leftrightarrow q - p_a \leq -z_a$. Similarly, for all $a \in \{i+1, ..., k\}$, $\hat{v} < \hat{d_a^-}$ imply that, $1 \leq p_a - q - z_a \Leftrightarrow q - p_a + 1 \leq -z_a$. Thus,

$$v = \hat{v} - qT + \sum_{a=1}^{k} \alpha_a z_a - \sum_{a=1}^{i} \alpha_a z_a - \sum_{a=i+1}^{k} \alpha_a z_a$$

Substitute the lower bounds for $-z_a$ as described above

$$v \geq \hat{v} - qT + \sum_{a=1}^{k} \alpha_a z_a + \sum_{a=1}^{i} \alpha_a (q - p_a) + \sum_{a=i+1}^{k} \alpha_a (q - p_a + 1)$$

Substituting $\sum_{a=1}^{k} \alpha_a = T$

$$v \geq \hat{v} + \sum_{a=1}^{k} \alpha_a z_a - \sum_{a=1}^{i} \alpha_a p_a - \sum_{a=i+1}^{k} \alpha_a p_a + \sum_{a=i+1}^{k} \alpha_a$$

and $\sum_{a=i+1}^{k} \alpha_a = \hat{d_k^-} - \hat{d_i^-}$, we get

$$v \geq \hat{v} + \sum_{a=1}^{k} \alpha_a (z_a - p_a) + \hat{d_k^-} - \hat{d_i^-}$$

Since $0 < \hat{v} - \hat{d_i^-}$,

$$v \geq \hat{d_k^-} + \sum_{a=1}^{k} \alpha_a (z_a - p_a)$$

which concludes the proof.

2.5.1 Chain cutting planes for CPF

Consider the cycle periodicity formulation of the PESP and an instance with constraint graph $G = (N, A, d^-, d^+)$ and cycle basis B. Furthermore, consider a particular vertex $\varepsilon \in N$ with C_ε being the set of k cycles in the cycle basis B, such that ε is visited by each of the cycles in C_ε. I.e. $C_\varepsilon = \bigcup_j \{c_j : \varepsilon \in N(c_j)\}$, where $N(c)$ is the set of nodes visited by the cycle c.

Now introduce into G a loop a for each cycle c in C_ε with $d_a^- = d_c^-$ and $d_a^+ = d_c^+$, so that $v_a - z_a T \in [d_c^-, d_c^+]$. Since a is a loop, $v_a = 0$. Let A' denote the set of k new loops generated in this way. If we let $z_a = -y_c$, we get $y_c T \in [d_c^-, d_c^+]$, and this new set of constraints does therefore not restrict the original problem, these constraints are already part of the formulation. Let the set of loops A' be indexed according to (2.8) and let the cycles in C_ε have the same indexation. Applying the valid inequality (2.9) for the set of edges A' yields,

$$\hat{d_k^-} + \sum_{a \in A'} \alpha_a \left(z_a - p_a \right) \leq 0$$

where $\hat{d_a^-}$, α_a, p_a are defined as above, assuming that the edges (and cycles) are ordered . Hence, the vertex $\varepsilon \in N$, defines the valid inequality

$$\hat{d_k^-} \leq \sum_{c \in C} \alpha_c (y_c + p_c) \tag{2.10}$$

In general, efficient chain cutting planes (2.10), are found by considering vertices for which the number of visiting cycles $|C|$ is large. However, if for some $c \in C$ the value of the integer variable y_c is fixed, the instance cannot be restricted further wrt. y_c. Also, cycles $c \in C$ for which $\alpha_c = 0$ do not contribute to the cut. Therefore, we may state more precisely, though informally, that

> an efficient chain cutting plane is obtained by considering a vertex ε in the constraint graph, such that the number of cycles in $C' = \{c \in C : \alpha_c \neq 0, \lceil d_c^- / T \rceil \neq \lfloor d_c^+ / T \rfloor\}$ is large.

2.6 Sequencing and Matching

In the following, the concepts of sequencing and matching of periodic events will be discussed, and important results that are used in timetable planning are derived.

2.6.1 Sequencing of Events

Consider an oriented cycle c with k edges, such that c^+ is the set of edges in c with orientation along c and c^- is the set of edges in c oriented against the orientation of c. Let $\{a_1, a_2, ..., a_k\}$ be the set of edges in c, and let ε_i denote the initial node of a_i if a_i is in c^+, otherwise the final node, such that if the cycle is traversed in the positive direction starting at ε_1, the nodes of the cycle will be visited in the order $\varepsilon_1, \varepsilon_2, ..., \varepsilon_k$. In figure 2.5 an example with $k = 5$ is shown.

The nodes in c are said to be cyclically sequenced in the order $\varepsilon_1, \varepsilon_2, ..., \varepsilon_k$ if and only if, ([22])

$$\begin{gathered} 0 \leq \pi_{i+1} - \pi_i,\ i = 1, ..., k-1 \\ \pi_1 - \pi_k \leq 0 \end{gathered}$$

If strict inequality holds, the nodes are said to be proper cyclically sequenced in the order $\varepsilon_1, \varepsilon_2, ..., \varepsilon_k$.

In terms of the tensions, this can be stated as

$$\begin{array}{l} 0 \leq v_a,\ a \in c^+ \setminus \{a_k\} \cup c^- \cap \{a_k\} \\ v_a \leq 0,\ a \in c^- \setminus \{a_k\} \cup c^+ \cap \{a_k\} \end{array}$$

Figure 2.5 shows a proper cyclic sequencing of five events.

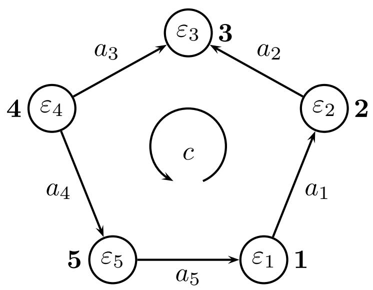


Figure 2.5: Cyclically sequencing five events for $T = 5$. Numbers in bold denote a periodic potential of the event, such that the events are proper cyclically sequenced in the order $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$.

In the following we assume that the tensions v are in the interval $]-T, T[$, and that the nodes in c are proper cyclically sequenced.

Since for all $a \in c^-$ it holds that $-T < v_a < 0$, we have that $z_a = -1$ for all $a \in c^- \setminus \{a_k\}$. Also, since $0 < v_a < T$ for all $a \in c^+$, we have that $z_a = 0$ for all $a \in c^+ \setminus \{a_k\}$. For a_k, we get $z_{a_k} = -1$ if $a_k \in c^+$ and $z_{a_k} = 0$ if $a_k \in c^-$.

For a set of proper cyclically sequenced events, the integer values corresponding to any pair of those events may therefore be fixed a priori, reducing the complexity of the PESP instance.

Summing the tensions along the cycle c, we get

$$\sum_{a \in c^+} z_a - \sum_{a \in c^-} z_a = - \sum_{a \in c^- \setminus \{a_k\}} z_a + \sum_{a \in c^+ \cap \{a_k\}} z_a = |c^- \setminus \{a_k\}| - |c^+ \cap \{a_k\}| \quad (2.11)$$

Now, either $a_k \in c^+$ or $a_k \in c^-$. However, in both cases we get,

$$\sum_{a \in c^+} z_a - \sum_{a \in c^-} z_a = |c^-| - 1 \tag{2.12}$$

Conversely, for a given directed cycle c with c^+ being the set of edges directed along the orientation of the cycle and c^- being the set of edges directed opposite to the orientation of the cycle, assume that (2.12) holds and that $z_a \in \{-1, 0\}$ for all edges $a \in c$. Then the following holds

$$-|c^-| \leq \sum_{a \in c^-} z_a = \sum_{a \in c^+} z_a + 1 - |c^-| \leq 1 - |c^-|$$

where the lower bound is obtained when $z_a = -1$ for all $a \in c^-$ and the upper bound is obtained when $z_a = 0$ for all $a \in c^+$.

Therefore, $\sum_{a \in c^-} z_a$ can take exactly two values $-|c^-|$ or $1 - |c^-|$. In the first case ($\sum_{a \in c^-} z_a = -|c^-|$), $z_a = -1$ for all $a \in c^-$ and for exactly one edge a^* in c^+, $z_{a^*} = -1$. Otherwise ($\sum_{a \in c^-} z_a = 1 - |c^-|$), $z_a = 0$ for all $a \in c^+$ and for exactly one edge a^* in c^-, $z_{a^*} = 0$. Label the nodes in c, such that $\varepsilon^+_{a^*} = \varepsilon_1$ if $a^* \in c^+$ and $\varepsilon^+_{a^*} = \varepsilon_k$ if $a^* \in c^-$. If a^* is in c^+ (first case), we have that

$$\begin{gathered}\pi_{\varepsilon_a^-} \leq \pi_{\varepsilon_a^+}, \, \forall a \in c^+ \setminus \{a^*\} \\ \pi_{\varepsilon_a^+} \leq \pi_{\varepsilon_a^-}, \, \forall a \in c^- \bigcup \{a^*\}\end{gathered}$$

otherwise, a^* is in c^- (second case), and we have

$$\begin{gathered}\pi_{\varepsilon_a^-} \leq \pi_{\varepsilon_a^+}, \, \forall a \in c^+ \bigcup \{a^*\} \\ \pi_{\varepsilon_a^+} \leq \pi_{\varepsilon_a^-}, \, \forall a \in c^- \setminus \{a^*\}\end{gathered}$$

If the nodes are labeled, such that a^* connects ε_1 and ε_k, this is equivalent to

$$\begin{gathered}0 \leq \pi_{i+1} - \pi_i, \, i = 1, ..., k-1 \\ \pi_1 - \pi_k \leq 0\end{gathered}$$

Hence, the nodes in c are cyclically sequenced.

Therefore, the nodes in c are cyclically sequenced in the order $\varepsilon_1, \varepsilon_2, ..., \varepsilon_k$ if and only if equation 2.12 holds. Or, in terms of the cycle integers y, the nodes in c are cyclically sequenced in the order $\varepsilon_1, \varepsilon_2, ..., \varepsilon_k$ if and only if

$$y_c = |c^-| - 1 \tag{2.13}$$

2.6.2 Matching four events

Matching of events is a powerful concept and may be modelled within the framework of the PESP. First matchings for a small subgraph of four events is considered, before extending to the general case.

Suppose four events $\varepsilon_1, ..., \varepsilon_4$ and six edges make up a subgraph G' of the constraint graph of a PESP instance, as shown in figure 2.6 (a). Assume that, a_{42} and a_{31} have *symmetric* bounds $[s^-, T - s^-]$. An ordered pair of events are said to be matched, if and only if, the tension between them is in some interval $[d^-, d^+]$.

Now we want to find a (maximum) matching M of the bipartite subgraph consisting of the events $\varepsilon_1, ..., \varepsilon_4$ and the constraints $a_{12}, a_{14}, a_{32}, a_{34}$, such that $\vartheta_{ij}, \vartheta_{kl} \in [d^-, d^+]_T$, if and only if, $M = \{a_{ij}, a_{kl}\}$. In other words, we want to impose the constraints

$$\vartheta_{12}, \vartheta_{34} \in [d^-, d^+] \quad \text{xor} \quad \vartheta_{14}, \vartheta_{32} \in [d^-, d^+] \tag{2.14}$$

relating to each of the two possible matchings.

Suppose, that the matching $\{a_{12}, a_{34}\}$ is chosen, then $\vartheta_{12}, \vartheta_{34} \in [d^-, d^+]_T$ and furthermore constraint a_{42} ensures $\vartheta_{32} \in [d^- + s^-, d^+ + T - s^-]_T$ and $\vartheta_{14} \in [d^- - (T - s^-), d^+ - s^-]_T = [d^- + s^-, d^+ + T - s^-]_T$. Similarly, if the matching $\{a_{32}, a_{14}\}$ is chosen, we have $\vartheta_{32}, \vartheta_{14} \in [d^-, d^+]_T$ and $\vartheta_{12}, \vartheta_{34} \in [d^- + s^-, d^+ + T - s^-]_T$.

Hence, constraint a_{42} ensures, that each event in $\{\varepsilon_1, \varepsilon_3\}$ is matched with at most one event from $\{\varepsilon_2, \varepsilon_4\}$. That is, e.g. $\vartheta_{34} \in [d^-, d^+]_T$ imply $\vartheta_{32} \in [d^- + s^-, d^+ + T - s^-]_T$. Similarly, constraint a_{31} ensures that each of the two events $\varepsilon_1, \varepsilon_3$ cannot be matched with the same event in $\{\varepsilon_2, \varepsilon_4\}$. That is, e.g. $\vartheta_{12} \in [d^-, d^+]_T$ imply $\vartheta_{32} \in [d^- + s^-, d^+ + T - s^-]_T$.

Also, we must require all events to be matched (if a maximum matching is

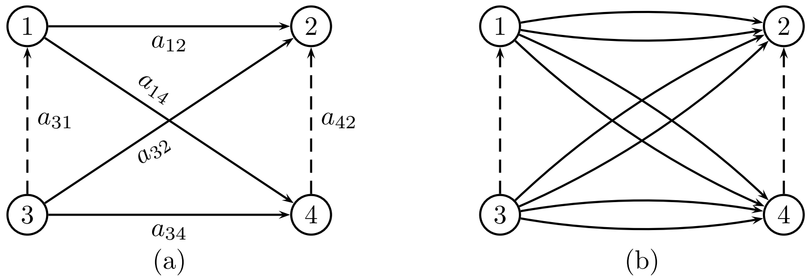


Figure 2.6: Subgraph ensuring matching of the four events 1, 2, 3, 4. In (b) each disjunctive constraint (solid edges) in (a) is replaced by two conjunctive constraints.

desired). To this end, we may assume

$$d^+ + T - s^- < d^- + 2s^- \iff d^+ - d^- + T < 3s^-$$

This will ensure that, if an event ε_i for $i = 1, 3$ is not matched with ε_j for $j = 2, 4$, i.e. $\vartheta_{ij} \in [d^- + s^-, d^+ + T - s^-]_T$, then constraint a_{42} ensures $\vartheta_{ik} \notin [d^- + s^-, d^+ + T - s^-]_T$, for $k = 2, 4$ and $k \neq j$, and therefore ε_i is matched with ε_k. Similarly, this also holds for all $i = 2, 4$ and $j = 1, 3$ with a_{31} as the separating constraint. In fact, this means that the events $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ either occur in the sequence $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ or in the sequence $\varepsilon_1, \varepsilon_4, \varepsilon_3, \varepsilon_2$.

