Mortgage Loan Portfolio Optimization Using Multi Stage Stochastic Programming

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Abstract

We consider the dynamics of the Danish mortgage loan system and propose several models to reflect the choices of a mortgagor as well as his attitude towards risk. The models are formulated as multi stage stochastic integer programs, which are difficult to solve for more than 10 stages. Scenario reduction and LP relaxation are used to obtain near optimal solutions for large problem instances. Our results show that the standard Danish mortgagor should hold a more diversified portfolio of mortgage loans, and that he should rebalance the portfolio more frequently than current practice.

1 Introduction

1.1 The Danish mortgage market

The Danish mortgage loan system is among the most complex of its kind in the world. Purchase of most properties in Denmark is financed by issuing fixed–rate callable mortgage bonds based on an annuity principle. It is also possible to raise loans, which are financed through issuing non–callable short term bullet bonds. Such loans may be refinanced at the market rate on an ongoing basis. The proportion of loans financed by short–term bullet bonds has been increasing since 1996. Furthermore it is allowed to mix loans in a mortgage loan portfolio, but this choice has not yet become popular.

Callable mortgage bonds have a fixed coupon throughout the full term of the loan. The maturities are 10, 15, 20 or 30 years. The borrower has a call

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option, i.e. he can redeem the mortgage loan at par at each term during the lifetime of the loan. When the interest rates are low the call option can be used to obtain a new loan with less interest payment in exchange for an increase in the amount of outstanding debt. The borrower has also a delivery option. When the interest rates are high this option can be used to reduce the amount of outstanding debt, in exchange for paying higher interest rate payments. There are both fixed and variable transaction costs associated with exercising any of these options.

Non–callable short–term bullet bonds are used to finance the adjustable–rate loans. The bonds' maturities range from one to eleven years and the adjustable–rate loans are offered as 10, 15, 20 or 30–year loans. Since 1996 the most popular adjustable–rate loan has been the loan financed by the one–year bond. From 2001, however, there has been a new trend, where demand for loans financed by bullet bonds with 3 and 5–year maturities has risen substantially.

1.2 The mortgagor's problem

It is known on the investor side of the financial markets that investment portfolios should consist of a variety of instruments in order to decrease risk while maintaining profitability. The portfolio is also rebalanced regularly to take best advantage of the moves in the market.

The portfolio diversification principle and rebalancing is, however, not common in the borrower side of the mortgage market. Most mortgagors finance their loans in one type of bond only. Besides they do not always rebalance their loan when good opportunities for this have arisen.

There are two major reasons for the mortgagors reluctance to better taking advantage of their options (that they have fully paid for) through the life time of the mortgage loan.

1. The complexity of the mortgage market makes it impossible for the average mortgagor to analyze all the alternatives and choose the best.

2. The mortgage companies do not provide enough quantitative advice to the individual mortgagor. They only provide general guidelines, which are normally not enough to illuminate all different options and their consequences.

The complexity of the mortgage loan system makes it a non–trivial task to decide on an initial choice of mortgage loan portfolio and on finding a continuing plan to readjust the portfolio optimally. There exists as of today no functional optimization model to provide decision support for the individual mortgagor on his choice of loan.

We assume in the following that the reader is familiar with the dynamics of a mortgage loan market such as the Danish one, as well as the basic ideas behind the mathematical modeling concept of stochastic programming.

The Danish mortgagor's problem has been introduced by Nielsen and Poulsen (N&P, [12]). They have developed a two factor term structure model for generating scenarios and on top of it have built a multi–stage stochastic program to find optimal loan strategies. The article, however, does not describe the details necessary to have a functional optimization model, and it does not differentiate between different types of risks in the mortgage market. The main contribution of this article is to make Nielsen & Poulsen's model operational by reformulating parts of their model and adding new features to it.

We reformulate the Nielsen & Poulsen model in section 2. In section 3 we model different options available to the Danish mortgagor, and in section 4 we model mortgagor's risk attitudes. Here we consider both market risk and wealth risk.

In the basic model we incorporate fixed transaction costs using binary variables. We use a non–combining binomial tree to generate scenarios in a 11 stage problem. This results in 51175 binary variables, making some versions of the problem extremely challenging to solve. Dupačová, Gröwe–Kuska, Heitsch and Römisch (Scenred, [7, 8]) have modeled the scenario reduction problem as a set covering problem and solved it using several heuristic algorithms. We review these algorithms in section 5 and use them in our implementation to reduce the size of the problem and hereby reduce the solution times. Another approach to getting shorter solution times is proposed in section 6, where we solve an LP–approximated version of the problem. In section 7 we discuss and comment on our numerical results and we conclude the article with suggestions for further research in section 8. We use GAMS (General Algebraic Modeling Language) to model the problem and CPLEX 9.0 as the underlying MP and MIP solver. For scenario reduction we use the GAMS/SCENRED module (scenred manual, [9]).

The obtained results show that the average Danish mortgagor would benefit from his option of mixing loans in his loan portfolio. Likewise he should readjust the portfolio more often than is the case today. The developed model and software can also be used to develop new loan portfolio products. Such products will consider the individual customer inputs such as budget constraints, risk profile, expected lifetime of the loan, etc.

Even though we consider the Danish mortgage loan marked, the problem is universal and the practitioners in any mortgage loan system should be able to use the models developed in this article, possibly with minor modifications.

2 The basic model

In this section we develop a risk–neutral optimization model which finds a mortgage loan portfolio with the minimum expected total payment. Consider the scenario tree in figure (1). We assume that we have such a tree at hand with information on price and coupon rate for all mortgage bonds available at each node together with the realization probability of a given node at a given time.
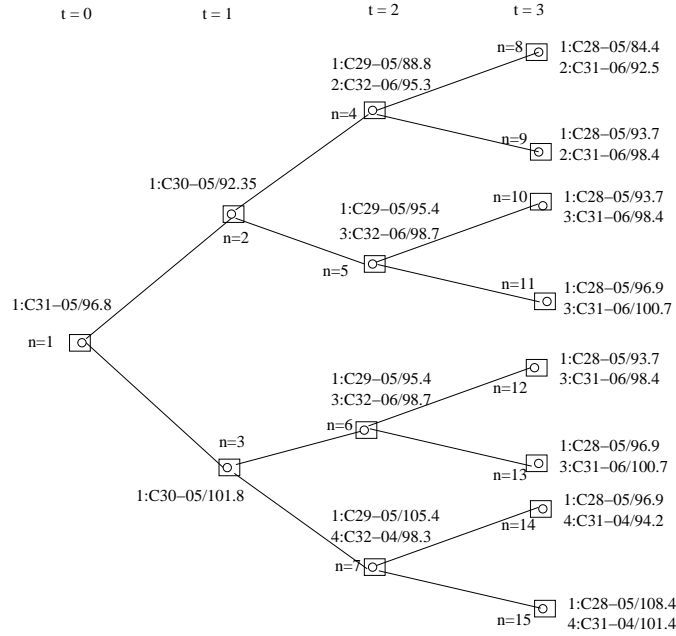

Figure 1: A binomial scenario tree, representing our expectation of future bond prices and coupon rates. All bonds are callable fixed–rate bonds.

