Intensity based registration
Cubic B-splines

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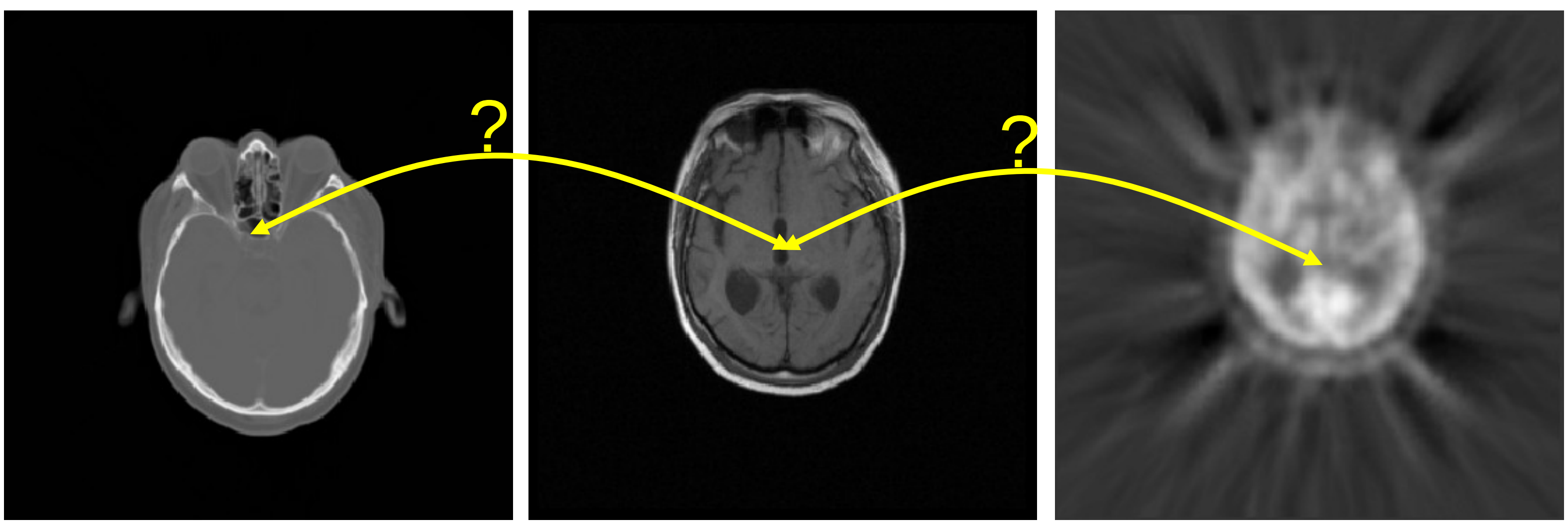
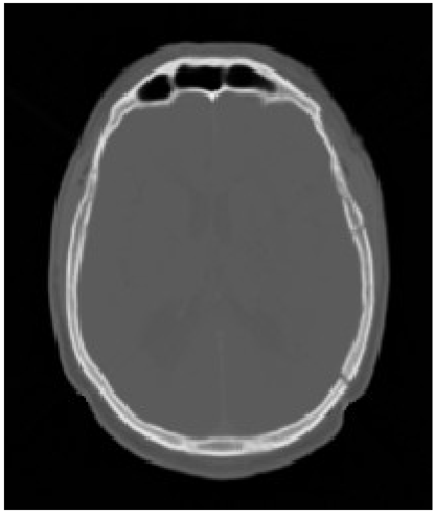
Image registration

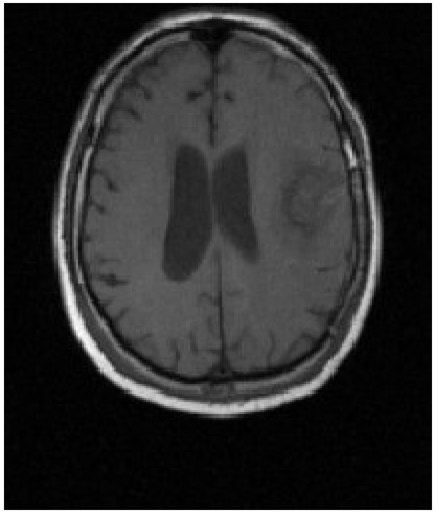
Determine a geometrical transformation that aligns points in an image with corresponding points in other image(s)