Now we can rewrite 2.14 as

$$\vartheta_{ij} \in [d^-, d^+]_T \cup [d^- + s^-, d^+ + T - s^-]_T \quad \text{for } i = 1, 3\, j = 2, 4 \tag{2.15}$$

Assuming that the two periodic intervals $[d^-, d^+]_T$ and $[d^- + s^-, d^+ + T - s^-]_T$ are disjoint, equations (2.15) may be expressed in terms of conjunctive span constraints

$$\vartheta_{ij} \in [d^-, d^+ + T - s^-]_T \cap [d^- + s^-, d^+ + T]_T \quad \text{for } i = 1, 3\, j = 2, 4 \tag{2.16}$$

Therefore, replace each disjunctive constraint a_{ij} for $i = 1, 3\, j = 2, 4$ in figure 2.6 (a) by two constraints a'_{ij} and a''_{ij} with span intervals $[d^-, d^+ + T - s^-]$ and $[d^- + s^-, d^+ + T]$, respectively, as shown in figure 2.6 (b).

2.6.3 Matching $2n$ events

In the following, the matching constraints from the previous section is generalised. We consider two disjunct sets of n events I and J. I and J each constitute a clique, where each edge in these two cliques have span interval $[s^-, s^+]$, and $s^+ = T - s^-$. Also for each pair of events $(i, j) \in I \times J$ the tension between i and j must be in the interval $[d^-, d^+] \cup [d^- + s^-, d^+ + s^+]$. This is modelled by the two edges a'_{ij} and a''_{ij} with span intervals $[d^-, d^+ + s^+]$ and $[d^- + s^-, d^+ + T]$, respectively.

Let ϑ_{ij} be the periodic tension of a'_{ij}. Now, we say that a pair of events $(i, j) \in I \times J$ is matched, or $(i, j) \in M$ for some matching M if and only if ϑ_{ij} is in the interval $[d^-, d^+]$.

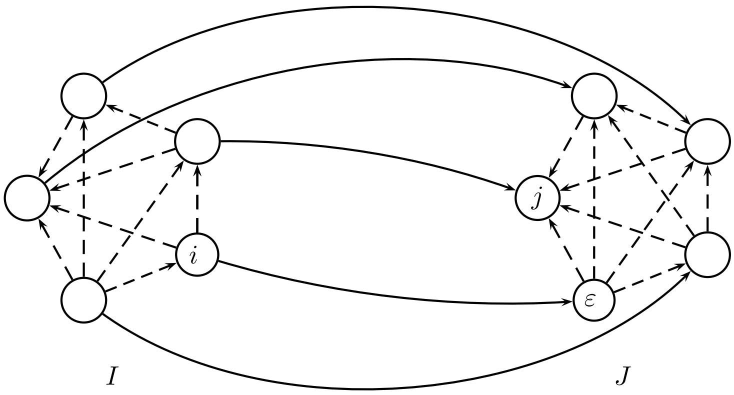


Figure 2.7: Subgraph for matching 10 events. For each pair $(i, j) \in I \times J$ a matching constraint exists. For clarity, only the matching constraints in some maximum matching M (solid arrows) are shown. The separation constraints (dashed arrows) form two cliques together with I and J.

For a maximum matching M, it must hold that each event $i \in I$ is matched with exactly one event $j \in J$, and that each event $j \in J$ is matched with exactly one event $i \in I$.

Suppose, that $i \in I$ and $j \in J$ are matched, i.e. $\vartheta_{ij} \in [d^-, d^+]$, then for any event $k \in I$, the clique constraints between the events of I ensures that $\vartheta_{kj} \in [d^- + s^-, d^+ + s^+]$ and hence (k, j) is not in the matching. Similarly, the clique constraints between events in J ensures, that for any event $k \in J$, $\vartheta_{ik} \in [d^- + s^-, d^+ + s^+]$ and hence (i, k) is not in the matching. So, each event $i \in I$ is matched with at most one event $j \in J$.

To ensure a maximum matching, we must also require that each event $i \in I$ is matched with at least one event $j \in J$ and that each event $j \in J$ is matched with at least one event $i \in I$.

$$d^+ + s^+ < d^- + ns^- \tag{2.17}$$

is a sufficient condtion to ensure that each event $i \in I$ is matched with at least one event $j \in J$.

Proof. Consider the events $i \in I$ and $j \in J$, such that i and j are not matched, i.e. $\vartheta_{ij} \in [d^- + s^-, d^+ + s^+]$. For some event $k \neq j \in J$, it must hold that

$$\vartheta_{ik} + yT = \vartheta_{ij} + \sum_{a \in P_{jk}^+} \vartheta_a - \sum_{a \in P_{jk}^-} \vartheta_a \tag{2.18}$$

for some integer y and all paths P_{jk} from j to k in Q, where P_{jk}^+ is the set of edges in P_{jk} along the direction of P_{jk} and P_{jk}^- is the set of edges with orientation opposite to P_{jk}. Choosing P_{jk} as the path visiting each node in J exactly once, and summing the lower bounds along the edges in $a_{ij} \cup P_{jk}$ we obtain a lower bound on $y_c T$

$$d^- + s^- + ps^- - q(T - s^-) - \vartheta_{ik} \leq y_c T \tag{2.19}$$

where p is the number of edges in P_{jk}^+ and q is the number of edges in P_{jk}^-. Since, $p + q = n - 1$, we get

$$d^- + ns^- \leq \vartheta_{ik} + (y_c + q)T \tag{2.20}$$

The assumption is that $d^+ + s^+ < d^- + ns^-$, yielding

$$d^+ + s^+ < \vartheta_{ik} + (y_c + q)T \tag{2.21}$$

Similarly, summing the upper bounds along the edges in $a_{ij} \cup P_{jk}$ we obtain an upper bound on $y_c T$

$$y_c T \le d^+ + s^+ + p(T - s^-) - qs^- - \vartheta_{ik} = d^+ + (p+1)T - ns^- - \vartheta_{ik} \quad (2.22)$$

Assuming $d^+ + s^+ < d^- + ns^-$, which is equivalent to $d^+ - ns^- < d^-(T - s^-)$, we get

$$\vartheta_{ik} + (y_c - p)T < d^- + s^- \quad (2.23)$$

Equation (2.21) and (2.23) imply, that the periodic tension of a_{ik} is not in the interval $[d^- + s^-, d^+ + s^+]$, hence a_{ik} is in the matching.

A similar result holds for $i \in J$, and $j \neq k \in I$.

2.6.4 An objective function for matching constraints

In many cases, it is desirable to minimise or maximise the sum of the tension of only those edges that are in the matching. In the following, I will explain how this can be accomplished for the matching subgraph $G' = (I \cup J, A')$. This has been shown by Peeters [22] for a subgraph of four nodes.

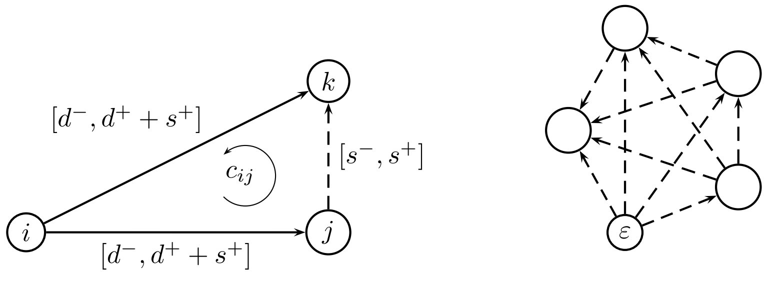
Consider some cycle $c_{ij} = \{a'_{ij}, a_{jk}, a'_{ik}\}$ visiting $i \in I$, $j \in J$, and $k \neq j \in J$ in that order, as shown in figure 2.8 (a), such that $c^+_{ij} = \{a'_{ij}, a_{jk}\}$ is the set of edges oriented along the direction of c_{ij} and $c^-_{ij} = \{a'_{ik}\}$ is the set of edges oriented against the direction of c_{ij}.

The integer

$$y_c = \sum_{a \in c^+_{ij}} \vartheta_a - \sum_{a \in c^-_{ij}} \vartheta_a \quad (2.24)$$

is associated with c_{ij}. Bounds on $y_{c_{ij}}$ are obtained by applying (2.7).

$$\left\lceil \frac{-(d^+ - d^-) - (s^+ - s^-)}{T} \right\rceil \le y_{c_{ij}} \le \left\lfloor \frac{(d^+ - d^-) + 2s^+}{T} \right\rfloor \quad (2.25)$$



(a) (b)

Figure 2.8: (a) The cycle c_{ij} with events $i \in I$ and $j, k \in J$. (b) The clique Q of events J and separation constraints. The edges of Q are directed such that no edge is incident to $\varepsilon \in J$.

Since $(d^+ + s^+) - (d^- + s^-) < T$ and $T < (d^+ + s^+) - (d^- + s^-) + T$, we have that $y_{c_{ij}}$ are binary,

$$0 \le y_{c_{ij}} \le 1 \tag{2.26}$$

Now according to (2.13), the events in c_{ij} are cyclically sequenced in the order i, j, k if and only if $y_{c_{ij}} = |c_{ij}^-| - 1 = 0$. It follows from (2.26), that $y_{c_{ij}} = 1$ otherwise.

Now, consider two events $\varepsilon \in J$ and $\varepsilon' \in I$ such that $a_{\varepsilon'\varepsilon}$ is in some maximum matching M. Next, reverse all edges in Q incident to ε, as shown in figure 2.8 (b). Note that since $s^+ = T - s^-$, the span intervals on the reversed constraints are $[s^- - T, -s^-]$ which is equivalent to $[s^-, T - s^-]$.

For each pair $(i, j) \in I \setminus \{\varepsilon'\} \times J \setminus \{\varepsilon\}$ consider the cycle $c_{ij} = \{a_{i\varepsilon}, a_{\varepsilon j}, a_{ij}\}$. Since $a_{i\varepsilon} \notin M$ we have that $y_{c_{ij}} = 1$. Equation 2.6 yields,

$$\vartheta_{ij} = \vartheta_{i\varepsilon} + \vartheta_{\varepsilon j} - T \tag{2.27}$$

Summing the tensions of the edges in the matching M, we get

$$Z = \vartheta_{\varepsilon'\varepsilon} + \sum_{a_{ij} \in M \setminus \{a_{\varepsilon'\varepsilon}\}} \vartheta_{i\varepsilon} + \vartheta_{\varepsilon j} - T \tag{2.28}$$

$$= \sum_{i \in I} \vartheta_{i\varepsilon} + \sum_{j \in J \setminus \{\varepsilon\}} \vartheta_{\varepsilon j} - (n-1)T \tag{2.29}$$

Chapter 3

Periodic Railway Timetabling

3.1 The Planning Process

Scheduling of ressources in a railway system is traditionally done hierarchically from the planning of the physical infrastructure to vehicle and crew scheduling. This subdivision of tasks is due to different planning horizons and the complexity of the tasks. Also the geographical implications of the problems influence the planning process. Wheras some tasks are local in nature (platform assignment, shunting), other tasks are done on a global level (e.g. lineplanning, timetabling, vehicle assignment). At each planning level, however, decisions interact with decisions made at the other levels, in particular the immediately adjacent levels.

The first step in the planning process is the planning of the physical infrastructure. At this level decisions concerning the allocation of new railway corridors and new stations to accomodate changing demography or travel patterns, as well as capacity enhancing measures, such as extra tracks etc., are taken. Decisions at this level are expensive, time consuming, and are not easily undone. Therefore the planning horizon is long, normally 5 - 20 years.

At the next level a lineplan is constructed that determines the general level of the service of the railway system. A (undirected) train line is a sequence of stations that a train must serve, starting and ending at the same terminal station. In a symmetric line, each station is served twice - once in each direction, such that the sequence of the stations is reversed for the opposite direction of the same line. A lineplan is a set of such lines. A good lineplan is characterized by low operational costs and high quality customer service. At this strategic level, low operational costs is obtained by minimising the estimated number of required vehicles, while high quality customer service is often obtained by maximising the estimated number of passengers with a direct connection.

Once a lineplan has been established, the basic timetable is constructed. That is, points in time are assigned to each arrival and departure event of the individual lines at each station. In periodic railway systems, each arrival and departure is repeated with a given time interval - the period length, e.g. one hour.

In the final phases of the planning process rolling stock and crew are scheduled and the routing and platform assignment of trains are determined. In rolling stock scheduling trips are generated and the specific rolling stock units are assigned to trips, such that requirements on e.g. maintenance are respected. Crew scheduling assigns the personnel to operate the train units and service passengers, while respecting the need for e.g. regular restperiods and demands on varied trip schedule for drivers.

Attempts has been made to integrate certain steps or aspects from different phases of the traditional planning process.

3.2 The Periodic Railway Timetabling Problem

Constructing a good timetable is a crucial step in the planning process to ensure low operational costs and high level passenger service, while operational constraints are satisfied according to known limitations of the infrastructure, rolling stock, and human factors.

