Statistics on Manifolds

PGA, Jacobi fields, and optimization

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Manifold Valued Statistics

For almost all problems involving non-linearity, (Riemannian) manifolds are useful modeling tools. But the loss of vector space structure means that the usual Euclidean space statistical operations must be redefined. For some operations this is easy, e.g. kNN, which relies solely on the metric structure. For others, it is considerably harder.

Regression, SVM, and PCA has been generalized to manifolds, but, even for these operations, many unsolved issues remain. One is how to compute the result of the operations. Often, problems appears as optimization problems involving variations of geodesics. This work considers such problems, how they can be solved without linearizing the manifold, and how the resulting computations allow the computation of Principal Geodesic Analysis, a generalization of PCA to manifolds.

Principal Geodesic Analysis, PGA, finds geodesics subspaces of a manifold which either maximizes the variance of the projection of a dataset to the subspaces or minimizes the reconstruction errors. In Euclidean space, we have

$$v^i = \text{argmax}_{\|v\|=1} \frac{1}{N}\sum_{j=1}^{N}\left(\left\langle x_j, v\right\rangle^2 + \sum_{l=1}^{i-1}\left\langle x_j, v^l\right\rangle^2\right)$$

using orthogonal projections. For manifolds, this translates, in the variance formulation of PGA, to

$$v^i = \text{argmax}_{\|v\|=1, v\in V_{i-1}^{\perp}} \frac{1}{N}\sum_{j=1}^{N} d(\mu, \pi_{S_v}(x_j))^2$$

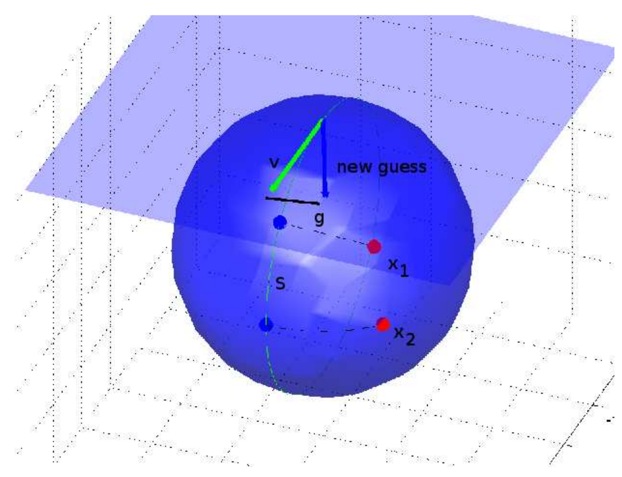
$$S_v = \text{Exp}_\mu(\text{span}\{V_{i-1}, v\})$$

which uses the manifold projection $\pi_{S_v}(x_j)$ of the data point x_j to the geodesic subspace S_v and manifolds distances by the metric $d(\cdot,\cdot)$.

Exact PGA Algorithm

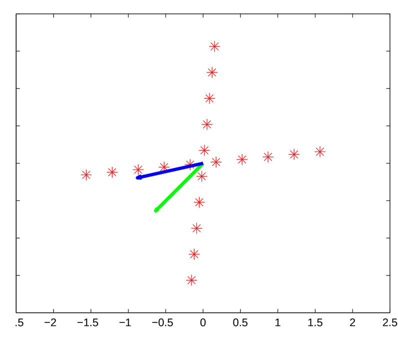
In [SLN10], we perform gradient descent or similar optimization methods on the cost function defining PGA. The hard part is differentiating $d(\mu, \pi_{S_v}(x_j))^2$, and, in particular, the projection $\pi_{S_v}(x_j)$, which itself is an optimization problem. Using the IVP's for the first and second derivative of the exponential map, we get this gradient.

An iteration of exact PGA:

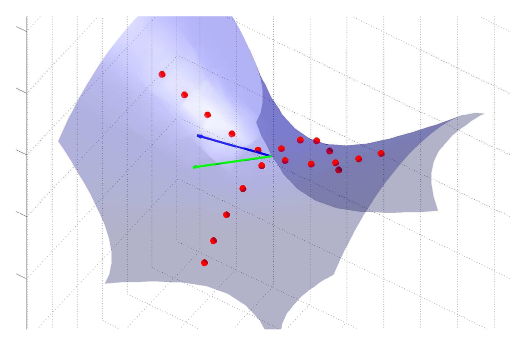


The computations are heavy, but the indicators can determine if the linearized algorithm (tangent space PCA) is sufficient.

