Thank you for inviting me to ThRaSH.
The Power of Tabulation Hashing

Mihai Pătrașcu and Mikkel Thorup

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Some of this work can be found in Proc. STOC’11.
Target

- Safe and simple hashing.
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- Guarantees akin to those of truely random hashing, yet easy to implement.
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- Safe and simple hashing.
- Guarantees akin to those of truly random hashing, yet easy to implement.
- Uniting theory and practice.
Applications of Hashing

Hash tables \((n\) keys and \(2n\) hashes: expect 1/2 keys per hash)

- chaining: follow pointers

\[
\begin{align*}
\times \rightarrow \bullet & \rightarrow a \rightarrow t \\
\rightarrow \bullet & \rightarrow v \\
\rightarrow \bullet & \rightarrow f \rightarrow s \rightarrow r
\end{align*}
\]
Applications of Hashing

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\[
\begin{align*}
X & \rightsquigarrow & a & \rightarrow & t & \rightarrow & x \\
& \rightsquigarrow & v \\
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- linear probing: sequential search in one array
- cuckoo hashing: search \(\leq 2\) locations, complex updates

\[
\begin{array}{c}
a \\
\bullet \\
\bullet \\
y \\
w \\
\bullet \\
\bullet \\
\end{array}
\quad \begin{array}{c}
\bullet \\
s \\
z \\
f \\
\bullet \\
r \\
\bullet \\
b \\
\end{array}
\]

\(x \rightsquigarrow y \rightsquigarrow w \rightsquigarrow x \rightsquigarrow b\)
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```
+---+---+---+---+
| a |   |   |   |
|   | b |   |   |
|   |   | c |   |
+---+---+---+---+

x ~~~
```

```
+---+---+---+---+
|   | s |   |   |
| z |   | f |   |
|   |   | r |   |
+---+---+---+---+