Such operational constraints impose restrictions on headway, train running- and stopping time, and turn around time at terminal stations. Additionally, constraints may be introduced that restrict waiting time between two connecting trains in order to ensure a high service level.

The set of constraints and possibly an objective function reflecting costs or passenger service define a *timetabling instance*. A feasible solution to the timetabling instance is a timetable that can be operated under the given circumstances.

The Periodic Railway Timetabling Problem is the problem of scheduling arrival and departure events of train lines at stations given a lineplan, such that each event is recurring with some interval T, and the requirements wrt. safety, running and stopping time etc. are respected.

An event may be characterised by a station, a line, a direction, the type of event (arrival or departure), and an index denoting which timeperiod it takes place. However, since each event is recurring with fixed interval, it is sufficient to schedule the events in some reference period. Let $\varepsilon_{arr}(s, l, \delta)$ and $\varepsilon_{dep}(s, l, \delta)$ denote the arrival respectively departure of line l in direction δ at station s in the reference period. Similarly, let $\pi_{arr}(s, l, \delta) \in [0, T[$ and $\pi_{dep}(s, l, \delta) \in [0, T[$ be the point in time, when $\varepsilon_{arr}(s, l, \delta)$ resp. $\varepsilon_{dep}(s, l, \delta)$ occurs. Also, let N be the set of all events in the reference period.

In the following, different types of constraints used in scheduling railway systems are identified. Connection constraints was introduced by Nachtigall in [21], running time, headway, and line synchronisation constraints was used by Odijk in [20], turnaround constraints was introduced by Lindner in [18], while constraints allowing for merging of lines at a terminal station [22] and matching of linesegments [13] was identified by Peeters respectively Liebchen and Möhring.

3.2.1 Dwell and running time constraints

Minimum (d^-) and maximum (d^+) dwell time of a line l at a station s in the direction δ, may be imposed by the constraint

$$\pi_{dep}(s, l, \delta) - \pi_{arr}(s, l, \delta) \in [d^-, d^+]_T$$

Similarly, minimum (r^-) and maximum (r^+) running time of a line l between two (adjacent) stations s_1 and s_2 in the direction δ, may be imposed by the constraint

$$\pi_{arr}(s_2, l, \delta) - \pi_{dep}(s_1, l, \delta) \in [r^-, r^+]_T$$

The minimum running time is composed of the actual minimum travel time of the train when driving at the maximum possible (or maximum allowed) speed, as well as a certain amount of slack time (or buffer time), so recovering from delays is possible.

3.2.2 Turnaround constraints

Limits (t^-, t^+) on the turnaround time of line l at station s, may be given by

$$\pi_{arr}(s, l, \delta) - \pi_{dep}(s, l, \delta') \in [t^-, t^+]_T$$

or

$$\pi_{dep}(s, l, \delta) - \pi_{arr}(s, l, \delta') \in [t^-, t^+]_T$$

where δ' denotes the opposite direction of δ. The first constraint type may be used, when turnaround occurs at a passenger platform and only arrival and departure events at s are given, while the latter constraint type may be used, when turnaround occurs at a shunting track and arrival and departure events at s are given for both directions. Turning of a train at a platform track usually requires a dedicated platform not used by other lines, which put a high demand on platform capacity. In figure 3.1, dwell, running, and turnaround constraints are shown for a single line. It is seen that, the northern terminal station (with four events) uses shunting at turnaround, while no shunting is required at the southern terminal.

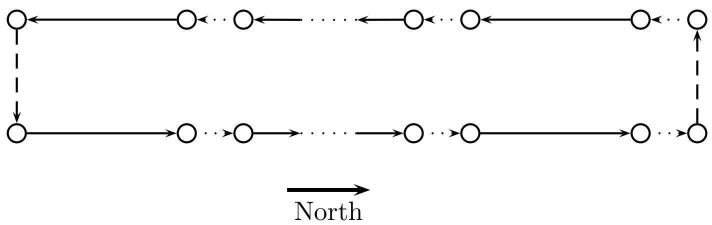


Figure 3.1: PESP subgraph for a single line. Each node represents either an arrival or a departure at a station on the line. Solid edges represent running time constraints, dotted edges represent dwell time constraints, while dashed edges represent turnaround constraints at the two terminal stations.

3.2.3 Headway constraints

Let a track segment q be a piece of (single) track between s_1 and s_2, with δ_1 being the direction from s_1 to s_2 and δ_2 the direction from s_2 to s_1. For each track segment q, a minimum headway h must be ensured between any pair of arrivals of two lines $l_1 \neq l_2$ in the same direction at the beginning of the track segment. That is,

$$\pi_{dep}(s_1, l_1, \delta_1) - \pi_{dep}(s_1, l_2, \delta_1) \in [h, T - h]_T$$

and

$$\pi_{dep}(s_2, l_1, \delta_2) - \pi_{dep}(s_2, l_2, \delta_2) \in [h, T - h]_T$$

must hold for any pair of lines $l_1 \neq l_2$ using q in the same direction.

Also it must hold, that the sequence of trains entering q must be equal to the sequence of trains leaving q, i.e. no train is allowed to overtake another train along q. If r^- and r^+ is the minimum and maximum running time resp. along q, then $r^+ - r^- < h$ ensures no overtaking along the track segment. However, if l_1 and l_2 uses the same platform track at s_2, we must also ensure, that no overtaking occurs at s_2, i.e. $(r^+ - r^-) + (d^+ - d^-) < h$. While r^-, d^-, and h are strict bounds dictated by infrastructure and the number of expected passengers, r^+ and d^+ may be chosen to accomodate this requirement.

To ensure no collissions between trains using the track segment q in opposite directions, assuming maximum running times r_1^+ in the direction δ_1 and r_2^+ in the direction δ_2, the following constraints for any two pair of lines $l_1 \neq l_2$ using q in the direction δ_1 and δ_2, respectively, is sufficient

$$\pi_{dep}(s_1, l_1, \delta_1) - \pi_{arr}(s_1, l_2, \delta_2) \in [h, T - (r_1^+ + r_2^+) - h]_T$$

Here, the lower bound ensures a minimum headway h at s_1 and the upper bound ensures a minimum headway h at s_2, as shown in figure 3.2. That the minimum headway at s_2 is ensured can be realised by reversing the constraints as shown in figure 3.3 and summing the lower bounds along the path $1, 2, 3, 4$, yielding

$$-r_1^+ - (T - r_1^+ - r_2^+ - h) - r_2^+ = h - T$$

Since the sum of tensions along the path $1, 2, 3, 4$ equals the tension $\pi_4 - \pi_1$ modulo T, we have

$$(h - T) \bmod T \leq (\pi_4 - \pi_1) \bmod T \tag{3.1}$$
$$\Leftrightarrow \quad h \leq (\pi_4 - \pi_1) \bmod T \tag{3.2}$$

If $r_i^+ = r_i^-$, for $i = 1, 2$, the condition is also necessary. Otherwise, some feasible timetables may be cut off if the actual running times are not at the maximum.

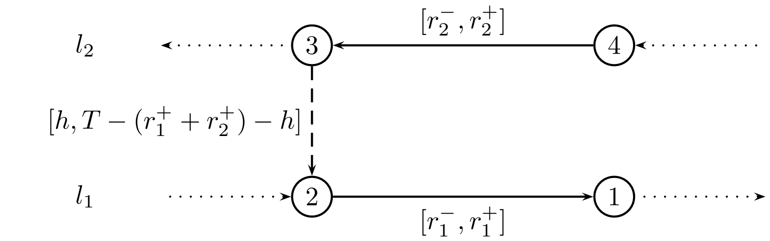


Figure 3.2: PESP constraints ensuring minimum headway h between two lines l_1 and l_2 using track segment q in opposite directions. The events 1 and 3 represents arrival of line l_1 and l_2 resp. at q, while 2 and 4 represents the departure of line l_1 and l_2, resp., from q.

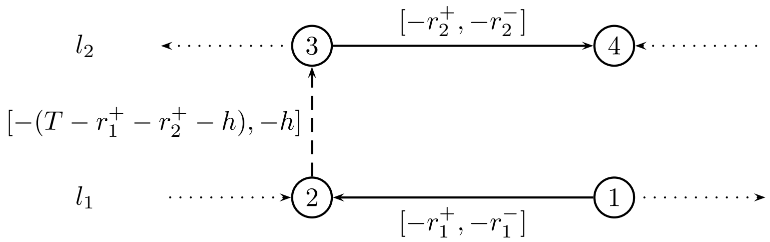


Figure 3.3: Reversing the constraints in figure 3.2.

These headway constraints may be generalised in that the track segment q does not necessarily have to begin and end at a station. E.g., a piece of track between two stations may be subdivided into several track segments by inserting extra nodes. This can be necessary to avoid overtaking along track segments, when the span of the running times are wide.

The constraints mentioned so far deals with fundamental properties of the railway system with respect to infrastructure and safety. A feasible timetable may be constructed using these fundamental constraints only. However, other constraints may be added in order f.ex. to ensure a minimum level of passenger service or to allow the merging of lines at certain stations.

3.2.4 Connection constraints

At certain stations where many people are expected to change from one line to another, it may be attractive to ensure an upper limit on the waiting time of passengers changing lines. This is done by introducing connection constraints. Let the minimum connection time g^- reflect the expected time it takes to transfer from line l_1 in direction δ_1 to line l_2 in direction δ_2 at station s, that is the time it takes to get in and out of the trains plus walking time between platforms. Also, let the maximum connection time g^+ be the transfer time plus the maximum allowed waiting time for passengers transferring from line l_1 in direction δ_1 to line l_2 in direction δ_2 at station s. Then the connection constraints may be formulated as

$$\pi_{dep}(s, l_2, \delta_2) - \pi_{arr}(s, l_1, \delta_1) \in [g^-, g^+]_T$$

3.2.5 Line synchronisation

At stations served by two or more lines in the same direction and with at least one common successive station, an attractive timetable will have an even distribution of departures in that direction. An even distribution of the k lines $l_1, l_2, ..., l_k$ in the direction δ at station s, can be achieved by requiring that the minimum time between the departures of any two lines is T/k, i.e.

$$\pi_{dep}(s, l_i, \delta) - \pi_{dep}(s, l_j, \delta) \in \left[\frac{T}{k}, T - \frac{T}{k}\right]_T, \ \forall i < j \in \{1, 2, ..., k\}$$

More flexibility may be introduced, by allowing (small) deviations up to ϵ time units from the equal distribution,

$$\pi_{dep}(s, l_i, \delta) - \pi_{dep}(s, l_j, \delta) \in \left[\frac{T}{k} - \epsilon, T - \frac{T}{k} + \epsilon\right]_T, \ \forall i < j \in \{1, 2, ..., k\}$$

3.2.6 Merging of lines at terminals

Merging of two lines at a common terminal station may be allowed using the matching constraints described in section 2.6.2. Let l_1 and l_2 be the lines for which we want to allow merging at their common terminal station s, and let δ_1^- and δ_2^- be the direction of l_1 respectively l_2 upon entering s, and δ_1^+ and δ_2^+ be the direction of l_1 respectively l_2 when leaving s. Also, let t^- and t^+ be the minimum and maximum turnaround times at s, and assume that l_1 and l_2 are separated by synchronisation constraints at arrival to and departure from s:

$$\pi_{arr}(s, l_1, \delta_1^-) - \pi_{arr}(s, l_2, \delta_2^-) \in \left[s^-, s^+\right]_T$$
$$\pi_{dep}(s, l_1, \delta_1^+) - \pi_{dep}(s, l_2, \delta_2^+) \in \left[s^-, s^+\right]_T$$

where $s^+ = T - s^-$. Assuming, that the intervals $[t^-, t^+]$ and $[t^- + s^-, t^+ + s^+]$ are disjoint, we can apply the matching constraints (2.16),

$$\pi_{dep}(s, l_i, \delta_i^+) - \pi_{arr}(s, l_j, \delta_j^-) \in \left[t^-, t^+ + s^+\right]_T, \ \forall i, j \in \{1, 2\} \text{ and} \tag{3.3}$$
$$\pi_{dep}(s, l_i, \delta_i^+) - \pi_{arr}(s, l_j, \delta_j^-) \in \left[t^- + s^-, s^+ + T\right]_T, \ \forall i, j \in \{1, 2\} \tag{3.4}$$

However, if we want to be able to get the correct number of train units needed to operate the resulting timetable, we need to sum the tensions for those edges, that describe the resulting matching as explained in section 2.6.4. This requires, that $t^+ + s^+ < T$. If this is not fulfilled, one may insert two dummy events ε_1 and ε_2 *between* the arrival and departure events, replace (3.3-3.4) by

$$\pi(\varepsilon_i) - \pi_{arr}(s, l_j, \delta_j^-) \in \left[0, t^+ - t^- + s^+\right]_T, \ \forall i, j \in \{1, 2\} \text{ and} \tag{3.5}$$
$$\pi(\varepsilon_i) - \pi_{arr}(s, l_j, \delta_j^-) \in \left[s^-, s^+ - t^- + T\right]_T, \ \forall i, j \in \{1, 2\} \tag{3.6}$$

and insert two new constraints with tension fixed as the minimum turnaround-time

$$\pi_{dep}(s, l_i, \delta_i^+) - \pi(\varepsilon_i) \in \left[t^-, t^-\right]_T, \ \forall i \in \{1, 2\}$$

This is shown in figure 3.4.