In the basic model we only consider fixed–rate loans, i.e. loans where the interest rate does not change during the lifetime of the loan. For the sake of demonstration we consider an example with 4 stages, $t \in \{0, 1, 2, 3\}$, and 15 decision nodes, $n \in \{1, \cdots, 15\}$, with the probability p_n for being at the node n. In order to make our model independent of the underlying scenario structure we capture the dependency between any time step and its nodes in the set $tn(t, n)$, where $n \in \{2^t, \cdots, 2^{t+1} - 1\}$ for all t. Similarly we define a set $tree(n', n)$ to define a parent-child relationship.[1]

We want the basic model to be able to find an optimal portfolio of bonds

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[1]Most modeling languages facilitate an easy way to declare such sets. In GAMS for example one can define two sets t and n and then declare $tn(t, n)$ and $tree(n, n)$ as:
set tn(t,n) time node mapping / t0.(n1), t1.(n2*n3), t2.(n4*n7) /,
tree(n,n) parent child mapping / n1.(n2*n3), n2.(n4*n5), n3.(n6*n7) /;

out of a finite number of fixed–rate bonds available at any node. Consider the 4 bonds shown in figure (1). Each bond is represented as (Index:Type–Coupon/Price), so (3:C32–06/98.7) is a fixed–rate callable bond with maturity after 32 periods, a coupon rate of 6% and a price of 98.7. The letter C indicates that the bond is callable, so the prepayment price is never higher than par.

To generate bonds information we can use term structure and bond pricing theories (see [10, 11, 4] for an introduction to these topics and further references to articles.). It is also possible to use expert knowledge to predict possible bond prices in the future. A combination of theoretical pricing and expert information can also be used to generate such scenario trees. Nielsen and Poulsen (N&P, [12]) have proposed one way to generate a scenario tree for the Danish mortgagor problem.

Given that we have a scenario tree with N stages at hand we can now formulate the optimization problem. We first consider the parameters of the optimization model:

p_n: The probability of being at node n, $\quad \forall n \in \{1, \cdots, 2^N - 1\}$.
τ_n: Discount factor at node n.
IA: The initial amount of loan needed by the mortgagor.
r_{in}: Coupon rate for bond i at node n.
k_{in}: Price of bond i at node n.
$Callk_{in}$: Price of a callable bond i at node n. We have $Callk_{in} = \min\{1, k_{in}\}$ for callable bonds and $Callk_{in} = k_{in}$ for non–callable bonds.
γ: Tax reduction rate from interest rate payment.
β: Tax reduction rate from administration fees.
b: Administration fee rates given as a percentage of outstanding debt for bond i at node n.
η: Transaction fee rates for sale and purchase of bonds.
m: Fixed costs associated with refinancing.

Next we define the variables used in our model:

B_{tn}: Total net payment at node n, time t.
X_{itn}: Outstanding debt of bond i at node n, time t.
S_{itn}: Units sold of bond i at scenario n, time t.
P_{itn}: Units purchased of bond i at node n, time t.
A_{itn}: Principal payment of bond i at node n, time t.
$L_{itn} : \begin{cases} 1 & \text{if there are any fixed costs associated with bond } i, \text{ node } n, \text{ time } t. \\ 0 & \text{otherwise.} \end{cases}$

The multi stage stochastic integer model can now be formulated as follows:

$$\min \sum_{tn(t,n)} p_n \cdot \tau_n \cdot B_{tn} \tag{1}$$

$$\sum_i k_{i1} \cdot S_{i01} \geq IA \tag{2}$$

$$X_{i01} = S_{i01} \qquad \forall i \tag{3}$$

$$\sum_{tree(n',n)} \left(X_{i,t-1,n'} - A_{i,t-1,n'} - P_{itn} + S_{itn} - X_{itn} \right) = 0 \quad \forall i, tn(t\backslash 0, n\backslash 1) \tag{4}$$

$$\sum_i (k_{in} \cdot S_{itn}) = \sum_i (Callk_{in} \cdot P_{itn}) \qquad \forall tn(t\backslash 0, n\backslash 1) \tag{5}$$

$$A_{itn} = X_{itn} \left[\frac{r_{in}}{1 - (1 + r_{in})^{-N+t}} - r_{in} \right] \qquad \forall i, tn(t, n) \tag{6}$$

$$B_{tn} = \sum_i \Big(A_{itn} + r_{in} \cdot (1 - \gamma) X_{itn} +$$

$$b \cdot (1 - \beta) X_{itn} + \eta \cdot (S_{itn} + P_{itn}) + m \cdot L_{itn} \Big) \qquad \forall tn(t, n) \tag{7}$$

$$BigM \cdot L_{itn} - S_{itn} \geq 0 \qquad \forall i, tn(t, n) \tag{8}$$

$$X_{itn}, \ S_{itn}, P_{itn} \geq 0, \ L_{itn} \in \{0, 1\} \qquad \forall i, tn(t, n) \tag{9}$$

The objective is to minimize the weighted average payment throughout the mortgage period. The payment at each node is defined in equation (7) as the sum of principal payments, tax reduced interest payments, taxed reduced administration fees, transaction fees for sale and purchase of bonds and finally fixed costs for establishing new mortgage loans. The principal payment is defined in equation (6) as an annuity payment.

The dynamics of the model are formulated in constraints (2) to (5). Constraint (2) makes sure that we sell enough bonds to raise an initial amount, IA, needed by the mortgagor. In equation (3) we initialize the outstanding debt. Equation (4) is the balance equation, where the outstanding debt at any child node for any bond equals the outstanding debt at the parent node minus principal payment and possible prepayment (purchased bonds), plus possible sold bonds to establish a new loan. Notice that the model is independent of the underlying scenario tree. Use of the set $tree(n', n)$ in the balance equation (4) guarantees that any tree structure, defined in the set $tree(n', n)$ can be used to form the balance equations. This formulation of the model is very useful when we want to reduce the number of scenarios in the original scenario tree. Equation (5) is a cash flow equation which guarantees that the money used to prepay comes from the sale of new bonds.

Finally constraint (8) adds the fixed costs to the node payment, if we perform any readjusting of the mortgage portfolio. The $BigM$ constant might be set to a value slightly greater than the initial amount raised. If a too large value is used, numerical problems may arise.

3 Modeling mortgagor's options

The model of section 2 has three implicit assumptions which limit its applicability:

1. We assume that a loan portfolio is held by the mortgagor until the end of horizon.

2. We assume that all bonds are fixed–rate and callable, i.e. they can be prepaid at any time at a price no higher than 100.