x ~~~
```
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Sketching, streaming, and sampling:

- moment estimation:
  \[ F_2(\bar{x}) = \sum_i x_i^2 \]

We need $h$ to be $\varepsilon$-minwise independent:

\[ \Pr[h(x) < \min_h(S)] = 1 \pm \varepsilon |S| + 1 \]
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Sketching, streaming, and sampling:

- moment estimation: $F_2(\bar{x}) = \sum_i x_i^2$
- sketch $A$ and $B$ to later find $|A \cap B|/|A \cup B|

$$|A \cap B|/|A \cup B| = \Pr_{h}[\min h(A) = \min h(B)]$$

We need $h$ to be $\varepsilon$-minwise independent:

$$\forall x \notin S : \Pr[h(x) < \min h(S)] = \frac{1 \pm \varepsilon}{|S| + 1}$$
Applications of Hashing

Hash tables \((n \text{ keys and } 2n \text{ hashes: expect } 1/2 \text{ keys per hash})\)

- **chaining**: follow pointers.
- **linear probing**: sequential search in *one* array

Important outside theory. These simple practical hash tables often bottlenecks in the processing of data—substantial fraction of worlds computational resources spent here.
Carter & Wegman (1977)

We do not have space for truly random hash functions, but

A family \( \mathcal{H} = \{ h : [u] \rightarrow [b] \} \) is \( k \)-independent iff:

\[ \begin{align*}
\forall x \in u, & \; h(x) \text{ is uniform in } [b]; \\
\forall x_1, \ldots, x_k \in [u], & \; h(x_1), \ldots, h(x_k) \text{ are independent.}
\end{align*} \]
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\end{align*}
\]

Prototypical example: degree \( k - 1 \) polynomial

\[
\begin{align*}
u \text{ prime; } \\
\text{choose } a_0, a_1, \ldots, a_{k-1} \text{ randomly in } [u]; \\
h(x) = (a_0 + a_1 x + \cdots + a_{k-1} x^{k-1}) \mod u.
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We do not have space for truly random hash functions, but

A family $\mathcal{H} = \{ h : [u] \rightarrow [b] \}$ is $k$-independent iff:

$\quad \forall x \in u, h(x) \text{ is uniform in } [b]$;

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Prototypical example: degree $k - 1$ polynomial

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Many solutions for $k$-independent hashing proposed, but generally slow for $k > 3$ and too slow for $k > 5$. 
How much independence needed?

<table>
<thead>
<tr>
<th>Method</th>
<th>Equations</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
</tr>
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<tbody>
<tr>
<td>Chaining</td>
<td>$E[t] = O(1)$, $E[t^k] = O(1)$</td>
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Independence has been the ruling measure for quality of hash functions for 30+ years, but is it right?
Simple tabulation

- Simple tabulation goes back to Carter and Wegman’77.
Simple tabulation

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- Key $x$ divided into $c = O(1)$ characters $x_1, \ldots, x_c$,
e.g., 32-bit key as $4 \times 8$-bit characters.

Hash value $h(x) = R_1[x_1] \oplus \cdots \oplus R_c[x_c]$ where the $R_i$ are independent random tables:
char $\rightarrow$ hash values (bit strings)

- With 8-bit characters, each table $R_i$ has 256 entries and fit
in fast memory.

- Simple tabulation is the fastest 3-independent hashing
scheme.
- Not 4-independent:

$$h(a_1 a_2) \oplus h(a_1 b_2) \oplus h(b_1 a_2) \oplus h(b_1 b_2) = (R_1[a_1] \oplus R_2[a_2]) \oplus (R_1[a_1] \oplus R_2[b_2]) \oplus (R_1[b_1] \oplus R_2[a_2]) \oplus (R_1[b_1] \oplus R_2[b_2]) = 0.$$
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**How much independence needed? Wrong question**

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\(t = O\left(\frac{\lg n}{\lg \lg n}\right)\) w.h.p.

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New result: Despite its 4-dependence, simple tabulation suffices for all the above applications:

*One simple and fast hashing scheme for almost all your needs.*
How much independence needed? Wrong question

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One simple and fast hashing scheme for almost all your needs.

Knuth recommends simple tabulation but cites only 3-independence as mathematical quality.
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| Linear probing         | $\leq 5$        | $\geq 5$                              |
|                        | [Pagh², Ružič’07]| [PT ICALP’10]                        |
| Cuckoo hashing         | $O(\lg n)$     | $\geq 6$                              |
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| $F_2$ estimation       | $4$             | [Alon, Mathias, Szegedy’99]           |
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New result: Despite its 4-dependence, simple tabulation suffices for all the above applications:

One simple and fast hashing scheme for almost all your needs.

Knuth recommends simple tabulation but cites only 3-independence as mathematical quality. We prove that dependence of simple tabulation is not harmful in any of the above applications.
Chaining/hashing into bins

**Theorem** Consider hashing $n$ balls into $m \geq n^{1-1/(2c)}$ bins by simple tabulation. Let $q$ be an additional *query ball*, and define $X_q$ as the number of regular balls that hash into a bin chosen as a function of $h(q)$. Let $\mu = \mathbb{E}[X_q] = \frac{n}{m}$. The following probability bounds hold for any constant $\gamma$:

$$\Pr[X \geq (1 + \delta)\mu] \leq \left(\frac{e^\delta}{(1 + \delta)^{(1+\delta)}}\right)^{\Omega(\mu)} + m^{-\gamma}$$