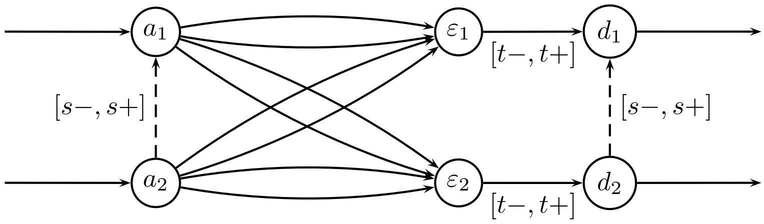


Figure 3.4: PESP subgraph allowing for merging of lines at a common terminal station s. The events a_1 and a_2 represents arrival of line l_1 and l_2 resp. at s, while d_1 and d_2 represents the departure of line l_1 and l_2 resp. The edges without labels correspond to the constraints defined by (3.5-3.6)

3.2.7 Matching of linesegments

So far, the lineplan we have considered was assumed to be given and fixed. However, the matching constraints discussed in section 2.6.3 may also be used to allow certain changes in the lineplan.

The principle is the same as for merging of lines at a terminal station as described previously. However, now we allow the lines to merge at a non-terminal station where the lines share a common track. The assumption is that to avoid collissions an arrival at the station is followed immediately by a departure. This is ensured by the headway constraints.

Consider the two lines l_1 and l_2 in figure 3.5 using the same track in each direction at their common station s. Two new lines l_1' and l_2' can be obtained, such that l_1' is defined by the linesegment of l_1 south of s and the linesegment of l_2 north of s, while l_2' is defined by the linesegment of l_2 south of s and the linesegment of l_1 north of s. Now, two different lineplans $L = \{l_1, l_2\}$ and $L' = \{l_1', l_2'\}$ are obtained covering the same track segments at the same frequencies. In finding an optimal timetable, it may be worthwhile to consider both lineplan L and L' in that one of the two may allow for a *better* timetable. E.g. lineplan L' may be cheaper than lineplan L to operate, requiring less trains.

This can be done by allowing l_1 and l_2 to merge in both directions. I.e. the dwell time constraints $[d^-, d^+]$ at s are replaced by the matching constraints

$$\pi_{dep}(s, l_i, \delta^+) - \pi_{arr}(s, l_j, \delta^+) \in \left[d^-, d^+ + s^+\right]_T \cap \left[d^- + s^-, s^+ + T\right]_T, \ \forall i, j \in \{1, 2\}$$
$$\pi_{dep}(s, l_i, \delta^-) - \pi_{arr}(s, l_j, \delta^-) \in \left[d^-, d^+ + s^+\right]_T \cap \left[d^- + s^-, s^+ + T\right]_T, \ \forall i, j \in \{1, 2\}$$

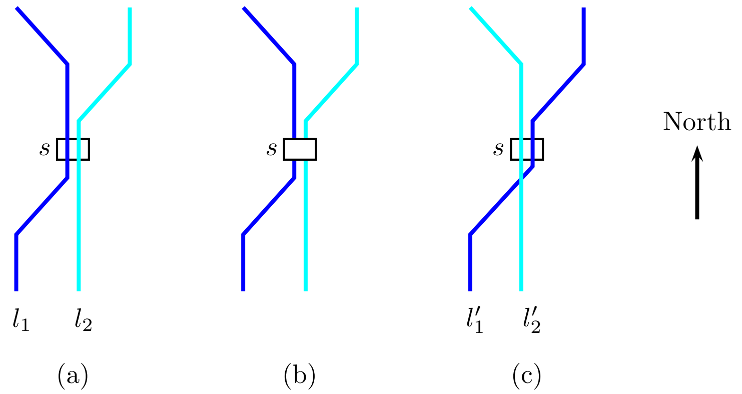


Figure 3.5: Merging of two lines at a non-terminal station s. Two lines l_1 and l_2 with a common station s (a). l_1 and l_2 split into four linesegments, such that each linesegment terminate at s (b). Possible matchings of the four linesegments resulting in the original lineplan $L = \{l_1, l_2\}$ (a) and an alternative lineplan $L' = \{l'_1, l'_2\}$ (c).

where $\delta^+ \neq \delta^-$ denote the two directions. The proper separation of arrivals and departures are enforced by the separation constraints

$$\pi_{arr}(s, l_1, \delta^+) - \pi_{arr}(s, l_2, \delta^+) \in \left[s^-, s^+\right]_T$$
$$\pi_{dep}(s, l_1, \delta^-) - \pi_{dep}(s, l_2, \delta^-) \in \left[s^-, s^+\right]_T$$

such that the assumption $d^+ + s^+ < d^- + 2s^-$ holds. However, to ensure a symmetric lineplan, i.e. that each line serves the same stations in each direction, we must require that if l_i merge with l_j in one direction, l_j must also merge with l_i in the opposite direction. This cannot be done within the framework of the PESP. However, additional constraints may be introduced.

Consider the cycles c^{δ^+} and c^{δ^-} in figure 3.6 (b). As shown in section 2.6.4, $y_{c_N}, y_{c_S} \in \{0, 1\}$. Also, if $y_{c_N} = 0$ no merging occurs in the northern direction since event 4 occur immediately after event 2. Similarly, if $y_{c_S} = 0$ no merging occurs in the southern direction. Hence, requiring that the resulting lineplan is symmetric is equivalent to requiring $y_{c_N} = y_{c_S}$.

In general, for a set $L = \{l_1, l_2, ..., l_k\}$ of k lines servicing a common non-terminal station s using the same track in each direction we may allow merging of the lines

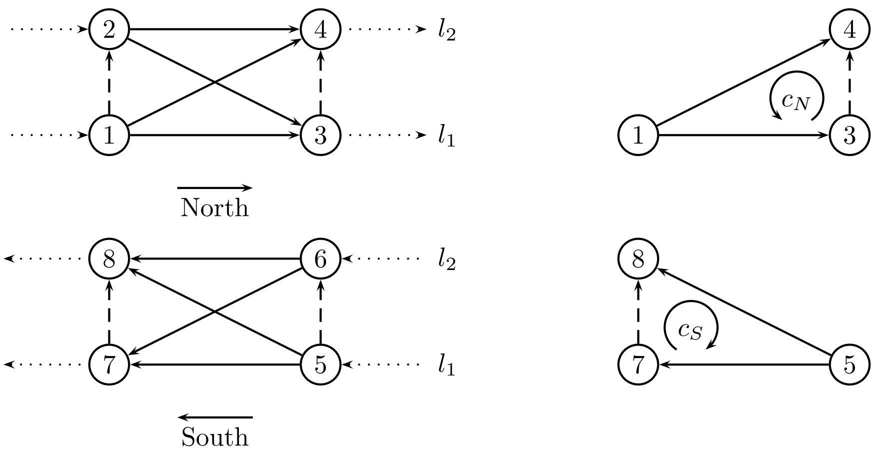


Figure 3.6: PESP subgraphs allowing for merging of lines at a common non-terminal station s (a). The events 1 and 2 represents arrival in the northern direction of line l_1 and l_2 resp. at s, while 3 and 4 represents the departures in the northern direction of line l_1 and l_2 respectively. Likewise, the events 5 and 6 represents arrivals in the southern direction of line l_1 and l_2 resp. at s, while 7 and 8 represents the respective departures. Dotted arcs represent running time constraints, while dashed arcs represent separation constraints and solid arcs represent matching constraints.

at s by replacing the dwell time constraints $[d^-, d^+]$ by the matching constraints

$$\pi_{dep}(s, l_i, \delta^+) - \pi_{arr}(s, l_j, \delta^+) \in \left[d^-, d^+ + s^+\right]_T \cap \left[d^- + s^-, s^+ + T\right]_T, \ \forall i, j \in L$$
$$\pi_{dep}(s, l_i, \delta^-) - \pi_{arr}(s, l_j, \delta^-) \in \left[d^-, d^+ + s^+\right]_T \cap \left[d^- + s^-, s^+ + T\right]_T, \ \forall i, j \in L$$

and ensuring the minimum separation of the lines, by the separation constraints

$$\pi_{arr}(s, l_i, \delta^+) - \pi_{arr}(s, l_j, \delta^+) \in \left[s^-, s^+\right]_T \ \forall i \neq j \in L$$
$$\pi_{dep}(s, l_1, \delta^-) - \pi_{dep}(s, l_2, \delta^-) \in \left[s^-, s^+\right]_T \ \forall i \neq j \in L$$

To ensure a symmetric lineplan, we consider, for each pair of lines $(l_i, l_j) \in L \times L$ and each direction $\delta \in \{\delta^+, \delta^-\}$, the cycles described in section 2.6.4. Define first the events $\varepsilon_i = \varepsilon_{arr}(s, l_i, \delta)$, $\varepsilon_j = \varepsilon_{dep}(s, l_j, \delta)$, and $\varepsilon_k = \varepsilon_{dep}(s, l_k, \delta)$.

Now let $c^\delta_{ijk} = \{a'_{ij}, a_{jk}, a'_{ik}\}$ be the cycle visiting the events ε_i, ε_j, and ε_k, in that order, and let a'_{ij} and a'_{ik} have span intervals $[d^-, d^+ + s^+]$, as shown in figure 3.7. Denote by J the set of departure events at station s in the direction δ

As shown in section 2.6.4, if a_{jk} is oriented along the orientation of c^{δ}_{ijk}, $y_{c^{\delta}_{ijk}} = 0$ if and only if the events i, j, k occur in the sequence i, j, k, otherwise $y_{c^{\delta}_{ijk}} = 1$.

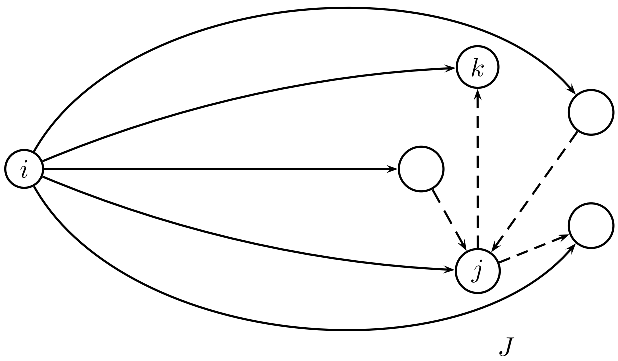


Figure 3.7: The event i and the events in J. Solid edges are matching constraints, while dashed edges symbolise separation constraints. To simplify the figure, only separation constraints with j as initial or final vertex are included.

Let $N^+_j \subset J$ be the set of nodes in J, that are in the out-neighbourhood of j and $N^-_j \subset J$ the set of nodes in J, that are in the in-neighbourhood of j. If $k \in N^+_j$ then $y_{c^{\delta}_{ijk}} = 0$ imply that line l_i and line l_k are not merged. Also, if $k \in N^-_j$ then $y_{c^{\delta}_{ikj}} = 1$ imply that line l_i and line l_k are not merged. Note that two lines l_i and l_j merge in some direction if and only if l_i only merge with l_j and l_j only merge with l_i. Hence, two lines l_i and l_j merge in the direction δ if and only if

$$\varphi^{\delta}_{ij} = \sum_{k \in N^+_j} y_{c^{\delta}_{ijk}} + \sum_{k \in N^-_j} \left(1 - y_{c^{\delta}_{ikj}}\right) = 0$$

We can therefore enforce the lineplan to be symmetric by only allowing merging of an ordered pair of lines in one direction if the same pair of lines also merge in the opposite direction. In terms of φ, this can be expressed as

$$\varphi_{ij}^{\delta^-} \leq M\varphi_{ji}^{\delta^+}$$
$$\varphi_{ij}^{\delta^+} \leq M\varphi_{ji}^{\delta^-}$$

where M is some integer greater than or equal to the number of lines at s.

3.2.8 Objective

The two most common objectives for the timetabing planning phase is to minimise the rolling stock required to operate the timetable and minimise total passenger waiting time in the system.

The first objective is easily obtained in the MIP formulation of the model, since the duration of each train cycle (dwell, running, and turnaround edges for each line) determines the number of train units required to operate each line. When merging between two lines is allowed, the turnaround edges are replaced in the objective function by the relevant matching and separation edges as described in section 2.6.4. The number of train units needed to operate the timetable is then given by

$$Z = \frac{1}{T} \sum_{a \in A'} \vartheta_a$$

where A' is the set of all running, dwell, and turnaround time edges.

Passenger waiting time for some connections defined by the set of connection constraints A'' can be simultaneously minimised by considering the contribution $w_a(\vartheta_a - d_a^-)$ of connection constraint a to the objective function. The weight w_a of that particular connection may for example be a function of the estimated number of passengers using the connection. Note, that $\vartheta_a - d_a^-$ denote the waiting time only and not the total transfer time (ϑ_a).

The combined objective function to be minimised is then

$$Z = \frac{1}{T} \sum_{a \in A'} \vartheta_a + \sum_{a \in A''} w_a(\vartheta_a - d_a^-)$$

3.3 Limitations of the Model

The Periodic Event Scheduling Problem can handle the most important aspects of traditional timetabling for periodic railway services. However, certain aspects may not be modeled using PESP. These include symmetry of timetables and different types of rolling stock equipment. Also, routing of passengers is not taken into account making it difficult to estimate e.g. average passenger waiting and travel time in the system.

According to practitioners [11] symmetry is an important property of periodic timetables, and is also to a large extent maintained in the 2006 timetable for S-train. However, timetable symmetry may not be modeled within the framework of the PESP. Recall that a timetable is symmetric if and only if the sum of the arrival time in one direction and the departure time in the opposite direction for the same line at the same station equals an integer multiple of the timeperiod. Liebchen shows in [11] that timetable symmetry can be ensured by introducing side constraints of the form

$$(\pi_{dep}(s, l, \delta) + \pi_{arr}(s, l, \delta')) \bmod T \leq 0$$

where δ' is the opposite direction of δ.

