The Power of Tabulation Hashing

Mihai Pătraşcu and Mikkel Thorup

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Some of this work can be found in Proc. STOC'11.

Target

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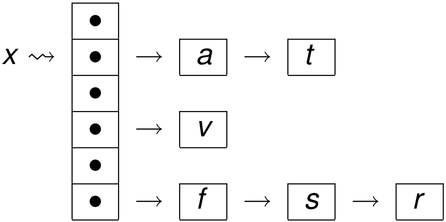
- Guarantees akin to those of truely random hashing, yet easy to implement.
- Uniting theory and practice.

Applications of Hashing $\rightsquigarrow$

Hash tables (n keys and $2n$ hashes: expect 1/2 keys per hash)

- chaining: follow pointers



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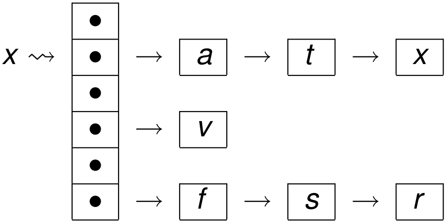

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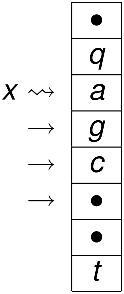

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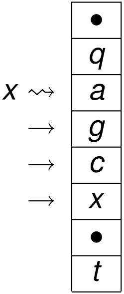

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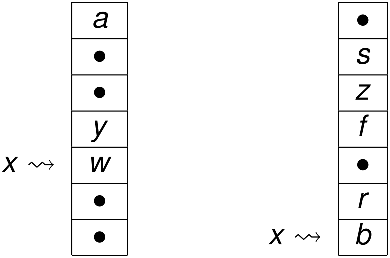

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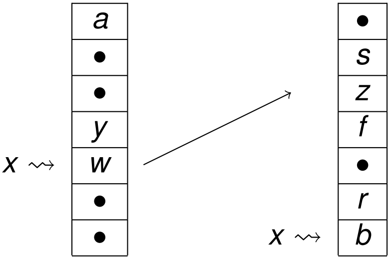

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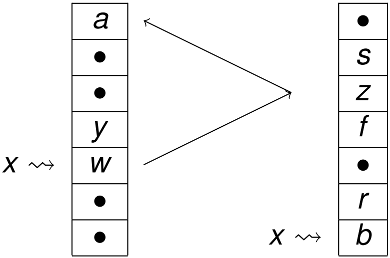

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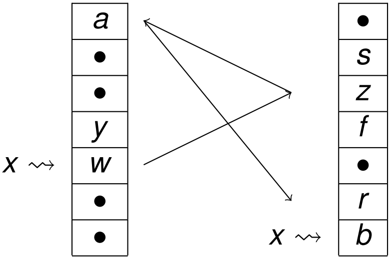

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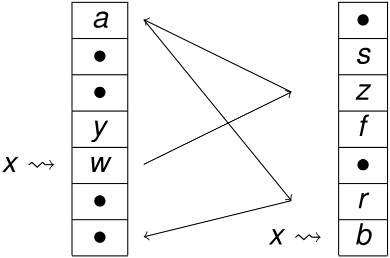

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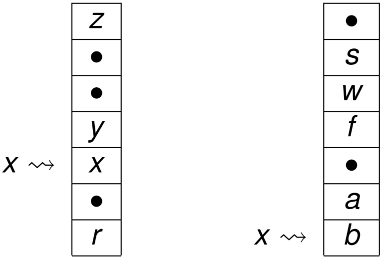

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Sketching, streaming, and sampling:

- moment estimation: $F_2(\bar{x}) = \sum_i x_i^2$
- sketch A and B to later find $|A \cap B| / |A \cup B|$

$$|A \cap B| / |A \cup B| = \Pr_h[\min h(A) = \min h(B)]$$

We need h to be ε-minwise independent:

$$(\forall) x \notin S : \qquad \Pr[h(x) < \min h(S)] = \frac{1 \pm \varepsilon}{|S| + 1}$$

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Important outside theory. These simple practical hash tables often bottlenecks in the processing of data—substantial fraction of worlds computational resources spent here.