Example: CT/MR

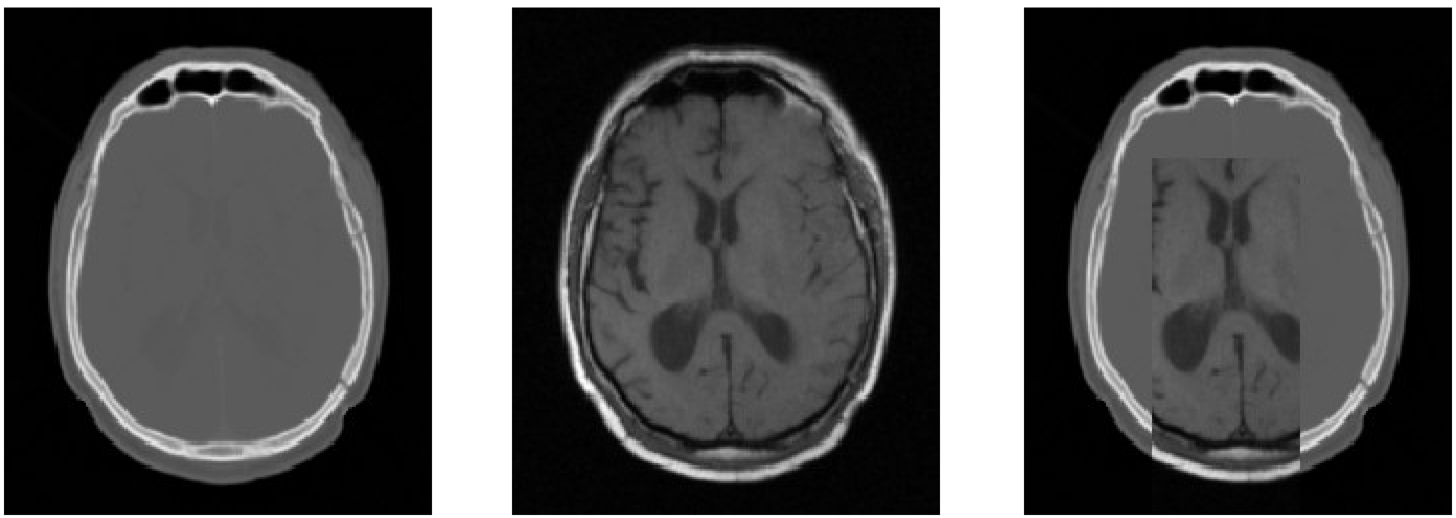
Before...





Example: CT/MR

...after



Elements in image registration

- Geometrical transformation $\mathbf{y}(\mathbf{x};\mathbf{w}) : \mathbb{R}^2 \to \mathbb{R}^2 \text{ or } \mathbb{R}^3 \to \mathbb{R}^3$
- Similarity measure $\mathcal{D}(\mathbf{w})$
- Regularization $\mathcal{S}(\mathbf{w})$
- Optimization algorithm $\mathcal{J}(\mathbf{w}) = \mathcal{D}(\mathbf{w}) + \alpha\mathcal{S}(\mathbf{w})$