Varying passenger demand during each day and during the week puts different requirements to the service provided by a railway service. To accomodate varying demands, usually different lineplans and timetables are constructed for different periods of the day and the week. For example, a high frequent service may be offered during rush hour when demand is high, while lower frequencies are offered during evening periods and Sundays. This model can be applied to any such planning period, and it is possible to construct optimal timetables for each period separately. However, the transition between two planning periods and between periods of operation and periods of no operation (e.g. at the beginning and end of the day, if no night train service is provided) are not considered. This problem of routing vehicles, such that the train units used in the current planning period are *ready* to operate the new timetable at the beginning of the next period, is known as the rolling stock circulation problem. This include taking some train units out of operation and/or inserting train units that are not currently in operation. The planning of rolling stock circulation is heavily dependent on the location of rolling stock depots and may require deadheading, i.e. moving train units not in operation between two points in the network.

From a passenger perspective, a very attractive timetable is one, that minimises the average passenger travel and/or waiting time. For this purpose reliable data for the route chosen by individual passengers is needed. These data may be estimated by statistical methods for a single timetable, or, if passengers are assumed always to choose the shortest route, by simply calculating the shortest path in an appropriate graph. However, when a new timetable is introduced these passenger flows are likely to change. Especially if also the lineplan changes, which requires some passengers to make more line changes, while others need to make less changes. To allow for *optimal* routing of passengers in the network in the MIP formulation, an extra set of variables denoting passenger flow on each of the edges in the PESP constraint graph is needed. Minimising the average passenger travel time in the network hence requires a quadratic objective function making even small instances difficult to solve.

A train unit may consist of various types and number of rolling stock. E.g., one train unit may consist of a locomotive and four train cars, while another train unit may consist of only two motorised train cars. The availability and difference in operating costs for different combination of rolling stock is not considered in the model.

Chapter 4

Timetable Construction for S-train

In this chapter a number of experiments will be performed to investigate the applicability of the PESP model to timetable construction for the Copenhagen S-train system. Two types of lineplans are considered. One with eleven lines each operated at 20 minute intervals and another with six lines each operated at 10 minute intervals. Is it possible to determine the minimum number of train units required in a reasonable amount of time? Furthermore, is it possible to allow merging of lines and matching of linesegments and still obtain solutions in reasonable time, and if so, does this improve the optimal solution? Finally, the effect of chain cutting planes on the solution time is evaluated.

4.1 The S-train Network

The S-train is a commuter train service for the greater Copenhagen area. It serves 85 stations along a total track length of 170 km [1]. The infrastructure is almost entirely double track, except for a 500 m track segment crossing a bridge between Værløse and Farum stations. The track network is star-shaped with a central track segment, three radial 'fingers' to the north of the central segment, three 'fingers' to the south and one (half-) circular track segment intersecting all 'fingers'. Figure (4.1) shows a schematic representation of the network. Stations have between two and five platform tracks.

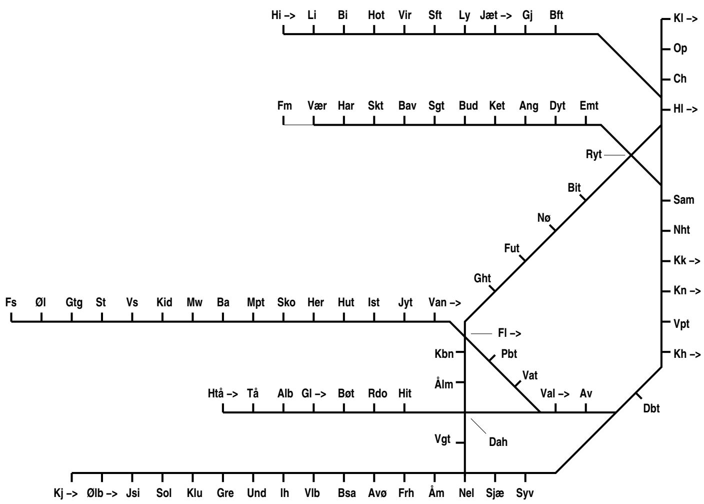


Figure 4.1: Schematic representation of the track layout. All tracksegments are double track except for a section between Farum(Fm) and Værløse(Vær). -> indicates that connections can be made to other train networks.

On an average day 240000 passengers uses the S-trains. The largest passenger station is Nørreport on the central track segment with app. 80000 passengers on a weekday [5].

The S-train network is part of a larger public transit transportation network comprising local-, metro-, regional-, and intercity trains, as well as busses. Connections between the other train networks and the S-train network are possible at certain stations marked by (->) in figure 4.1. Changing between S-train and intercity trains is possible at four stations, while connection to regional trains can be made at seven stations, to metro at three stations, and to local train operators at three stations.

The current lineplan is operated with a fixed frequency of 20 minutes for all lines. It consists of three main types of lines, namely core lines running all day Monday through Sunday, daytime lines running during the daytime period Monday through Saturday, and a single rush hour line with only a few departures during the morning rush hour period Monday through Friday. Furthermore, a few lines are extended during certain periods. The operating times are summarised in table (4.1). All lines except the rush hour line are symmetric, i.e. they serve the same stations in each direction, but in opposite order.

In the published timetable for 2005, there were seven core lines (A, B, C, E, F, F+, H), four daytime lines (A+, B+, Bx, H+) and one rush hour line (Ex). Line E was a daytime line to Hi, but turns in Ly outside the daytime period. Line F+ was a core line between Nel and Hl, but was extended to Kl during the weekend daytime. In 2006, line Bx was taken out, and line F+ was extended to Kl all days during daytime. Furthermore, line A+ was extended to Bud during Monday to Friday. The lines and line types for the 2005 and 2006 timetable are shown in table (4.2). Figure (4.2) shows the 2005 lineplan during morning rush hour.

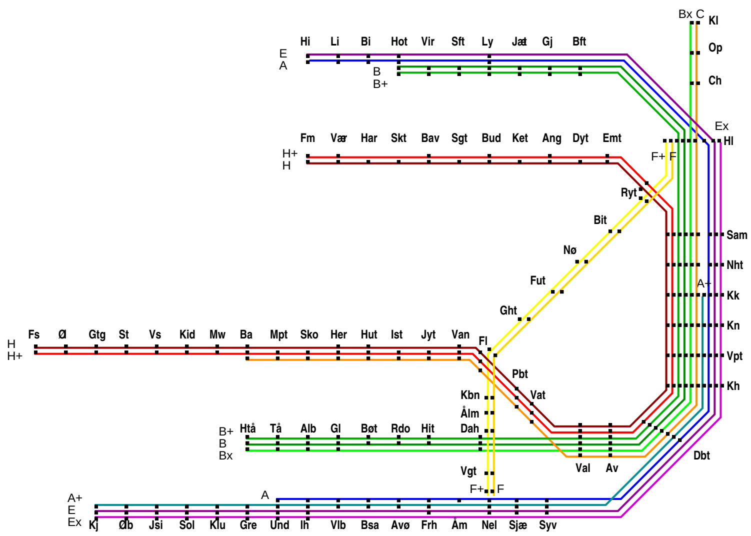


Figure 4.2: The 2005 lineplan during morning rush hour. A black square indicates whether the line is stopping at the respective station.

	Monday - Friday	Saturday	Sunday
Core	05:00 - 00:30	05:00 - 00:30	06:00 - 00:30
Daytime M-S	06:00 - 19:00	09:00 - 15:00	-
Daytime M-F	06:00 - 19:00	-	-
Rush hour	06:45 - 07:45	-	-

Table 4.1: Approximate operating times for line types.

Line type	Lines (2005)	Lines (2006)
Core	A B C E(Ly) F F+ H	A B C E(Ly) F F+ H
Daytime M-S	A+ B+ E(Hi) H+	A+(Kk) B+ E(Hi) H+
Daytime M-F	Bx	A+(Bud)
Rush hour	Ex	Ex

Table 4.2: Grouping of lines according to linetype in the 2005 and 2006 published timetables.

4.2 Model Assumptions

In the following sections, timetables will be constructed for the S-train network based on the model presented in section 3.2. Some assumptions regarding the network has been made.

4.2.1 Infrastructure

A new station Ny Ellebjerg (Nel) between Sjælør and Ellebjerg (Elb) is under construction and is to be ready by 2007. At that time Ellebjerg station will be closed. At Nel connections can be made between trains on the southern (Køge Bugt) track and the circular (Ringbane) track. Therefore Elb is replaced in the model by Nel and all lines are to stop in Nel in all scenarios. The running times to and from Nel on the southern track are assumed to be the same as to and from Elb (which is not entirely correct). The running times to and from Nel on the circular track are assumed to be the same as to and from the temporary Ny Ellebjerg station at Gl. Køge Landevej.

The single track segment between Fm and Vær is inserted with an assumed scheduled running time from Fm of 1 minute. Scheduled running time across the single track segment are assumed to be 0.4 minutes, i.e. trains are assumed to pass at a speed of 80 km/h. This running time is fixed in both directions for all scenarios.

The headway between two departures from the same station in the same direction is in practice minimum two minutes. The theoretical minimum is assumed to be 1.5 minutes. In the following a minimum headway of 2 minutes is used for experiments with 20 minute line frequencies, while a minimum headway of 1.9 minutes is used for experiments with 10 minute line frequencies.

4.2.2 Dwell, running, and turnaround times

Dwell times at stations and running times between stations are based on the published timetable for 2006 [6], which specifies scheduled departure times at each station for each line in half minutes, as well as scheduled dwell times (10, 20, or 30 seconds) for each station.

Minimum and maximum running times are found by decreasing respectively increasing the scheduled running time by a certain percentage of the scheduled time. In the case where the scheduled running time differs between lines on the respective track segment, the scheduled running time for an arbitrary line is chosen. The dwell times are assumed to be fixed and are chosen as the scheduled dwell time according to [6].

Minimum turnaround times for all scenarios are set to 6 minutes for platform turnaround and 10 minutes for turnaround when shunting is needed. Recall that platform turnaround occur while the train is parked at a track adjacent to a passenger platform. Maximum turnaround time is assumed to be 15 minutes for platform turnaround and 18 minutes when shunting is needed. These maximum times are designed to comply with the assumption for merging constraints to be disjunct, so that merging of lines at terminals easily can be implemented. Shunting is required for turnaround at Bud and Ba. Shunting is also used for turnaround of line F/F+ at Hl to ensure adjacency to connecting trains.

4.2.3 Objective

The objective of the timetable construction in the experiments is to minimise the required number of train units needed to operate the timetable. In section 4.5.3, however, a combination of rolling stock and passenger waiting time for some connections are minimised.

4.3 Computational Environment

The timetabling instance is modeled in C++, while the underlying PESP model is developed in GAMS. The PESP instance is solved using ILOG CPLEX 9.020. Computations are performed on a 1000 Mhz SUN Fire V440 with four processors and 8 GB RAM running Solaris 9.

4.3.1 CPLEX Parameters

Two configurations of CPLEX parameters has been used in the experiments. On small and easy timetable instances CPLEX was run with all parameters set to their default values. When solution time was long the CPLEX parameters *varsel* and *mipemphasis* was changed from their default settings. In particular, the variable selection method (*varsel*) was set to strong branching, while MIP emphasis (*mipemphasis*) was set to optimality.

When strong branching is selected a number of sub problems are solved at each node in the branch-and-bound tree to determine the best variable to branch on. This reduces the size of the branch-and-bound tree considerably.

Setting MIP emphasis to optimality causes the branch-and-bound algorithm to first process nodes in the branch-and-bound tree that have the best lower bound (for minimisation). This leads to a *shallow* branching tree with relatively few nodes necessary to be evaluated. Even though an optimal solution is often found faster for a large problem, the first feasible solution may take long to find. As default, MIP emphasis is set to balance optimality and feasibility. Therefore a depth-first search of the branch-and-bound tree is employed until a feasible solution is obtained, whereafter optimality is emphasised.

4.4 Experiments with Twenty Minute Interval Lineplan

In this section timetables are generated for a lineplan based on the 2005 lineplan, with line A+ extended to Bud as in the published timetable for 2006. This is based on the assumption that line Bx will be reinserted at some point, and line F+ therefore will be shortened to Hl during Monday to Friday. Line Ex is omitted, as this line is operated mainly in one direction (Kj - Hl). Transferring vehicles without passengers (deadheading) is necessary from Kh to Kj when merging with other lines are not allowed in order to maintain a circulation of rolling stock in the PESP. In practice, the vehicles operating Ex are probably taken out of operation at the end of the morning rush hour. Figure (4.3) shows the lineplan for the basic scenario.

Initially, strict synchronisation between lines are enforced pairwise, such that two lines form a 10 minute interval service at certain track segments. Table (4.3) shows where synchronisation is introduced.

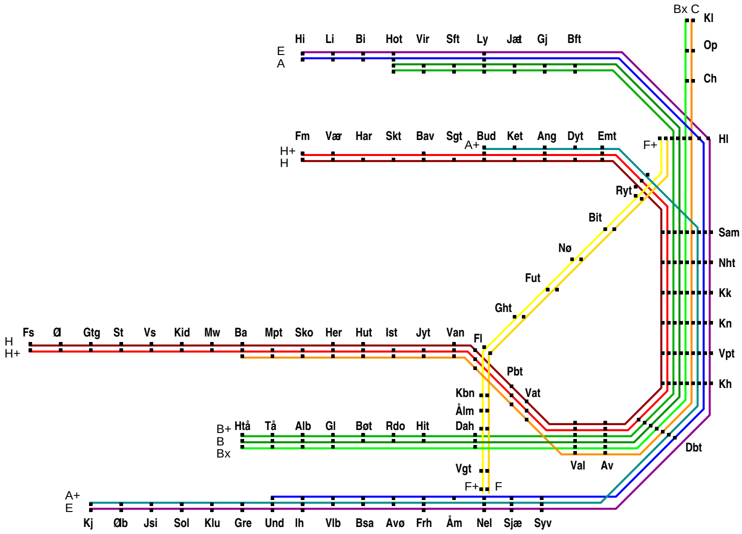


Figure 4.3: The lineplan for the basic scenario. A black square indicates whether the line is stopping at the respective station.