Carter & Wegman (1977)

We do not have space for truly random hash functions, but

A family $\mathcal{H} = \{h : [u] \rightarrow [b]\}$ is *k-independent* iff:

- $(\forall) x \in u$, $h(x)$ is uniform in $[b]$;
- $(\forall) x_1, \ldots, x_k \in [u]$, $h(x_1), \ldots, h(x_k)$ are independent.

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Prototypical example: degree $k - 1$ polynomial

- u prime;
- choose $a_0, a_1, \ldots, a_{k-1}$ randomly in $[u]$;
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Many solutions for k-independent hashing proposed, but generally slow for $k > 3$ and too slow for $k > 5$.

How much independence needed?

Chaining $\mathbf{E}[t] = O(1)$ $\mathbf{E}[t^k] = O(1)$ $t = O\left(\frac{\lg n}{\lg \lg n}\right)$ w.h.p.	2 $2k+1$ $\Theta\left(\frac{\lg n}{\lg \lg n}\right)$	
Linear probing	≤ 5 [Pagh^2, Ružić'07]	≥ 5 **[PT ICALP'10]**
Cuckoo hashing	$O(\lg n)$	≥ 6 [Cohen, Kane'05]
F_2 estimation	4 [Alon, Mathias, Szegedy'99]	
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Independence has been the ruling measure for quality of hash functions for 30+ years, but is it right?

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- Simple tabulation is the fastest 3-independent hashing scheme.
- Not 4-independent: $h(a_1 a_2) \oplus h(a_1 b_2) \oplus h(b_1 a_2) \oplus h(b_1 b_2)$

$$\begin{aligned} = \quad & (R_1[a_1] \oplus R_2[a_2]) \oplus (R_1[a_1] \oplus R_2[b_2]) \oplus \\ & (R_1[b_1] \oplus R_2[a_2]) \oplus (R_1[b_1] \oplus R_2[b_2]) \; = \; 0. \end{aligned}$$

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New result: Despite its 4-dependence, simple tabulation suffices for all the above applications:

One simple and fast hashing scheme for almost all your needs.

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We prove that dependence of simple tabulation is not harmful in any of the above applications.

Chaining/hashing into bins

Theorem Consider hashing n balls into $m \geq n^{1-1/(2c)}$ bins by simple tabulation. Let q be an additional *query ball*, and define X_q as the number of regular balls that hash into a bin chosen as a function of $h(q)$. Let $\mu = \mathbf{E}[X_q] = \frac{n}{m}$. The following probability bounds hold for any constant γ:

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\Omega(\mu)} + m^{-\gamma}$$

$$\Pr[X \leq (1-\delta)\mu] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\Omega(\mu)} + m^{-\gamma}$$

With $m \leq n$ bins, every bin gets

$$n/m \pm O\left(\sqrt{n/m}\log^c n\right).$$

keys with probability $1 - n^{-\gamma}$.

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Nothing like this lemma holds if we instead of simple tabulation assumed k-independent hashing with $k = O(1)$.

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Proof that for any positive constants ε, γ, if we hash n keys into m bins and $n \leq m^{1-\varepsilon}$, then all bins get less than $d = 2^{(1+\gamma)/\varepsilon}$ keys with probability $\geq 1 - m^{-\gamma}$.

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- Return $\{x\} \cup U'$ where U' independent subset of T'.

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- Recall $n \leq m^{1-\varepsilon}$.

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Short of time?

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- By lemma, w.h.p., each G distributes with $\leq d$ in each bin.

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- If the X_G were really independent, by Chernoff

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- Good enough for Chernoff bounds.