3. The mortgagor is assumed to be risk–neutral.

We will relax the first two assumptions in this section and the third in the following section.

The first assumption can be easily relaxed by introducing a constant T as the number of decision stages until the prepayment of the loan portfolio, such that $T \leq N$, where N is the horizon. The decision nodes represent only the first T stages, while the horizon remains as N. It means that the principal and interest payments are calculated with the horizon N, whereas we only consider the first T stages in the problem.

These changes mean that the outstanding debt at stage T is now a positive amount. We define this prepayment amount (PP_{Tn}) as:

$$PP_{Tn} = \sum_i (X_{iTn} - A_{iTn}) \qquad \forall n \in \{2^{T-1}, \cdots, 2^T - 1\} \qquad (10)$$

We add this equation to the model and we update the object function as follows:

$$\min \sum_{tn(t,n)} p_n \cdot \tau_n \cdot B_{tn} + \sum_{tn(t=T,n=2^{(T-1)})}^{tn(t=T,n=2^T-1)} p_n \cdot \tau_n \cdot PP_{tn}. \qquad (11)$$

With this object function we are now minimizing the weighted payment at all nodes plus the weighted average amount of prepayment at time T.

The problem with the second assumption is more subtle. Consider the scenario tree at figure (2), where two adjustable–rate loans have been added to our set of loans at time 0. Loan 5 (F1) is an adjustable–rate loan with annual refinancing and loan 6 (F2) is an adjustable–rate loan with refinancing every second year

For adjustable–rate loans (Fm–loans) the underlying m–year bond is completely refinanced every m years by selling another m–year bond. But unlike normal refinancing this kind of refinancing does not incur any extra fixed or
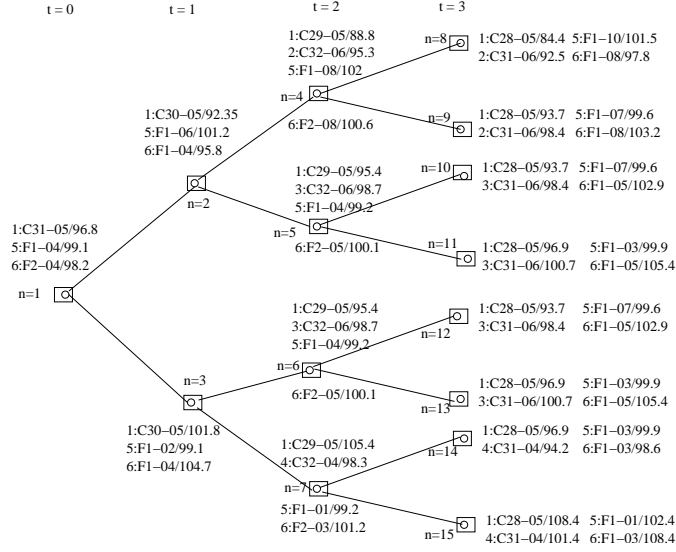
t = 0   t = 1   t = 2   t = 3

1:C29–05/88.8
2:C32–06/95.3
5:F1–08/102

n=8 1:C28–05/84.4 5:F1–10/101.5
     2:C31–06/92.5 6:F1–08/97.8

n=4
6:F2–08/100.6

n=9 1:C28–05/93.7 5:F1–07/99.6
     2:C31–06/98.4 6:F1–08/103.2

1:C30–05/92.35
5:F1–06/101.2
6:F1–04/95.8

1:C29–05/95.4
3:C32–06/98.7
5:F1–04/99.2

n=10 1:C28–05/93.7 5:F1–07/99.6
      3:C31–06/98.4 6:F1–05/102.9

n=2

n=5
6:F2–05/100.1

n=11 1:C28–05/96.9   5:F1–03/99.9
      3:C31–06/100.7 6:F1–05/105.4

1:C31–05/96.8
5:F1–04/99.1
6:F2–04/98.2

1:C29–05/95.4
3:C32–06/98.7
5:F1–04/99.2

n=12 1:C28–05/93.7 5:F1–07/99.6
      3:C31–06/98.4 6:F1–05/102.9

n=1

n=6
6:F2–05/100.1

n=13 1:C28–05/96.9   5:F1–03/99.9
      3:C31–06/100.7 6:F1–05/105.4

n=3

1:C29–05/105.4
4:C32–04/98.3

n=14 1:C28–05/96.9 5:F1–03/99.9
      4:C31–04/94.2 6:F1–03/98.6

1:C30–05/101.8
5:F1–02/99.1
6:F1–04/104.7

n=7
5:F1–01/99.2
6:F2–03/101.2

n=15 1:C28–05/108.4 5:F1–01/102.4
      4:C31–04/101.4 6:F1–03/108.4

Figure 2: A binomial scenario tree where both fixed–rate and adjustable–rate loans are considered.

variable transaction costs. We model that by using the same loan index for an adjustable–rate loan throughout the mortgage period. For example index 5 is used for the loan with annual refinancing, even though the actual bonds behind the loan change every year. Since we use the same index, the model does not register any actual sale or purchase of bonds when refinancing occurs. We should, however, readjust the outstanding debt given that the bond price is normally different from par. To take this into account we introduce the set i' of non–callable adjustable–rate loans. For these loans we use the following balance equation instead of equation (4).

$$
\sum_{tree(n',n)} \left(X_{i',t-1,n'} - A_{i',t-1,n'} - P_{i'tn} + S_{i'tn} - k_{i'n} \cdot X_{i'tn} \right) = 0 \quad \forall i', tn(t\backslash 0, n\backslash 1).
$$

(12)

Note that variables $P_{i'tn}$ and $S_{i'tn}$ remain 0 as long we keep an adjustable–rate loan i' in our mortgage portfolio. The outstanding debt in the child node is however rebalanced by multiplying the bond price.

When we consider the adjustable–rate loans we should remember that these loans are non–callable, so for prepayment purposes we have: $Callk_{i'n} = k_{i'n}$.

Another issue to be dealt with is that if a bond is not available for establishing a loan at a given node, we have to set the corresponding value of k_{in} to 0 to make sure that the bond is not sold at that node in an optimal solution.

8

For example bond (6:F1–04/95.8) at node 2 is not open for sale but only for prepayment.

4   Modeling risk

So far we have only considered a risk neutral mortgagor who is interested in the minimum weighted average of total costs. Most mortgagors however have an aversion towards risk. There are two kinds of risk which most mortgagors are aware of:

1. **Market risk**: In the mortgage market this is the risk of extra interest rate payment for a mortgagor who holds an adjustable–rate loan when interest rate increases, or the risk of extra prepayment for a mortgagor with any kind of mortgage loan when the interest rate decreases so the bond price increases.