4.4.1 Basic scenario

First timetables are constructed that explore the possibility of variable running times. Instances in which running times are assumed to be fixed are infeasible. Of course, instances with fixed running times equal to those in the schedule should be feasible. However, recall that for the same track segment different line may have different running times in the published timetable, whereas we assume lines to have the same bounds on running time along the same track segment. In the following timetables with running times of up to 1% and 5% deviation from the scheduled running time are considered.

When strict 10 minute synchronisation are enforced and no merging is allowed, PESP instances with 1124 events, 2033 constraints, and 910 fundamental cycles are obtained. The timetabling instance with up to 1% deviation is denoted instance 20a, while the timetabling instance with maximum 5% deviation is denoted instance 20b. Instance 20a yields a minimum rolling stock usage of 77 train units, while instance 20b yields a solution using 75 train units. Table 4.4 shows the rolling stock usage on the different lines for the latter timetable. The CPLEX calculation time is 3 s. and 54 s. respectively.

Synchronise lines	at stations
C Bx	Kl-Ch
B B+	All stations
F F+	All stations
A E	Hi-Hot
A+ E	Kj-Gre
H H+	Fs-Mw, Fm-Vær

Table 4.3: Synchronisation of lines.

Line	A	A+	B	B+	Bx	C	E	F	F+	H	H+	total
Train units	8	8	6	6	6	6	9	3	3	10	10	75

Table 4.4: Rolling stock usage on lines for basic twenty minute lineplan with running times deviating up to 5% from the scheduled running time.

4.4.2 Merging at terminals

Next, merging of lines are allowed at terminal stations between pairs of lines that both turnaround at that station. In particular, merging is allowed between A and E in Hil, H and H+ in Fm and Fs, B and B+ in Hot and Htå, C and Bx in Kl, F and F+ in Hl and Nel, as well as, between A+ and E in Kj. As two extra events and nine extra constraints are inserted for each of the nine merging possibilities, this results in an instance with 1142 events, 2114 constraints, and 973 fundamental cycles. Instance 20c and 20d refer to the timetabling instances with up to 1% respectively 5% deviation from the scheduled running time.

Allowing merging at these terminals reduced the required rolling stock to 74 for instance 20c, and 73 for instance 20d. That is a reduction of 3 respectively 2 train units compared to the results without merging (instance 20a and 20b). Table 4.5 shows that the reduction in rolling stock for instance 20d is due to merging of line A/A+/E and H/H+.

Line	A/A+/E	B/B+	Bx/C	F/F+	H/H+	total
Train units	24	12	12	6	19	73

Table 4.5: Rolling stock usage on lines for twenty minute lineplan allowing merging at terminals with running times deviating up to 5% from the scheduled running time (instance 20d).

Figure 4.4 shows a timetable graph for the southern track segment from Kj to Syv for the obtained timetable. The vertical axis represent time in the twenty minute timeperiod, while the vertical axis represent the physical location on the track network (the station). Each coloured linesegment represent either dwell time (vertical segments) or running time of the train lines. Hence, the end of each linesegment represent either an arrival or a departure of a train line at the given time and station. Note, that both directions of a train line are depicted, so that running time in the northern direction is represented by SW-NE oriented line segments, while running time in the southern direction is represented by SE-NW oriented line segments.

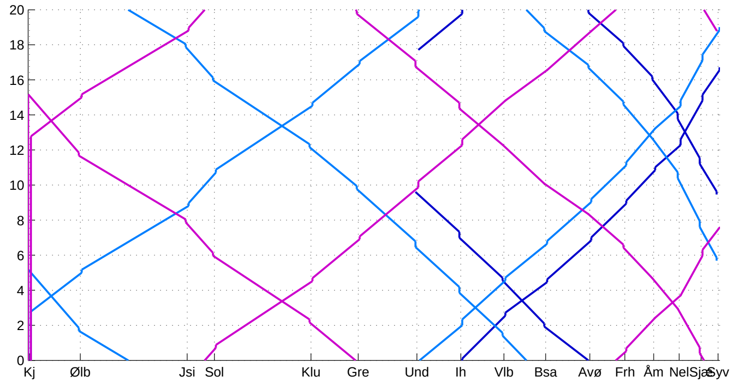


Figure 4.4: Visual representation of timetable when merging is allowed at terminals for the tracksegment Kj - Syv. It is seen that, merging of lines A+ (cyan) and E (purple) occur at Kj, as the time between arrival of line A+ and departure of line E (app. 8 min.) is in the interval [6,15].

4.4.3 Merging at Vesterport

Now, modification of the existing lineplan is allowed, by allowing merging of lines at Vpt on the central track segment. However, allowing merging between all lines, results in a timetabling instance which is too large to be solved. Therefore, four lines (Bx, C, H, H+) are chosen, for which merging is allowed. A span of +/-5% on the running times is used. After 20 hours of calculation time, optimality had not been proven by CPLEX. However, an optimal solution of 73 train units had been reached. (Best possible solution objective was 72.12). Hence, no reduction of rolling stock was obtained by allowing merging of only these four lines. This can be due to the relatively small number of lineplans (24) considered and the assumption (2.17) enforcing a minimum separation (¿4) between the lines allowed to merge.

For experiments, where merging is allowed between more than four lines, no integer solution was obtained within 20 hours of computation time.

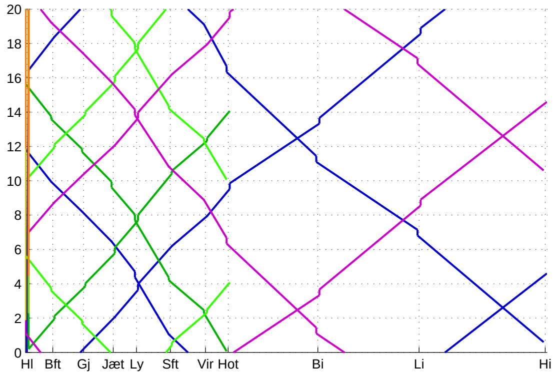


Figure 4.5: Visual representation of timetable when merging is allowed at terminals for the tracksegment Hl - Hi (b). It is seen that, lines A (blue) and E (purple) merge in Hi, while there is no merging between B (green) and B+ (light green) in Hot.

4.5 Experiments with Ten Minute Interval Lineplan

In this section, I will investigate a lineplan, in which all lines are operated at a fixed interval of ten minutes.

The motivation for introducing ten minute frequency of lines is first of all to reduce the number lines and thereby reduce the complexity of the timetabling instance to be solved, in the hope that solutions allowing merging of lines at a central track segment can be obtained. For the passengers, the result is a less complex lineplan and fewer departure times to remember.

The initial ten minute frequency lineplan is constructed to be close to the basic twenty minute frequency lineplan described previously. This is possible since many track segments in the basic lineplan are serviced by pairs of lines (e.g. B and B+ have the same route from Htå to Hot). However, since not all lines are *paired*, a choice has to be made as to which *single* lines are upgraded to ten minute frequency and which are omitted. Line C and A+ are chosen to be in the lineplan, while Bx is omitted. This results in a lineplan with increased frequency (one extra train each 20 minutes) on the segments Ba-Fl, Bud-Ryt, Dbt-Sam, and Und-Syv, but lower frequency (one train less each 20 minute) on the segment Htå-Dah. The remaining parts of the network has same frequency as in the basic 20 minute lineplan. The resulting lineplan is shown in figure 4.6. The stopping pattern of each line has been revised slightly.

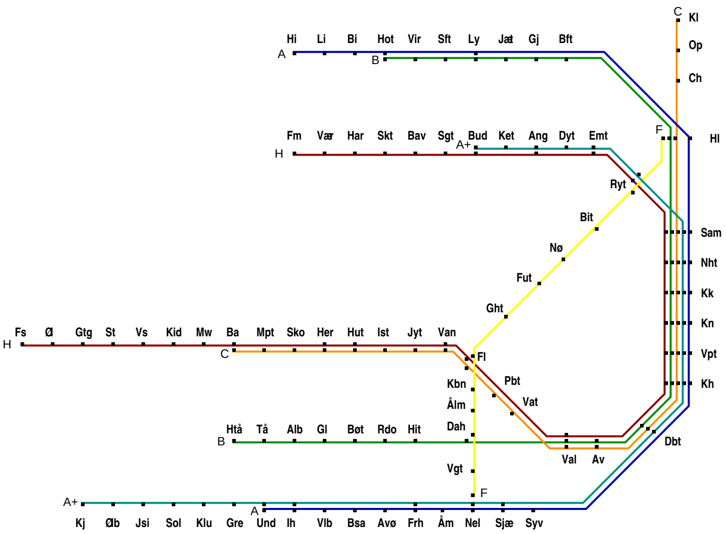


Figure 4.6: Lineplan with ten minute frequencies for all lines.

4.5.1 Basic scenario

In all instances a deviation of up to 5% from the scheduled running time is allowed, whereas dwelltime is fixed. Turnaround time is defined as in the previous section. No sycnchronisation between lines is enforced. Initially, no merging is allowed and minimum headway is set to 2 minutes as in the previous instances. This yields a PESP instance of 614 nodes, 827 edges, and 214 cycle basis edges.

Requiring a minimum headway of 2 minutes results in an infeasible timetabling instance (since the timeperiod is 10 minutes and 5 lines uses the central track segment, all lines must be separated by exactly 2 minutes at the central segment and therefore have fixed running time along the central track segment). Therefore, the minimum headway is reduced to 1.9 minutes in the following runs. With no merging of lines a timetable is obtained using 79 train units. The CPLEX solution time is 0.21 seconds. The rolling stock usage on the different lines is shown in table 4.6.

Line	A	A+	B	C	F	H	total
Train units	15	15	12	12	6	19	79

Table 4.6: Rolling stock usage on lines for basic ten minute lineplan.

4.5.2 Merging of lines

Now, we want to consider a set of lineplans allowing merging of the existing lines at Vesterport(Vpt) station. Five lines (all lines except line F) uses Vpt, and therefore $5! = 120$ lineplans are considered. The minimum headway is set to 1.9 minutes as in the previous instance.

After 20 hours of running CPLEX had still not determined the optimal solution. However, the absolute gap was only 1.074 train units, yielding an optimal value of either 76 or 77 train units. As the best possible solution is very close to 76, it is not likely that 76 train units allows a feasible integer solution. Another run may provide the evidence. The solution using 77 units gives a reduction of 2 train units compared to the scenario without line merging at Vpt, which would give a considerable reduction of operational costs.

In a second run, the MIP emphasis and the variable selection strategy in the CPLEX solution procedure is changed from their default values. The MIP emphasis is set to emphasize finding an optimal solution, while the variable selection strategy is set to strong branching. The optimal solution is found after app. 7.5 hours, yielding a rolling stock usage of 77 train units.

Figure 4.7 shows a visual representation of the resulting timetable for the central track segment. E.g. it is seen that the southern segment of line H (red) merges with the northern segment of line C (orange) in both directions, hence creating a *new* line between Fs and Kl. In figure 4.8 the resulting lineplan is depicted.

It should be noted that there are possibly many lineplans yielding a rolling stock usage of 77 train units. Extending the objective function to also minimise passenger travel and waiting time may thus result in a timetable with the same rolling stock usage, which is more attractive for passengers.

Currently, trains from Holte, Hillerød and Klampenborg may continue towards Ny Ellebjerg via Flintholm by changing to the circular track at Hellerup, hence bypassing the central tracksegment. The planned infrastructure at Ny Ellebjerg station, however, does not allow for trains on the circular track segment to continue on the southern track and vice versa. In the following such an extension of the infrastructure is considered. This allows for merging of lines at Hl and Nel stations, thereby allowing line F to be extended in both northern and southern

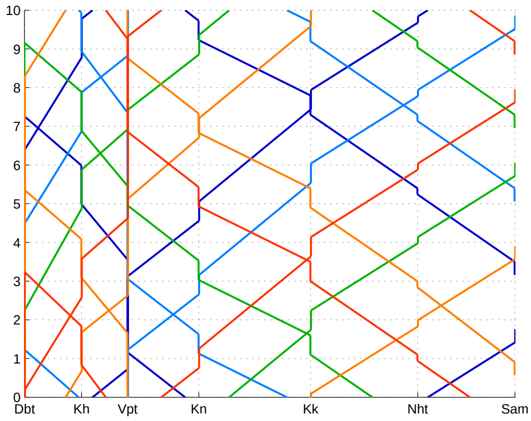


Figure 4.7: Visual representation of timetable with 10 minutes frequencies on all lines and merging at Vpt.

direction.

We now consider lineplan 10b and allow merging at Hl and Nel, but not at Vpt, hence an optimal timetable without merging requires 77 train units as obtained above. After 20 hours of calculation the best integer solution uses 77 train units with a lower bound of 76.4. Hence, the optimal schedule uses the same amount of train units (77) as the original lineplan, and no further reduction in cost was immediately obtained. However, allowing also merging at Vpt will enlarge the solution space and in that case the cheapest solution may comprise an extension of line F.

4.5.3 Connection constraints

So far, in the construction of timetables, passenger service level has not been considered. This means that the obtained timetable and lineplan, may force a relatively large number of passengers to change lines, lengthening their total travel time. Also, for passengers changing between two lines at some station, the waiting time may be very high. The latter problem may be reduced by introducing connection constraints, i.e. introducing an upper bound on the waiting time for certain important connections.