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W.h.p., the contribution X to given obeys Chernoff

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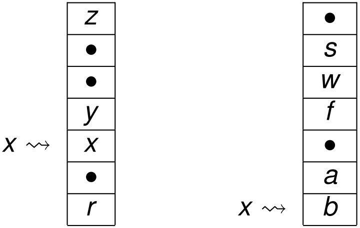
Similar story for linear probing.

Cuckoo hashing

Each key placed in one of two hash locations.



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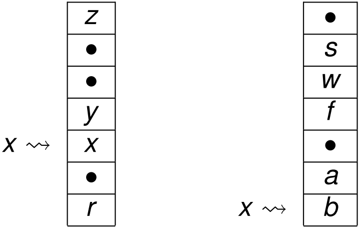

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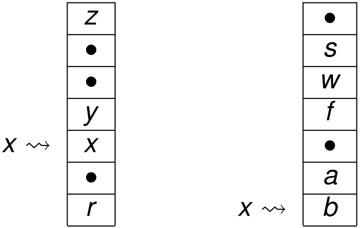

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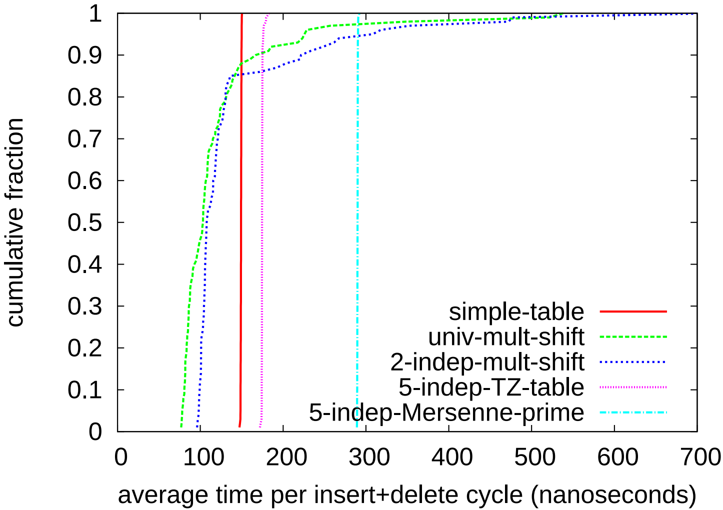
- Very delicate proof showing that obstruction can be used to code random tables R_i with few bits.

Speed

Hashing random keys		32-bit computer	64-bit computer
bits	hashing scheme	hashing time (ns)	
32	univ-mult-shift `(a*x)>>s`	1.87	2.33
32	2-indep-mult-shift	5.78	2.88
32	5-indep-Mersenne-prime	99.70	45.06
32	5-indep-TZ-table	10.12	12.66
32	simple-table	4.98	4.61
64	univ-mult-shift	7.05	3.14
64	2-indep-mult-shift	22.91	5.90
64	5-indep-Mersenne-prime	241.99	68.67
64	5-indep-TZ-table	75.81	59.84
64	simple-table	15.54	11.40

Experiments with help from Yin Zhang.

Robustness in linear probing for dense interval



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- Multiplicative hashing used in practice, but turns out to be very unreliable under typical denial-of-service (DoS) attacks based on consecutive IP addresses: systematic good performance 90% of the time, but systematic terrible performance 10% of the time [TZ'10].

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- Here we proved linear probing safe with good probabilistic performance for all input if we use simple tabulation.
- Simple tabulation also powerful for chaining, cuckoo hashing, and min-wise hashing:

  one simple and fast scheme for (almost) all your needs.

Work in progress: short range amortization with twisted tabulation

- With chaining and linear probing, each operation takes expected constant time, but out of $\sqrt{n}$ operations, some are expected to take $\tilde{\Omega}(\log n)$ time.

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- Twisted tabulation also need for biased sampling of weighted items, and any other context where we really care about few among many keys.

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