2. **Wealth risk**: In the mortgage market this is a potential risk which can be realized if the mortgagor needs to prepay the mortgage before a planned date or if he needs to use the free value of the property to take another loan. It can be measured as a deviation from an average outstanding debt at any given time during the life time of the loan.

We will in the following model both kinds of risk. To that end we use the ideas behind minmax optimization and utility theory with use of budget constraints.

4.1   The minmax criterion

An extremely risk averse mortgagor wants to pay least in the worst possible scenario. In other words if we define the maximum payment as MP then we have the following minmax criterion:

$$\min \quad MP, \tag{13}$$

$$MP \geq \sum_{tn(t,n \in NP_f)} B_{tn} \quad \forall f \in \{1, \cdots, 2^{T-1}\}, \tag{14}$$

where NP_f is a set of scenarios with $f \in \{1, \cdots, 2^{T-1}\}$. Each scenario is described as the unique path of nodes from the root of the tree to one of the

leaves. For the example given in figure 2 we have:

$$NP_1 = \{1, 2, 4, 8\}$$
$$NP_2 = \{1, 2, 4, 9\}$$
$$\dots$$
$$NP_8 = \{1, 3, 7, 15\}$$

4.2   Utility function

Instead of minimizing costs we can define a utility function, which represents a saving and maximize it. Nielsen and Poulsen (N&P [12]) suggest a concave utility function with the same form as in sketch (3).
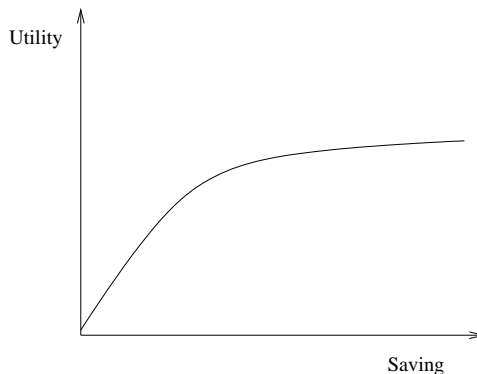

Figure 3: A concave utility function. An increase of an already big saving is not as interesting as an increase of a smaller saving.

The decreasing interest for bigger savings is based on the idea that bigger savings are typically riskier than small savings. Nielsen and Poulsen suggest a logarithmic object function, which can be formulated as follows:

$$\max \sum_{tn(t,n)} p_n \cdot \log(\tau_n \cdot (B_{tn}^{max} - B_{tn})), \qquad (15)$$

where B_{tn}^{max} is the maximum amount a mortgagor is willing to pay. Nielsen and Poulsen fix B_{tn}^{max} to a big value so that the actual payment will never rise above this level.

Adding this non–linear object function to our stochastic binary problem makes the problem extremely challenging to solve. There are no effective general purpose solvers for solving large mixed integer non–linear programs (see Bussieck and Pruessner, [6]). There are three ways of circumventing the problem: Either we use a linear utility function in conjunction with budget

constraints (mip) or relax the binary variables and solve the non–linear problem (nlp) or both (lp). We demonstrate the first approach in the following and comment on the second and third approach in section 6.

Instead of maximizing the logarithm of the saving at each node we can simply maximize the saving: $B_{tn}^{max} - B_{tn}$. If B_{tn}^{max} is so large that the saving is always positive, then we are in effect minimizing the weighted average costs similar to the risk neutral case presented in section 2. However if we allow the saving to be negative at times and add a penalty to the objective function whenever we get a negative saving, we can introduce risk aversion into the model. For this reason we need to have a good estimate for B_{tn}^{max}. The risk neutral model can be solved to give us these estimates. Then we can use the following objective function and budget constraints.

$$\max \sum_{tn} \Big(p_n \cdot \tau_n \big((B_{tn}^{max} - B_{tn}) - PR_{tn} \cdot BO_{tn} \big) \Big) \tag{16}$$

$$B_{tn}^{max} + BO_{tn} - B_{tn} \geq 0 \qquad\qquad \forall tn(t,n) \tag{17}$$

$$BO_{tn} \leq BO_{tn}^{max} \qquad\qquad \forall tn(t,n). \tag{18}$$

We allow crossing the budget limit in constraint (17) by introducing the slack variable BO_{tn}. This value will then be penalized by a given penalty rate (PR_{tn}) in the objective function (16). The penalty rate can for example be a high one time interest rate for taking a bank loan. The budget overflow (BO_{tn}) is then controlled in constraint (18) where the overflow is not allowed to be greater than a maximum amount BO_{tn}^{max}.

4.3 Wealth risk aversion

So far we have only considered the market risk or the interest rate risk. In the following we will model the other important risk factor in the mortgage market, namely the wealth risk.

Wealth risk is the risk that the actual outstanding debt becomes bigger than the expected outstanding debt at a given time during the life time of the loan. For example selling a 30–year bond at a price of 80, we have a big wealth risk given that a small fall in the interest rate can cause a considerable increase in the bond price, which means a considerable increase in the amount of the outstanding debt.

We consider the deviation from the average outstanding debt, which we define as DX_{tn}:
$$DX_{tn} = \overline{X}_t - \sum_i X_{itn}, \quad \forall tn(t,n),$$

where $\overline{X}_t$ is the average outstanding debt for a given time t:

$$\overline{X}_t = \sum_{i,tn(t,n)} p_n \cdot X_{itn}, \quad \forall t.$$

A positive value of DX_{tn} means that we have a saving and a negative value means a loss as compared to the average outstanding debt $\overline{X}_t$. We introduce a surplus variable XS_{tn} to represent the amount of saving and a slack variable XL_{tn} to represent the amount of loss:

$$\left(\overline{X}_t - \sum_i X_{itn}\right) - XS_{tn} + XL_{tn} = 0 \qquad \forall tn(t,n), \tag{19}$$

To make the model both market risk and wealth risk averse we update the objective function with weighted values of XS_{tn} and XL_{tn} as follows:

$$\max \sum_{tn(t,n)} p_n \cdot \tau_n \left((B_{tn}^{max} - B_{tn}) - PR_{tn} \cdot BO_{tn} + PW_n \cdot XS_{tn} - NW_n \cdot XL_{tn} \right), \tag{20}$$

where PW_n is short for positive weight and can be used to encourage savings and NW_n, short for negative weight, is there to penalize a loss as compared to the average outstanding debt. If we set $PW_n = NW_n$, it means that the model is indifferent towards wealth risk. On the other hand $PW_n < NW_n$, means that the model is wealth risk averse, since it penalizes a potential loss harder than it encourages a potential saving.

5 Scenario reduction

Since the number of scenarios grows exponentially as a function of time steps the stochastic binary model is no longer tractable when we have more than 10 time steps. For an 11–stage model we have the scenario tree in figure (4).