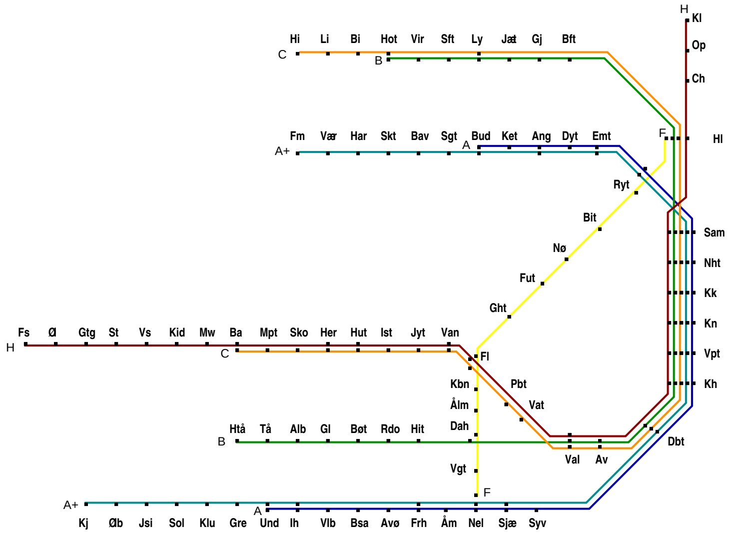


Figure 4.8: Visual representation of minimum cost lineplan with 10 minutes frequencies on all lines.

We want to investigate whether the minimum cost lineplan 10b obtained in section 4.5.2 can accomodate a timetable with small changing times for some previously defined connections. To ensure low travel times between any two stations on the same track segment, connection constraints are inserted between the *fast* line, and the *normal* line on each finger (except the western finger), e.g. in Und the waiting time when changing from line A+ to line A in the northern direction is limited to 3 minutes, assuming a minimum transfer time of 1 minute. At stations where the radial lines intersect the circular line F, slightly higher waiting times (4 minutes) are allowed, still assuming a minimum transfer time of 1 minute. To avoid restricting the instance too much, only two to four connection constraints are introduced at each station. Table 4.7 summarises the connection constraints introduced.

Introducing these connection constraints results in a timetable using 80 train units instead of 77, which is a significant increase in cost. Table 4.8 shows where the cost is imposed, and gives an indication of which constraints are *too* tight. F.ex. the cost on line A is increased by one which is likely to be due to one of the connection constraints involving line A causing it to run slower along certain track segments or have increased turnaround time at one of the terminals.

If the trade-off between passenger waiting time and rolling stock usage is known,

Station	From		To		Min	Max
Und	A+	north	A	north	1	4
Und	A	south	A+	south	1	4
Ba	H	north	C	north	1	4
Ba	C	south	H	south	1	4
Bud	A+	south	A	south	1	4
Bud	A	north	A+	north	1	4
Hot	C	south	B	south	1	4
Hot	B	north	C	north	1	4
Nel	A+	north	F	north	1	5
Nel	F	south	A+	south	1	5
Dah	B	north	F	north	1	5
Dah	F	south	B	south	1	5
Fl	H	north	F	north	1	5
Fl	F	south	H	south	1	5
Fl	H	north	F	south	1	5
Fl	F	south	H	north	1	5
Ryt	A+	south	F	south	1	5
Ryt	F	north	A+	north	1	5
Hl	C	south	F	south	1	5
Hl	F	north	C	north	1	5

Table 4.7: Connection constraints

Line	A	B	C	A+	F	H	total
w/o. connection const.	11	12	16	17	6	15	77
w. connection const.	12	13	16	17	6	16	80

Table 4.8: Rolling stock usage on lines for ten minute lineplan with and without connection constraints.

it is possible to find an optimal timetable, wrt. this service-cost trade-off. As an example, in the following a maximum waiting time of eight minutes is assumed for all connection constraints in table 4.8, i.e. each connection constraint have span interval $[1, 9]$. Each train unit is assumed to have a weight (or cost) of 10. First assume that each minute a passenger must wait for a connecting train (including minimum transfer time) have weight 0.1. The resulting timetable uses the minimum number of train units (77), and the total waiting time on the 20 connections is 52 minutes (not considering the number of passengers using each connection), giving an average waiting time of 2.6 minutes. Increasing the weight of passenger waiting time to 1 results in a timetable using 79 train units, but the total waiting time is reduced to 30 minutes, giving an average waiting time of 1.5 minutes. The results are summarised in table 4.9.

Weight		Train	Transfer time	Waiting time	
train unit	connections	units	total	total	avg
10.0	0.1	77	72	52	2.6
10.0	1.0	79	50	30	1.5

Table 4.9: Rolling stock usage and passenger waiting time for different weights on passenger waiting time (Weight, connections) at 20 predefined connections with a minimum transfer time of 1 minute and maximum waiting time of 8 minutes.

If the number of transferring passengers at each connection is known, different weights on transfer time may be assumed, allowing for a more accurate estimation of average passenger waiting time.

4.6 Evaluation of Chain Cutting Planes

In this section the chain cutting planes developed in section 2.5 is evaluated based on the effect on solution time. A few timetabling instances obtained from the previous examples from S-train are solved with and without the chain cutting planes.

In table 4.10, the optimisation time, CPLEX iterations, and branch-and-bound nodes are shown for three instances with and without the use of chain cutting planes. They clearly show that the chain cuts have no effect on the solution of the instance. In fact, it seems that the solution space is not reduced at all, since the number of CPLEX iterations and the number of branch-and-bound nodes is the same with and without chain cuts.

inst.	$\|N\|$	$\|A\|$	$\|C\|$	opt. time	iterations	BB nodes
1	1124	2033	910	58 (61)	136712 (136712)	3429 (3429)
2	1124	2028	905	28 (27)	60824 (60824)	2940 (2940)
3	1142	2114	973	390 (386)	689633 (689633)	22257 (22257)
4	616	939	324	27280 (26716)	15077831 (15077831)	349124 (349124)

Table 4.10: The effect of chain cuts. CPLEX optimisation time (s), iterations, and number of nodes in the branch and bound tree for some timetabling instances with chain cutting planes. Figures in parenthesis are without chain cutting planes. Instance 1-3 are for 20 minute interval lineplan without merging at a non-terminal station, while instance 4 is for a ten minute interval lineplan, in which merging is allowed at Vpt. CPLEX parameters were set to default for instance 1-3 and to the non-default configuration (section 4.3.1) for instance 4.

The fact that the chain cutting planes have no impact on the solution of the instance is probably due to the structure of the timetabling instances, in which the value of the bounds are relatively close to each other. This is likely to result in many cycle integers with the same lower bound (2.7). In particularly, two cycles consisting of many headway constraints (with the same lower and upper bounds) are likely to have the same cycle integer bounds. If for some vertex many of the lower bounds on the cycle integers associated with the fundamental cycles passing through the vertex are equal, the respective chain cutting plane is not *tight*. In particular, if all the lower bounds are equal (e.g. 0), the chain cutting plane does not impose any restrictions to the problem at all. For all the instances, it holds that many of the cycle integer lower bounds are equal, and the cuts obtained therefore impose little or no restriction of the solution space.

4.7 Summary and Conclusion on Experiments

Based on the S-train network, a number of lineplans and timetabling instances have been created and solutions have been obtained using CPLEX to solve the underlying PESP instance.

Instances with a timeperiod of 20 minutes and 11 lines were easily solved as long as the lineplan was fixed. However, when considering several lineplans (by allowing merging at a central tracksegment), the solution time increased drastically. An optimal solution was found when considering 24 different lineplans by merging of four lines. This solution could also have been obtained using enumeration as the number of lineplans is small.

When reducing the total number of lines to six (and thereby the total number of constraints) by doubling the line frequency, a solution to a timetabling instance, considering 120 lineplans, was obtained in reasonable time (7.5 hours). Allowing merging of lines at Vpt decreased the optimal solution by two train units compared to the fixed lineplan first considered. A solution considering 144 lineplans (allowing merging at Nel and Hl) was obtained in less than 20 hours, but did not result in a further reduction of train units.

From the experiments shown, it is clear that most of the operational requirements for constructing periodic timetables can be modeled using the Periodic Event Scheduling Problem. Furthermore, good solutions minimising total rolling stock usage as well as estimated passenger waiting time on selected connections can be found for relatively large railway networks in reasonable time.

Also, advanced modelling possiblities makes it possible to integrate aspects of other planning phases, such as lineplanning and rolling stock circulation into the timetabling process, at least for networks of limited size.

Chapter 5

Conclusion

5.1 Summary

In this report, the Periodic Event Scheduling Problem and its Cycle Periodicity formulation is described and important properties and modelling possibilities are investigated. In particular, it has been shown how sequencing and matching of events can be modelled by PESP constraints. Furthermore, a class of cutting planes for the CPF has been derived.

Secondly, the problem of constructing periodic railway timetables is addressed, and it is shown how the most important aspects of timetabling can modelled using the PESP. Furthermore, advanced planning possibilities (i.e. merging of lines) is introduced by employing matching of events.

Finally, a number of timetables are constructed for the Copenhagen S-train service. Merging of lines was first employed between two lines at terminal stations. Secondly, merging was used at a central station, breaking up the initial lineplan and creating a larger solution space. Finally, constraints on the passenger waiting time on selected connections are introduced, and the sum of these waiting times are incorporated into the objective function.

5.2 Further Activities

The modelling and experimentation described in this report is, of course, limited. Other modelling possibilities and experiments are relevant in the context of timetabling, some of which has previously been described in the litterature. Also, further research in optimisation methods tailored for the specific application will be interesting in order to be able to solve large problems arising from the integration of planning phases.

5.2.1 Model improvement

In the timetabling model presented in section 3.2, slack time is implicitly taken into account, as it is incorporated into the minimum running time d^-. It is therefore fixed for each track segment between stations. In theory, it may be that a small reduction in the slack at certain track segments or terminals (turnaround), will allow for a significantly cheaper timetable, e.g. by saving one train unit. As a reduction in slack probably incur a less robust timetable, one must consider the trade-off between cost and robustness. What consequences does a reduction in slack at some track segment have on the robustness of the entire system? If these two questions can be answered, one may incorporate the slack into the objective function, penalising timetables with possibly very low cost, but also low robustness. Alternatively, one may allow a different distribution of the slack over several track segments and/or lines.

Timetable symmetry may be introduced into the model, however, exceeding the PESP framework. How would ensuring symmetric timetables affect the solutions wrt. e.g. rolling stock usage, and how would the solution time be affected?

5.2.2 Minimising passenger travel time

As mentioned in section 3.3, minimising passenger travel time when constructing timetables is difficult. In the following, a heuristic to minimise expected total passenger travel time in a railway network is outlined.

The method assumes that origin-destination data describing the average number of passenger travelling from any station to any other station in the railway network is available for the duration of the planning period only (e.g. rush hour, if a timetable is constructed for rush hour only).

First the PESP constraint graph G associated with the timetabling instance is constructed. Secondly, a graph F is constructed that defines the possible movements of passengers. F consists of all nodes and all train dwell- and running time constraints in G, as well as connection constraints allowing passengers to change between lines at stations. In F is also inserted for each station two nodes, representing departure from respectively arrival to that station. Edges are inserted from each departure node to all train departure nodes at the respective station. Similarly, edges are inserted from all train arrival nodes to the respective passenger arrival node at each station.

In each iteration of the heuristic, shortest paths in F for each unordered pair of stations from passenger departure node to passenger arrival node of the respective stations, are calculated. This makes it possible to estimate the number of passengers *using* each edge of PESP constraint graph and assign weights to the PESP problem accordingly. Then, the PESP is solved yielding an optimal timetable given the fixed passenger flow. At each iteration, the passenger flow is calculated based on the travel times given by the solution to the PESP instance solved in the previous iteration. In the first iteration, the routing of passengers may done using minimum travel times, i.e. lower bounds of the PESP graph, yielding a lower bound to the total passenger travel time.

Having one passenger departure node for each station and letting travel time to and from these nodes be 0 assumes that passengers plan their trip to arrive at the station exactly in time to catch the *best/fastest* connection. This is very unlikely due to arrival times of connecting buses or trains. However, an arbitrary discrete distribution of arriving passengers may be assumed by replacing each single passenger departure node in F by an arbitrary number of nodes, each representing a specific point in time in the reference timeperiod $[0, T[$, and each connected to all departing lines at the station.

The heuristic is halted after a certain number of iterations or when convergence of the obtained solutions is observed.

Although common when assigning traffic flows in road network models, the method has not, to my knowledge, been applied when designing timetables for public transportation. Also, it is not known whether the method will result in solutions that are converging.

5.2.3 Efficiency measures

For large timetabling instances, solving the MIP formulation using CPLEX is still too time consuming for most practical purposes. Several measures to improve efficiency of the solution procedure may be investigated.

Liebchen et. al has done a study on the effect of some CPLEX parameters. However a comprehensive study may be needed for each specific timetabling problem to determine the optimal combination of parameters, since this may depend on the specific structure of the instance under consideration.

The impact of choosing a *good* cycle basis has been studied intensively, in particular by Liebchen and Peeters. However, it is still not clear how the optimal cycle basis should be constructed. E.g., is it worth to consider not only generalised fundamental cycle basis, but the entire domain of integral cycle bases, in order to get a good MIP formulation? The trade-off is longer cycle basis calculation time.

In this project, the cycle integer bounds (2.7) are introduced only for cycles in the cycle basis. However, in principle these cycle cutting planes can be introduced for all cycles in the constraint graph. For which (possibly all) cycles should the cycle cutting planes be enforced, and what is the impact on solution time?

A further investigation of the chain cutting planes (2.10) would be both relevant and interesting, especially in relation to the structure of the constraint graph. Which types of PESP instances (if any) benefit from chain cutting planes, and which do not?

The chain cutting planes are based on cycle integer lower bounds. Possibly, similar cutting planes may be developed based on upper bounds. If so, will these have any impact on the solution space?