$$\Pr[X \leq (1 - \delta)\mu] \leq \left(\frac{e^{-\delta}}{(1 - \delta)^{(1-\delta)}}\right)^{\Omega(\mu)} + m^{-\gamma}$$

With $m \leq n$ bins, every bin gets $n/m \pm O\left(\sqrt{n/m \log^c n}\right)$ keys with probability $1 - n^{-\gamma}$. 
Hashing into many bins

**Lemma** If we hash $n$ keys into $n^{1+\Omega(1)}$ bins, then all bins get $O(1)$ keys w.h.p.
Hashing into many bins

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Nothing like this lemma holds if we instead of simple tabulation assumed $k$-independent hashing with $k = O(1)$. 
Hashing into many bins

**Lemma** If we hash $n$ keys into $n^{1+\Omega(1)}$ bins, then all bins get $O(1)$ keys w.h.p.

**Proof** that for any positive constants $\varepsilon, \gamma$, if we hash $n$ keys into $m$ bins and $n \leq m^{1-\varepsilon}$, then all bins get less than $d = 2^{(1+\gamma)/\varepsilon}$ keys with probability $\geq 1 - m^{-\gamma}$.
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**Claim 1** Any set $T$ contains a subset $U$ of $\log_2 |T|$ keys that hash independently.
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- Let \( i \) be character position where keys in \( T \) differ.
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- Reduce \( T \) to \( T' \) removing all keys \( y \) from \( T \) with \( y_i = a \).
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- The hash of \( x \) is independent of the hash of \( T' \) as only \( h(x) \) depends on \( R_i[a] \).
Hashing into many bins

**Lemma** If we hash $n$ keys into $n^{1+\Omega(1)}$ bins, then all bins get $O(1)$ keys w.h.p.

**Proof** that for any positive constants $\varepsilon, \gamma$, if we hash $n$ keys into $m$ bins and $n \leq m^{1-\varepsilon}$, then all bins get less than $d = 2^{(1+\gamma)/\varepsilon}$ keys with probability $\geq 1 - m^{-\gamma}$.

**Claim 1** Any set $T$ contains a subset $U$ of $\log_2 |T|$ keys that hash independently.

- Let $i$ be character position where keys in $T$ differ.
- Let $a$ be least common character in position $i$ and pick $x \in T$ with $x_i = a$
- Reduce $T$ to $T'$ removing all keys $y$ from $T$ with $y_i = a$.
- The hash of $x$ is independent of the hash of $T'$ as only $h(x)$ depends on $R_i[a]$.
- Return $\{x\} \cup U'$ where $U'$ independent subset of $T'$. 
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**Claim 1** Any set $T$ contains a subset $U$ of $\log_2 |T|$ keys that hash independently—if $|T| \geq d$ then $|U| \geq (1 + \gamma)/\varepsilon$. □
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Claim 2 The probability that there exists $u = (1 + \gamma)/\epsilon$ keys hashing independently to the same bin is $m^{-\gamma}$. 
Hashing into many bins

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- There are \( \binom{n}{u} < n^u \) sets \( U \) of \( u \) keys to consider.
- By independence, \( U \) hash to one bin with probability \( m^{u-1} \).
- Recall \( n \leq m^{1-\varepsilon} \).
Hashing into many bins

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Basic proof pattern with $m \geq n^{1-1/(2c)}$ bins
Basic proof pattern with \( m \geq n^{1-1/(2c)} \) bins

- Deterministic partition key set \( S \) into groups \( G \) that are mutually “independent”, each of size \( \leq n^{1-1/c} \leq m^{1-\varepsilon} \).
Basic proof pattern with $m \geq n^{1-1/(2c)}$ bins

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- Let $X_G \leq d$ be contribution to fixed bin, and $X = \sum_G X_G$. 

$\Pr\left[X \geq (1+\delta)\mu\right] \leq \left(\frac{e\delta(1+\delta)}{1+\delta}\right)\frac{\mu}{d}$

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- Let $X_G \leq d$ be contribution to fixed bin, and $X = \sum_G X_G$.
- If the $X_G$ were really independent, by Chernoff

$$
\Pr[X \geq (1 + \delta)\mu] \leq \left(\frac{e^{\delta}}{(1 + \delta)(1 + \delta)}\right)^{\mu/d}
$$

$$
\Pr[X \leq (1 - \delta)\mu] \leq \left(\frac{e^{-\delta}}{(1 - \delta)(1 - \delta)}\right)^{\mu/d}
$$
Recursive partition into “independent” groups

Define position character \((i, a)\) in key \(x\) iff \(x_i = a\).
Recursive partition into “independent” groups

Define position character \((i, a)\) in key \(x\) iff \(x_i = a\).
Let \((i, a)\) be least common position character among keys in \(S\)
and \(G_{(i,a)} \subseteq S\) be the group of keys using it.

Claim \(|G_{(i,a)}| \leq n^{1-1/c}\).
▶ For each position \(i \in [c]\), we have \(n^{1/c}\) characters used
▶ so claim false implies \(S\) in hypercube of size \(< (n^{1/c})^c = n\).
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Claim \(|G_{(i,a)}| \leq n^{1-1/c}\). □

Recursively, we group \(S \setminus G_{(i,a)}\) and hash all position characters in \(S\) excluding \((i, a)\).
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Recursive partition into “independent” groups

Define position character $(i, a)$ in key $x$ iff $x_i = a$. Let $(i, a)$ be least common position character among keys in $S$ and $G_{(i,a)} \subseteq S$ be the group of keys using it.

Claim $|G_{(i,a)}| \leq n^{1-1/c}$. □

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- Particularly, it is fixed which keys from $G_{(i,a)}$ end in same bin. By Lemma, w.h.p., at most $d$ in every bin.