As of today there are no general purpose solvers which can solve stochastic integer problems of this size in a reasonable amount of time. Notice however that a great number of nodes in the last 3-4 time steps have such a close distance that a reduction of nodes for these time steps might not effect the first–stage results. We are in other words interested in finding a way to optimally reduce the number of scenarios. If we get the same first stage result for a reduced and a non–reduced problem, it suffices to solve the reduced problem, and then at each step resolve the problem until horizon. In that case the final result of solving any of the two problems will be the same. The reason for this is that we initially only implement the first stage solution. As the time passes by and we get more information we have to solve the new problem and implement the new first stage solution each time.
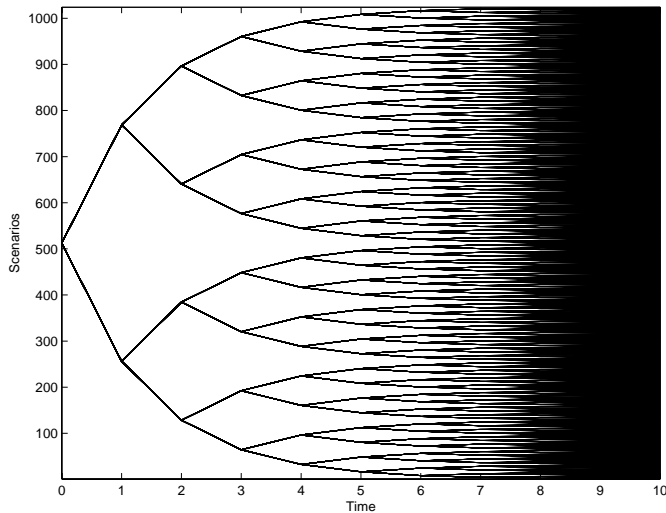
Figure 4: A binomial scenario tree with 11 stages.

Nicole Gröwe–Kuska, Holger Heitsch, Jitka Dupačová and Werner Römisch (see [7, 8]) have defined the scenario reduction problem (SRP) as a special set covering problem and have solved it using heuristic algorithms.

The authors behind the SCENRED articles have in cooperation with "GAMS Software GmbH" and "GAMS Development Corporation", developed a number of C++ routines, SCENRED, for optimal scenario reduction in a given scenario tree. Likewise they have developed a link, GAMS/SCENRED, which connects the GAMS program to the SCENRED module (see [9]). The scenario tree in figure 5 is obtained after using the fast backward algorithm of the GAMS SCENRED module for a 50% relative reduction, where the relative reduction is measured as an average of node reductions for all time step. If we for example remove half of the nodes at the last time step, we get a 50% reduction for that time step only. Then we measure the reduction percentages for all other time steps in the same way. The average of these percentages corresponds to the relative reduction (see [7, 8]). In our example the number of scenarios is reduced from 1024 to 12.

We use GAMS/SCENRED and SCENRED modules for scenario reduction, and compare the results with those found by solving the LP–relaxed non–reduced problem.

6 LP relaxation

Whenever we refinance the mortage portfolio we need to pay a variable and a fixed transaction cost. The variable cost is $100 \cdot \eta$ percent of the sum of
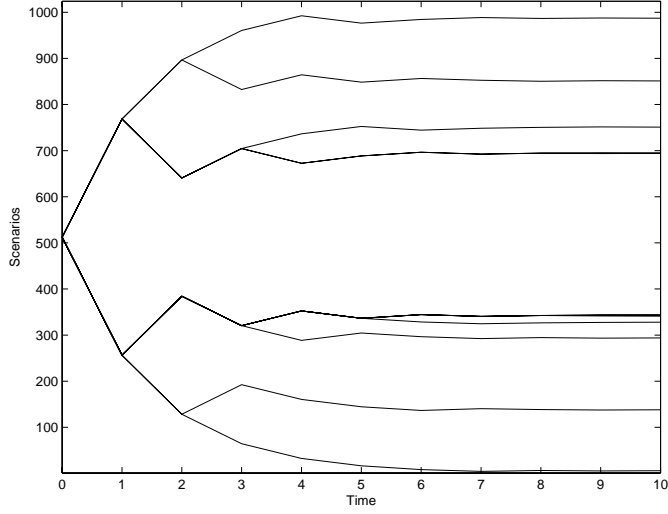
Figure 5: A binomial scenario tree with 11 stages after a 50% scenario reduction using the fast backward algorithm of the SCENRED module in GAMS.

the sold and purchased amount of bonds and the fixed cost is simply DKK m (see constraint 7 and 8). The binary variables in the problem (1 to 9) are due to incorporation of fixed costs m. The numeric value of these fixed costs is about DKK 2500 whereas $\eta = 0.15\%$. While the value of the variable transaction costs decreases as the time passes by, the fixed costs remain the same. Therefore, if we simply drop the fixed costs or just add a small percentage to the variable transaction costs, we normally get a different first stage solution than what we get if we solve the problem with the actual fixed transaction costs. We therefore suggest an iterative updating scheme for the variable transaction costs, so that we can approximate the fixed costs without using binary variables. We do that by iteratively solving the LP problem k times as follows.

We define a ratio ψ_{itn}^k and initialize it to $\psi_{itn}^0 = 0$. The ratio ψ_{itn}^k can then be used in the definition of a node payment (7) in the $k + 1$st iteration as follows:

$$
\begin{aligned}
B_{tn} = \sum_i \Big(& A_{itn} + r_{in} \cdot (1 - \gamma)X_{itn} + \\
& b \cdot (1 - \beta)X_{itn} + \eta \cdot (S_{itn} + P_{itn}) + \psi_{itn}^{k+1} \cdot S_{itn} \Big) \qquad \forall tn(t, n) \qquad (21)
\end{aligned}
$$

Solving the LP problem at each iteration k we get S_{itn}^{*k} as the optimal value of the sold bonds at the kth iteration. Before each iteration $k > 0$, the ratio ψ_{itn}^k is then updated according to the following rule:

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$$\psi_{itn}^k = \begin{cases} \frac{m}{S_{itn}^{*k}} & \forall i, tn(t,n) \quad \text{if } S_{itn}^{*k} > 0, \\ \psi_{itn}^{k-1} & \text{otherwise.} \end{cases} \tag{22}$$

This brings us to our approximation scheme for an LP relaxation of the problem:

1. Drop the fixed costs and solve the LP relaxed problem.

2. Find the ratios ψ_{itn} according to (22).

3. Incorporate the ratios in the model so that DKK m is added to the objective function for each purchased bond, given the same solution as the one in the last iteration is obtained. Solve the problem again.

4. Stop if the solution in iteration $k+1$ has not changed more than α percent as compared to the solution in iteration k. Otherwise go to step 5.

5. Update ψ_{itn} according to (22).

6. Repeat from step 3.

Our experimental results show that for $\alpha \simeq 2\%$ we find near optimal solutions which have similar characteristics to the solutions from the original problem with the fixed costs.