5.3 Conclusion

The objective of this thesis has in general been to investigate the PESP model and its application to timetable construction for periodic railway systems, and in particular to develop timetables for S-train using the PESP. The project has shown that it is possible to construct timetables, minimising the number of required train units for the full S-train network (excluding line Ex) using the CPF formulation of the PESP in reasonable time. Furthermore, for a smaller network it is possible to integrate aspects of lineplanning extending the original PESP model to obtain timetables of lower cost in reasonable time. However, further research especially in the area of solution methods and adjustment of solution methods for specific instances may be necessary to solve larger problems and make it possible to facilitate greater integration of planning phases.

Appendix A

Stations

The following table shows the station abbrevations and full name, as well as, the required dwell time at each station.

Abbreviation	**Station**	**Dwell time (s)**
Alb	Albertslund	20
Li	Allerød	20
Avø	Avedøre	10
Ålm	Ålholm	10
Åm	Åmarken	10
Bav	Bagsværd	10
Ba	Ballerup	20
Bft	Bernstorffsvej	10
Bi	Birkerød	20
Bit	Bispebjerg	10
Bsa	Brøndby Strand	10
Bøt	Brøndbyøster	20
Bud	Buddinge	10
Ch	Charlottenlund	10
Dah	Danshøj	10
Dbt	Dybbølsbro	10
Dyt	Dyssegård	10
Elb	Ellebjerg	10

Emt	Emdrup	10
Av	Enghave	10
Fm	Farum	0
Fl	Flintholm	30
Fs	Frederikssund	0
Frh	Friheden	10
Fut	Fuglebakken	10
Gj	Gentofte	10
Gtg	Gl. Toftegård	10
Gl	Glostrup	20
Gre	Greve	10
Ght	Grøndal	10
Har	Hareskov	10
Hl	Hellerup	30
Her	Herlev	20
Hi	Hillerød	0
Hot	Holte	20
Htå	Høje Taastrup	0
Und	Hundige	20
Hut	Husum	10
Hit	Hvidovre	20
Ih	Ishøj	20
Ist	Islev	10
Jæt	Jægersborg	20
Jsi	Jersie	10
Jyt	Jyllingevej	10
Klu	Karlslunde	10
Kbn	KB Hallen	10
Ket	Kildebakke	10
Kid	Kildedal	10
Kl	Klampenborg	0
Kh	København H	60
Kj	Køge	0
Vat	Langgade	10
Ly	Lyngby	20
Mpt	Malmparken	10
Mw	Måløv	10
Nht	Nordhavn	10
Nø	Nørrebro	20
Kn	Nørreport	30
Nel	Ny Ellebjerg	0
Op	Ordrup	10
Ølb	Ølby	10
Øl	Ølstykke	10

Kk	Østerport	30
Pbt	Peter Bangs Vej	10
Rdo	Rødovre	20
Ryt	Ryparken	20
Sjæ	Sjælør	20
Skt	Skovbrynet	10
Sko	Skovlunde	10
Sol	Solrød Strand	10
Sft	Sorgenfri	10
Sgt	Stengården	10
St	Stenløse	10
Sam	Svanemøllen	20
Syv	Sydhavn	10
Tå	Taastrup	20
Val	Valby	30
Vlb	Vallensbæk	10
Ang	Vangede	10
Van	Vanløse	20
Vær	Værløse	10
Vs	Veksø	10
Vpt	Vesterport	30
Vgt	Vigerslev Allé	10
Vir	Virum	10

Appendix B

Running times

The following table shows the running time between pairs of station in each direction according to the current schedule, as well as, the +/-5% band. Direction = 0 denote southern direction, while direction = 1 denote northern directions.

From	**To**	**Direction**	**Scheduled runtime (min)**	**-5%**	**+5%**
Htå	Tå	1	2.17	2.06	2.28
Tå	Htå	0	2.00	1.90	2.10
Tå	Alb	1	2.67	2.53	2.80
Alb	Tå	0	3.17	3.01	3.33
Alb	Gl	1	2.67	2.53	2.80
Gl	Alb	0	2.67	2.53	2.80
Gl	Bøt	1	2.17	2.06	2.28
Bøt	Gl	0	2.67	2.53	2.80
Bøt	Rdo	1	1.67	1.58	1.75
Rdo	Bøt	0	1.67	1.58	1.75
Rdo	Hit	1	1.67	1.58	1.75
Hit	Rdo	0	1.67	1.58	1.75
Hit	Dah	1	1.17	1.11	1.23
Dah	Hit	0	1.17	1.11	1.23
Dah	Val	1	2.00	1.90	2.10
Val	Dah	0	1.67	1.58	1.75
Val	Av	1	1.83	1.74	1.93

Av	Val	0	2.50	2.38	2.63
Av	Dbt	1	2.33	2.22	2.45
Dbt	Av	0	2.33	2.22	2.45
Hl	Sam	0	2.67	2.53	2.80
Sam	Hl	1	2.50	2.38	2.63
Sam	Nht	0	1.83	1.74	1.93
Nht	Sam	1	1.67	1.58	1.75
Nht	Kk	0	2.00	1.90	2.10
Kk	Nht	1	1.83	1.74	1.93
Kk	Kn	0	1.50	1.43	1.58
Kn	Kk	1	2.50	2.38	2.63
Kn	Vpt	0	1.50	1.43	1.58
Vpt	Kn	1	1.50	1.43	1.58
Vpt	Kh	0	1.50	1.43	1.58
Kh	Vpt	1	1.00	0.95	1.05
Kh	Dbt	0	1.33	1.27	1.40
Dbt	Kh	1	2.50	2.38	2.63
Fs	Øl	1	5.33	5.07	5.60
Øl	Fs	0	6.00	5.70	6.30
Øl	Gtg	1	2.83	2.69	2.98
Gtg	Øl	0	2.33	2.22	2.45
Gtg	St	1	1.83	1.74	1.93
St	Gtg	0	1.83	1.74	1.93
St	Vs	1	3.33	3.17	3.50
Vs	St	0	3.33	3.17	3.50
Vs	Kid	1	2.83	2.69	2.98
Kid	Vs	0	2.83	2.69	2.98
Kid	Mw	1	2.33	2.22	2.45
Mw	Kid	0	2.33	2.22	2.45
Mw	Ba	1	3.17	3.01	3.33
Ba	Mw	0	2.83	2.69	2.98
Ba	Mpt	1	1.83	1.74	1.93
Mpt	Ba	0	2.17	2.06	2.28
Mpt	Sko	1	1.33	1.27	1.40
Sko	Mpt	0	1.83	1.74	1.93
Sko	Her	1	2.17	2.06	2.28
Her	Sko	0	2.33	2.22	2.45
Her	Hut	1	1.83	1.74	1.93
Hut	Her	0	1.67	1.58	1.75
Hut	Ist	1	1.83	1.74	1.93
Ist	Hut	0	1.83	1.74	1.93
Ist	Jyt	1	1.83	1.74	1.93
Jyt	Ist	0	1.33	1.27	1.40
Jyt	Van	1	1.67	1.58	1.75

Van	Jyt	0	1.33	1.27	1.40
Van	Fl	1	1.00	0.95	1.05
Fl	Van	0	1.17	1.11	1.23
Fl	Pbt	1	1.33	1.27	1.40
Pbt	Fl	0	1.00	0.95	1.05
Pbt	Vat	1	1.83	1.74	1.93
Vat	Pbt	0	1.83	1.74	1.93
Vat	Val	1	1.50	1.43	1.58
Val	Vat	0	1.33	1.27	1.40
Hi	Li	0	6.17	5.86	6.48
Li	Hi	1	6.00	5.70	6.30
Li	Bi	0	4.17	3.96	4.38
Bi	Li	1	4.67	4.43	4.90
Bi	Hot	0	4.67	4.43	4.90
Hot	Bi	1	3.67	3.48	3.85
Hot	Vir	0	2.33	2.22	2.45
Vir	Hot	1	1.67	1.58	1.75
Vir	Sft	0	1.83	1.74	1.93
Sft	Vir	1	1.83	1.74	1.93
Sft	Ly	0	3.17	3.01	3.33
Ly	Sft	1	2.33	2.22	2.45
Ly	Jæt	0	1.67	1.58	1.75
Jæt	Ly	1	1.67	1.58	1.75
Jæt	Gj	0	1.83	1.74	1.93
Gj	Jæt	1	1.67	1.58	1.75
Gj	Bft	0	1.83	1.74	1.93
Bft	Gj	1	1.83	1.74	1.93
Bft	Hl	0	2.00	1.90	2.10
Hl	Bft	1	1.83	1.74	1.93
Kl	Op	0	2.33	2.22	2.45
Op	Kl	1	2.50	2.38	2.63
Op	Ch	0	1.83	1.74	1.93
Ch	Op	1	1.83	1.74	1.93
Ch	Hl	0	3.00	2.85	3.15
Hl	Ch	1	2.83	2.69	2.98
Fm	Fbk	0	1.00	0.95	1.05
Fbk	Fm	1	1.00	0.95	1.05
Fbk	Vær	0	1.93	1.84	2.03
Vær	Fbk	1	2.60	2.47	2.73
Vær	Har	0	2.83	2.69	2.98
Har	Vær	1	2.83	2.69	2.98
Har	Skt	0	2.33	2.22	2.45
Skt	Har	1	2.33	2.22	2.45
Skt	Bav	0	2.33	2.22	2.45

Bav	Skt	1	1.83	1.74	1.93
Bav	Sgt	0	1.83	1.74	1.93
Sgt	Bav	1	1.83	1.74	1.93
Sgt	Bud	0	1.83	1.74	1.93
Bud	Sgt	1	1.83	1.74	1.93
Bud	Ket	0	1.83	1.74	1.93
Ket	Bud	1	1.83	1.74	1.93
Ket	Ang	0	1.83	1.74	1.93
Ang	Ket	1	1.83	1.74	1.93
Ang	Dyt	0	1.83	1.74	1.93
Dyt	Ang	1	1.83	1.74	1.93
Dyt	Emt	0	1.83	1.74	1.93
Emt	Dyt	1	1.83	1.74	1.93
Emt	Ryt	0	2.67	2.53	2.80
Ryt	Emt	1	2.33	2.22	2.45
Ryt	Sam	0	2.17	2.06	2.28
Sam	Ryt	1	2.17	2.06	2.28
Kj	Ølb	1	2.33	2.22	2.45
Ølb	Kj	0	3.50	3.33	3.68
Ølb	Jsi	1	3.83	3.64	4.03
Jsi	Ølb	0	3.83	3.64	4.03
Jsi	Sol	1	1.83	1.74	1.93
Sol	Jsi	0	1.83	1.74	1.93
Sol	Klu	1	3.83	3.64	4.03
Klu	Sol	0	3.83	3.64	4.03
Klu	Gre	1	2.33	2.22	2.45
Gre	Klu	0	2.33	2.22	2.45
Gre	Und	1	2.67	2.53	2.80
Und	Gre	0	2.83	2.69	2.98
Und	Ih	1	2.17	2.06	2.28
Ih	Und	0	2.17	2.06	2.28
Ih	Vlb	1	2.33	2.22	2.45
Vlb	Ih	0	2.17	2.06	2.28
Vlb	Bsa	1	1.83	1.74	1.93
Bsa	Vlb	0	2.33	2.22	2.45
Bsa	Avø	1	2.33	2.22	2.45
Avø	Bsa	0	1.83	1.74	1.93
Avø	Frh	1	1.83	1.74	1.93
Frh	Avø	0	1.83	1.74	1.93
Frh	Åm	1	1.83	1.74	1.93
Åm	Frh	0	1.83	1.74	1.93
Åm	Nel	1	1.33	1.27	1.40
Nel	Åm	0	1.83	1.74	1.93

Nel	Sjæ	1	2.17	2.06	2.28
Sjæ	Nel	0	2.33	2.22	2.45
Sjæ	Syv	1	1.33	1.27	1.40
Syv	Sjæ	0	1.67	1.58	1.75
Syv	Dbt	1	2.83	2.69	2.98
Dbt	Syv	0	2.33	2.22	2.45
Hl	Ryt	0	2.17	2.06	2.28
Ryt	Hl	1	2.00	1.90	2.10
Ryt	Bit	0	1.83	1.74	1.93
Bit	Ryt	1	2.17	2.06	2.28
Bit	Nø	0	1.17	1.11	1.23
Nø	Bit	1	1.33	1.27	1.40
Nø	Fut	0	1.33	1.27	1.40
Fut	Nø	1	1.17	1.11	1.23
Fut	Ght	0	1.33	1.27	1.40
Ght	Fut	1	1.33	1.27	1.40
Ght	Fl	0	1.50	1.43	1.58
Fl	Ght	1	1.83	1.74	1.93
Fl	Kbn	0	1.83	1.74	1.93
Kbn	Fl	1	1.50	1.43	1.58
Kbn	Ålm	0	0.83	0.79	0.88
Ålm	Kbn	1	1.33	1.27	1.40
Ålm	Dah	0	1.17	1.11	1.23
Dah	Ålm	1	0.83	0.79	0.88
Dah	Vgt	0	1.33	1.27	1.40
Vgt	Dah	1	1.67	1.58	1.75
Vgt	Nel	0	1.50	1.43	1.58
Nel	Vgt	1	1.33	1.27	1.40

Appendix C

Turnaround times

The bounds on the turnaround times used (in minutes) are summarised in the following table.

Station	**Type**	**Min. time**	**Max. time**
Htå	platform	6	15
Htå	shunting	10	18
Hot	platform	6	15
Hot	shunting	10	18
Kl	both	6	15
Ba	platform	6	15
Ba	shunting	10	18
Fs	both	6	15
Fm	both	6	15
Bud	both	10	18
Und	both	6	15
Kj	both	6	15
Nel	both	6	15
Hl	platform	6	15
Hl	shunting	10	18
Hi	platform	6	15
Kk	both	6	15

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