Differences in tagents space:



Differences on manifold:



We see several interesting non-Euclidean effects: the minimizing residual and maximizing variance formulations are not equivalent, variance can decrease when including more principal components, the greedy definition of PGA results in weak performance, etc.

Experiment: Human Poses

The pose of a human body can be represented by spatial coordinates of the end-effectors: joints and end-points of bones. Since the length of bones is constant, the poses will reside on a $(3k-b)$-dimensional implicitly represented manifold $M = F^{-1}(0)$. Here

$$F^i(x) = \|e_{i_1} - e_{i_2}\|^2 - l_i^2\,,$$

where e_{i_1} and e_{i_2} denote the coordinates of the end-effectors and l_i the constant length of the ith bone.

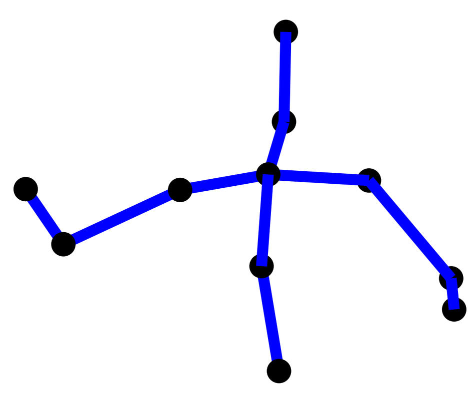
In [SLHN10], we perform linearized and exact PGA and get differences as follows:

Princ. comp.:	1	2	3	4
angle (°):	5.30	2.19	1.82	1.21
approx. sq. res.:	2.43	1.17	0.43	0.10
exact sq. res.:	2.41	1.18	0.44	0.11
difference:	0.05	-0.01	-0.01	-0.01
difference (%):	0.5	-0.6	-2.3	-13.3

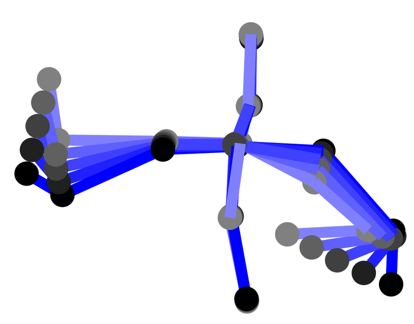
Furthermore, we can predict these differences using indicators.

Camera output:

A recorded pose:



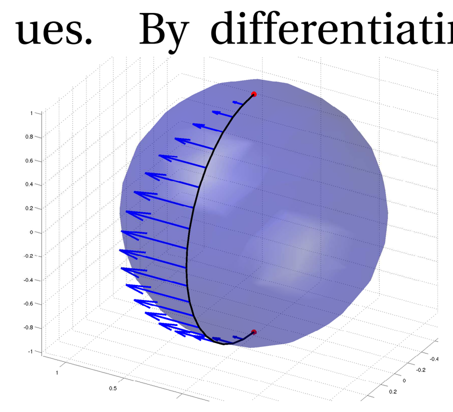
First principal component of eight poses:



The human pose manifold is relatively curved and the recorded data show large variation. This gives notable differences between the two methods. For other datasets (e.g. bicycle chain shape manifolds), the differences are neglible showing that linearization is a good approximation for these cases.

Exponential Map, Jacobi Fields, and Derivatives

Jacobi fields arises from the varion of the initial velocity of geodesics. Geodesics can be described by ODE's in which the initial velocity appear as initial values. By differentiating the initial



values, we get ODE's generating Jacobi fields, and second derivaties of the exponential map. Theses ODE's are fundamental in computing exact PGA.

Furthermore, Jacobi fields allow estimation of the sectional curvature of the manifold by the equation

$$\|J_t\| = t - \frac{1}{6}K_{q_0}(\sigma)t^3 + O(t^4)$$

Upper bounds for the injectivity radius, the minimum length of nonminimizing geodesics, can also be computed using vanishing Jacobi fields.

The presented research is joint work with François Lauze, Søren Hauberg, and Mads Nielsen.

References

[SLHN10] Stefan Sommer, Francois Lauze, Søren Hauberg, and Mads Nielsen, Manifold valued statistics, exact principal geodesic analysis and the effect of linear approximations, ECCV 2010 (Heraklion, Greece), Lecture Notes in Computer Science, vol. 6316, Springer, Heidelberg, 2010, pp. 43–56.

[SLN10] Stefan Sommer, François Lauze, and Mads Nielsen, The differential of the exponential map, jacobi fields, and exact principal geodesic analysis, Submitted. (2010).