7   Numerical results

We consider an 11 stage problem, starting with 3 callable bonds and 1 1–year bullet bond at the first stage. We then introduce 7 new bonds every 3 years. An initial portfolio of loans has to be chosen at year 0 and it may be rebalanced once a year the next 10 years. We assume that the loan is a 30–year loan and that it is prepaid fully at year 11.

The 24 callable bonds used in our test problem are seen in table 1. The table only presents the average coupon rate and price for these bonds at the date of issue. Besides these 24 callable bonds we use a 1–year non–callable bullet bond, bond 25. The effective interest rate on this bond is about 2% to start with. Using a BDT tree (see [5, 3]) with the input given in table 2 the effective rate can increase to 21% or decrease to slightly under 1% at the 10th year. The BDT tree has also been used for estimating the prices and rates of the 24 callable bonds during the life time of the mortgage loan.

A practical problem arises when writing the GAMS tables containing the stochastic data. The optimization problem is a path dependent problem,

Bond nr.	rate	Average start price	Date of issue	Date of maturity
1	6%	103.06	3/10-02	3/10-35
2	5%	98.5	3/10-02	3/10-35
3	4%	89.4	3/10-02	3/10-35
4	9%	107.33	3/10-05	3/10-38
5	8%	103.16	3/10-05	3/10-38
6	7%	103.09	3/10-05	3/10-38
7	6%	100.51	3/10-05	3/10-38
8	5%	94.01	3/10-05	3/10-38
9	4%	84.55	3/10-05	3/10-38
10	3%	74.46	3/10-05	3/10-38
11	9%	105.4	3/10-08	3/10-41
12	8%	101.98	3/10-08	3/10-41
13	7%	100.3	3/10-08	3/10-41
14	6%	96.19	3/10-08	3/10-41
15	5%	89.5	3/10-08	3/10-41
16	4%	80.74	3/10-08	3/10-41
17	3%	71.32	3/10-08	3/10-41
18	9%	104.41	3/10-11	3/10-44
19	8%	100.9	3/10-11	3/10-44
20	7%	98.51	3/10-11	3/10-44
21	6%	94.07	3/10-11	3/10-44
22	5%	87.49	3/10-11	3/10-44
23	4%	79.25	3/10-11	3/10-44
24	3%	70.26	3/10-11	3/10-44

Table 1: The callable bonds used as input to the problem.

whereas the BDT tree is path independent. GAMS is not well suited for such programming tasks as mapping the data from a combining binomial tree (a lattice) to a non–combining binomial tree. A general purpose programming language is better suited for this task. We have used VBA to generate the input data to the GAMS model, and we have run the GAMS models on a sun solaris machine.

The purpose of our tests can be summarized as the following:

1. Validating the 4 versions of our model, i.e. we investigate if the models represent the actual dynamics of the Danish mortgage market.

2. Observing the effects of using the GAMS/SCENRED module.

3. Trying our LP approximation on the problem.

For each of these objectives we consider all four versions of the model and compare the results.

Maturity (Year)	Yield (%)	Yield Volatility (%)		Maturity (Year)	Yield (%)	Yield Volatility (%)
1	2.23%	–		16	4.87%	17.25%
2	2.35%	32.20%		17	4.93%	17.00%
3	2.73%	32.10%		18	4.99%	16.85%
4	3.08%	29.50%		19	5.05%	16.75%
5	3.41%	27.00%		20	5.11%	16.70%
6	3.68%	25.00%		21	5.16%	16.65%
7	3.92%	23.00%		22	5.21%	16.60%
8	4.12%	22.00%		23	5.25%	16.56%
9	4.30%	20.90%		24	5.29%	16.52%
10	4.44%	20.10%		25	5.34%	16.48%
11	4.56%	19.40%		26	5.37%	16.45%
12	4.62%	18.80%		27	5.40%	16.42%
13	4.68%	18.30%		28	5.43%	16.39%
14	4.74%	17.90%		29	5.46%	16.36%
15	4.80%	17.55%		30	5.49%	16.34%

Table 2: The input term structure to the BDT model.

7.1 The original stochastic MIP problem

Figure (6) shows the solutions found for the first three stages of the problem for all four instances of our model, namely the risk neutral model, the minimax model, the model with interest rate risk aversion with budget constraints and finally the model with interest rate and wealth risk aversion with budget constraints. Notice, however, that no feasible solution could be found for the model with interest rate and wealth risk aversion with budget constraints within a time limit of 10 hours.

A full prescription of the solution with all 11 stages will not contribute to a better understanding of the dynamics of the solution, which is why we present the solution to the first three stages of the problem only. It is though enough to give us an indication of the behaviour of each solution. In the risk neutral case we start by taking a 1–year adjustable–rate loan. If the interest rate increases after a year, the adjustable–rate loan is prepaid by taking a fixed–rate loan. Even if it means an increase in the amount of the outstanding debt, it proves to be a profitable strategy since if the rates increase again in the next stage we can reduce the amount of outstanding debt greatly by refinancing the loan to another fixed–rate loan with a higher price. The minmax strategy chooses a fixed–rate loan with a price close to par to start with. This loan is not refinanced until the 9th stage of the problem.

The risk neutral and the minimax model represent the two extreme mortgagors as far as the risk attitude is concerned. The third model reflects a
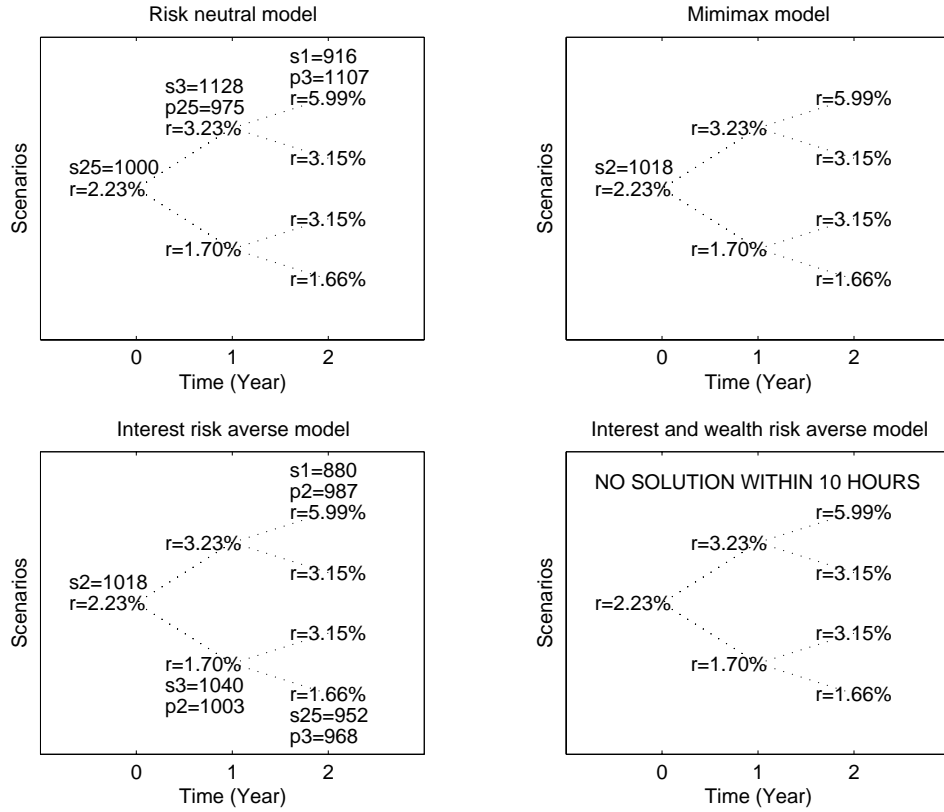
Figure 6: Presentation of the solutions for the first 3 stages of the problem. Variable s is for sale and p for purchase and the units are given in 1000 DKK, so $s3 = 1128$ means that the mortgagor should sell approximately 1.128.000 DKK at the given node. The short rates from the BDT tree are indicated using the letter r.