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Now we randomly pick \(R_i[a]\) finalizing hashing of group \(G_{(i, a)}\).
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- But always \(\mathbb{E}[X_{G(i,a)}] = |X_{G(i,a)}| / m\). Moreover \(X_{G(i,a)} \leq d\).
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- But always \(\mathbb{E}[X_{G(i,a)}] = |X_{G(i,a)}|/m\). Moreover \(X_{G(i,a)} \leq d\).
- Good enough for Chernoff bounds.
Chernoff with $m \geq n^{1-1/(2c)}$ bins

W.h.p., the contribution $X$ to given obeys Chernoff

$$\Pr[X \geq (1 + \delta)\mu] \leq \left(\frac{e^{\delta}}{(1 + \delta)(1+\delta)}\right)^{\mu/d}$$

$$\Pr[X \leq (1 - \delta)\mu] \leq \left(\frac{e^{-\delta}}{(1 - \delta)(1-\delta)}\right)^{\mu/d}$$
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Similar story for linear probing.
Cuckoo hashing

Each key placed in one of two hash locations.

\[
\begin{array}{c}
\text{z} \\
\bullet \\
\bullet \\
y \\
x \\
\bullet \\
r
\end{array}
\quad
\begin{array}{c}
\bullet \\
\text{s} \\
\text{w} \\
f \\
\bullet \\
a \\
b
\end{array}
\quad
\begin{array}{c}
\bullet \\
\text{x} \\
x \\
x \\
\bullet \\
\bullet \\
\bullet
\end{array}
\]

**Theorem** With simple tabulation Cuckoo hashing works with probability \(1 - \tilde{\Theta}(n^{-1/3})\).
Cuckoo hashing

Each key placed in one of two hash locations.

\[
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  \bullet \\
  \bullet \\
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  \bullet \\
  r \\
\end{array}
\quad
\begin{array}{c}
  \bullet \\
  s \\
  w \\
  f \\
  \bullet \\
  a \\
  b \\
\end{array}
\quad
\begin{array}{c}
  x \mapsto \\
  x \mapsto \\
\end{array}
\]

**Theorem** With simple tabulation Cuckoo hashing works with probability \(1 - \tilde{\Theta}(n^{-1/3})\).

- For chaining and linear probing, we did not care about a constant loss, but obstructions to cuckoo hashing may be of just constant size, e.g., 3 keys sharing same two hash locations.
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\end{array} \\
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\bullet \\
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\bullet \\
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Theorem With simple tabulation Cuckoo hashing works with probability \( 1 - \tilde{\Theta}(n^{-1/3}) \).

- For chaining and linear probing, we did not care about a constant loss, but obstructions to cuckoo hashing may be of just constant size, e.g., 3 keys sharing same two hash locations.
- Very delicate proof showing that obstruction can be used to code random tables \( R_i \) with few bits.
### Speed

<table>
<thead>
<tr>
<th>Hashing random keys</th>
<th>32-bit computer</th>
<th>64-bit computer</th>
</tr>
</thead>
<tbody>
<tr>
<td>bits</td>
<td>hashing scheme</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>univ-mult-shift</td>
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<tr>
<td>32</td>
<td>(a*x) &gt;&gt; s</td>
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<tr>
<td>32</td>
<td>2-indep-mult-shift</td>
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<td>32</td>
<td>5-indep-Mersenne-prime</td>
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<tr>
<td>32</td>
<td>5-indep-TZ-table</td>
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<td>simple-table</td>
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<td>64</td>
<td>simple-table</td>
<td>15.54</td>
</tr>
</tbody>
</table>

Experiments with help from Yin Zhang.
Robustness in linear probing for dense interval

![Graph showing cumulative fraction vs. average time per insert+delete cycle (nanoseconds)]
Pitch for theory in case of linear probing

- Multiplicative hashing used in practice, but turns out to be very unreliable under typical denial-of-service (DoS) attacks based on consecutive IP addresses: systematic good performance 90% of the time, but systematic terrible performance 10% of the time [TZ’10].
Pitch for theory in case of linear probing

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- Problems in randomized algorithms like hashing hard to detect for practitioners. Hard for them to know if bad performance is from being unlucky, or because of systematic problems.
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- Linear probing had gotten a reputation for being fastest in practice, but sometimes unreliable needing special protection against bad cases.
- Here we proved linear probing safe with good probabilistic performance for all input if we use simple tabulation.
- Simple tabulation also powerful for chaining, cuckoo hashing, and min-wise hashing:
  
  one simple and fast scheme for (almost) all your needs.
Work in progress: short range amortization with twisted tabulation

- With chaining and linear probing, each operation takes expected constant time, but out of $\sqrt{n}$ operations, some are expected to take $\tilde{\Omega}(\log n)$ time.
Work in progress: short range amortization with twisted tabulation

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- With truly random hash function, we handle every window of $\log n$ operations in $O(\log n)$ time w.h.p.

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- So far, no technique is known that can make any such separation between deterministic and randomized solutions for any data structure problem.