mortgagor with a risk attitude between the first two mortgagors. The solution to this model guarantees that the mortgagor will not pay more than what his budget allows at any given node. Table (3) indicates the difference in the characteristics of the solutions for the three different models.

The risk neutral model gives the lowest average total cost. The standard deviation from this average cost is, however, rather high. The minimax model has a much smaller standard deviation. This higher level of security against variation has though an average cost of about 72000 DKK. The third model has reduced the risk considerably without having increased the total average cost with more than about 7000 DKK.

For the sake of budget constraints in model 3 and 4 we use the following constants:

```
SCALAR BMAX    /570842/; // An average level of payment for a scenario
```

Model type	Total costs	Std. dev.	max	min	time
1 - Risk neutral	1.281.857	92.289	1.502.042	1.004.583	276 s
2 - Minimax	1.353.713	19.729	1.374.183	1.117.084	10 h
3 - Int. rate risk averse	1.288.405	66.019	1.431.857	1.005.412	10 h
4 - Int./Wealth risk averse	No solution found within 10 h.				

Table 3: Comparison of the four strategies for the original problem.

```
SCALAR IBMAX    /711015/; // An average level of prepayment at year 11
SCALAR BOMAX    /50000/;  // Maximum extra payment for a scenario
SCALAR IBOMAX   /100000/; // Maximum extra prepayment at year 11
```

These constants are chosen after considering the average payments and the standard deviations from these in the risk neutral model.

The major problem with these solutions is the computing time taken to find near optimal solutions by CPLEX. Except for the first model, we cannot find solutions within 1% of a lower bound after 10 hours of CPU time. For the fourth model no feasible solution is found at all. In the following we will see the results found for the reduced problem.

7.2 The reduced stochastic MIP problem

After reducing the number of scenarios from 1024 to 12 we get the solutions in figure (7).

Except for the risk neutral model we do not obtain the same first stage solutions as we saw for the original problem. It seems that the reduced problem has a more optimistic view of the future as compared to the original model. This can also be seen in table (4) where the total costs are considerably less than the ones for the original problem.

Model type	Total costs	Std. dev.	max	min	time
1 - Risk neutral	1.169.173	49.765	1.274.079	1.064.525	12 s
2 - Minimax	1.187.938	0.00	1.187.938	1.187.938	52.2 s
3 - Int. rate risk averse	1.171.926	24.270	1.229.897	1.136.655	300 s
4 - Int./Wealth risk averse	1.172.479	25.610	1.229.742	1.128.412	300 s

Table 4: Comparison of the four strategies for the reduced problem.

By testing the scenario reduction algorithms for different levels of reduction on our problem we notice that the reduced problem has almost always an overweight of scenarios with lower interest rates. This explains the more optimistic view of the future given by scenario reduction. Even much less aggressive scenario reductions do not guarantee us the same initial solutions as found for the original problem. What we need is therefore a method to optimally reduce the number of scenarios while the tree remains balanced.
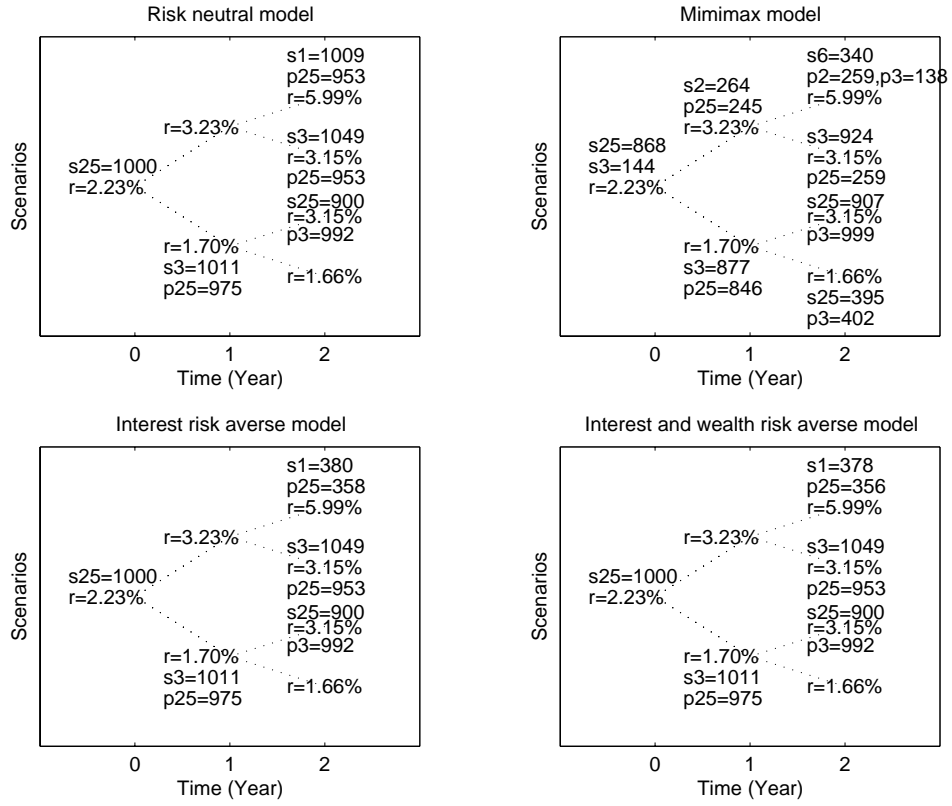
Figure 7: Presentation of the solutions for the first 3 stages of the reduced problem. Units are given in 1000 DKK.

For the purposes of this article however, we continue with the reduced tree found by GAMS/SCENRED. The reason for this is our focus on the optimization part of the problem in this article. We suggest that the market practitioners experiment with different reduced scenario trees ranging from very pessimistic to optimistic ones to find robust solutions for each risk category of mortgagors.

We can see in table (4) that the behaviour of the solutions for the different models is similar to that of the original problem. Notice also that we get a feasible solution here for the fourth model with interest rate and wealth risk aversion.

For the sake of budget constraints in model 3 and 4 we use the following constants:

```
SCALAR BMAX     /565915/;    // An average level of payment for a scenario
SCALAR IBMAX    /601983/;    // An average level of prepayment at year 11
SCALAR BOMAX     /50000/;    // Maximum extra payment for a scenario
```

```
SCALAR IBOMAX   /35000/;    // Maximum extra prepayment at year 11
```

7.3   The reduced and LP–approximated problem

When we use our LP–approximation algorithm on this problem we get the
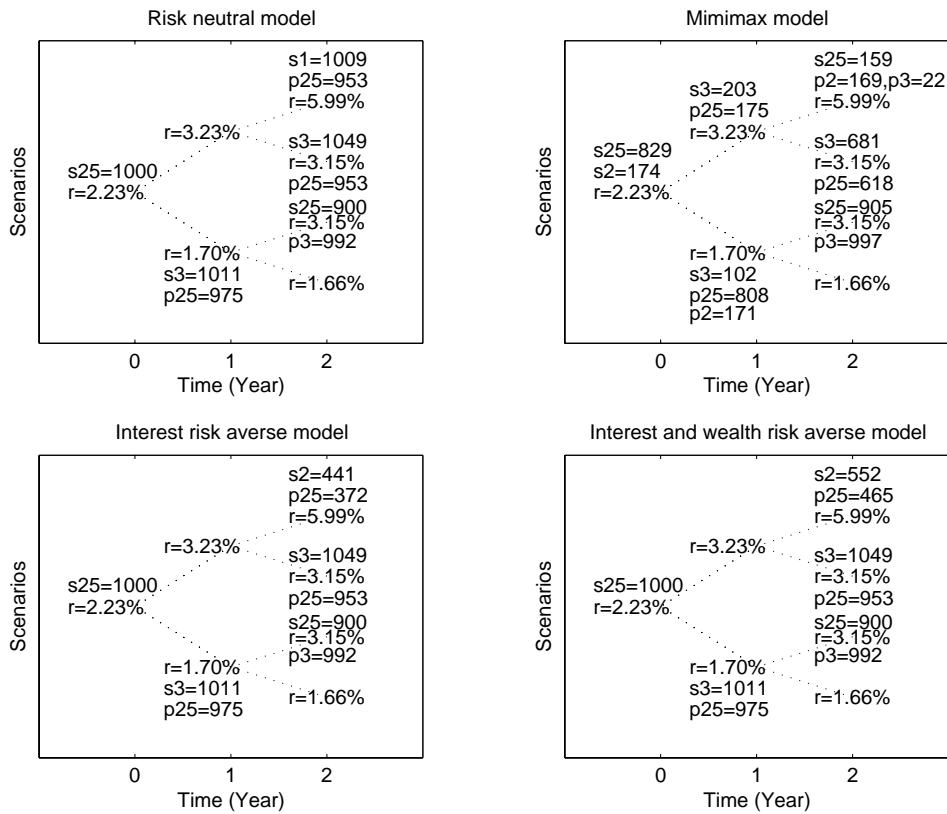solution as presented in table (5) and figure (8).

Figure 8: Presentation of the solutions for the first 3 stages of the LP approximated
reduced problem. Units are given in 1000 DKK.

The algorithm uses 10–18 runs for the different problems to find solutions
which are over all less than 2% different from the solutions found in the last
iteration.

It is important to point out that simply dropping the fixed costs results in
solutions which deviate considerably from the problems with the fixed costs,
whereas approximating the fixed costs using our algorithm gives very similar
results as found by the MIP model.

Model type	Total costs	Std. dev.	max	min	time
1 - Risk neutral	1.169.147	49.775	1.274.078	1.064.524	25 s
2 - Minimax	1.179.654	11.150	1.185.795	1.154.602	22 s
3 - Int. rate risk averse	1.172.364	26.436	1.239.168	1.130.196	28 s
4 - Int./Wealth risk averse	1.174.038	29.128	1.249.520	1.131.185	44 s

Table 5: Comparison of the four strategies for the reduced problem with LP approximation.

7.4   Comments on results

The results presented in this section are in agreement with the financial arguments used in the Danish mortgage market. Even though the original problem is hard to solve we have shown that useful results can be found by solving the reduced problems. The reduced scenario trees represented a more optimistic prediction of the future, but the results found are still quite useful. In practice the mortgage portfolio manager should try several scenario trees with different risk representations as an input to the model. This way the optimization model can be used as an analytical tool for performing "what–if" analyses on a high abstraction level.

8   Conclusions

We have developed a functional optimization model that can be used as the basis for a quantitative analysis of the mortgagors decision options. This model in conjunction with different term structures or market expert opinions on the development of bond prices can assist market analysts in the following ways:

Decision support: Instead of calculating the consequences of the single loan portfolios for single interest rate scenarios, the optimization model allows for performing "what if" analysis on a higher level of abstraction. The analyst can provide the system with different sets of information such as the presumed lifetime of the loan, budget constraints and risk attitudes. The system then finds the optimal loan portfolio for each set of input information.

Product development: Traditionally, loan products are based on single bonds or bonds with embedded options. In some mortgage markets such as the Danish one it is allowed to mix bonds in a mortgage portfolio and there are even some standard products which are based on mixing bonds. The product P_{33} is for example a loan portfolio where 33% of the loan is financed in 3–year non–callable bonds and the rest in fixed–rate callable bonds. These mixed products are currently not popular since the rationale behind exactly this kind of mix is not well argued. The optimization model gives the possibility to tailor mixed products that, given a set of requirements, can be

argued to be optimal for a certain mortgagor.

The greatest challenge in solving the presented models is on decreasing the computing times. We have experimented with scenario reduction (scenred, [7, 8, 9]) and we have suggested an LP approximation method to reduce the solution times while maintaining solution quality. It is, however, an open problem to develop tailored exact algorithms such as decomposition algorithms (see [1, 2]) to solve the mortgagors problem. Another approach for getting real time solutions is to investigate different heuristic algorithms or make use of parallel programming (see [13, 14]) to solve the problem.

Integration of the two disciplines of mathematical finance and stochastic programming combined with use of the state of the art software has a great potential, which has not yet been realized in all financial markets in general and in mortgage companies in particular. There is a need for more detailed and operational models and high performing easy to use accompanying software to promote use of the mathematical models with special focus on stochastic programming.

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