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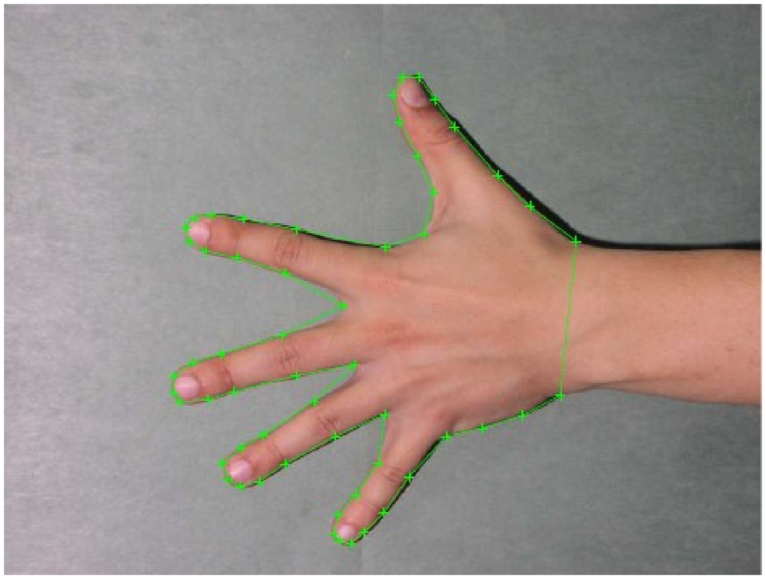
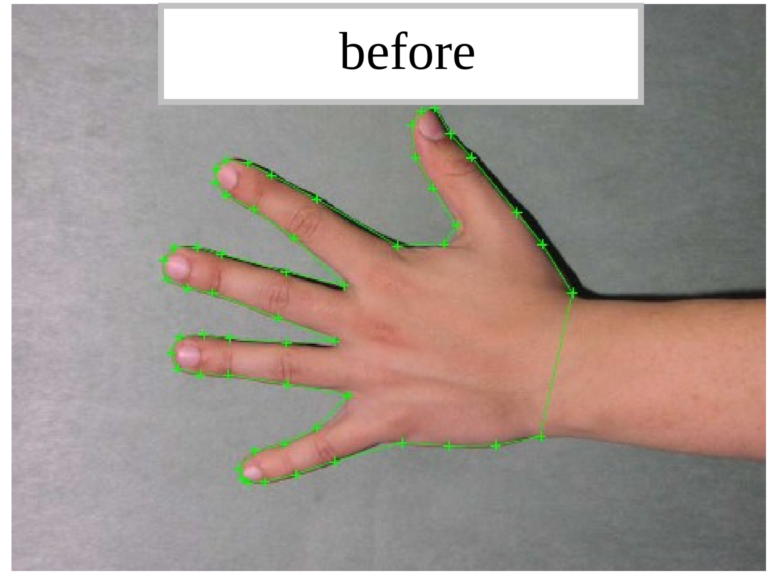
Landmark-based

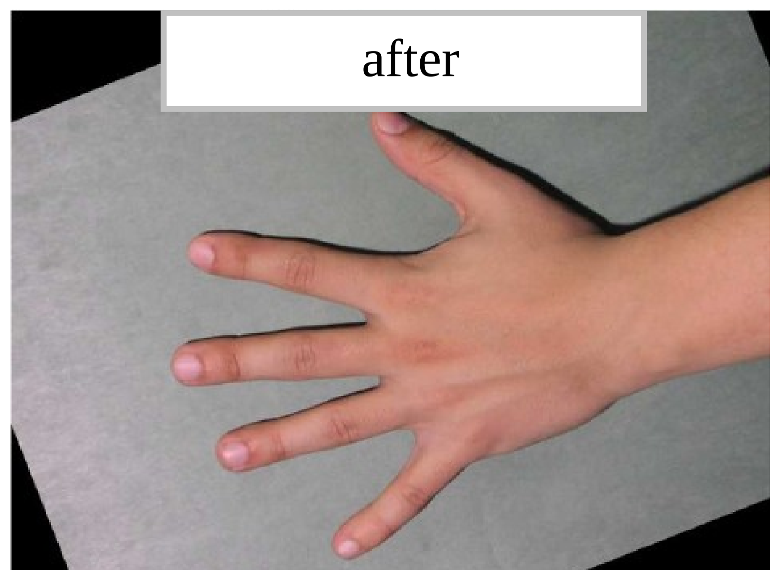
Match two corresponding point sets defined in two images: $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^2 \text{ or } \mathbb{R}^3$

$$\mathcal{D}(\mathbf{w}) = \sum_{i=1}^{N} \|\mathbf{y}(\mathbf{x}_i;\mathbf{w}) - \mathbf{y}_i\|^2$$

Rigid landmark based registration







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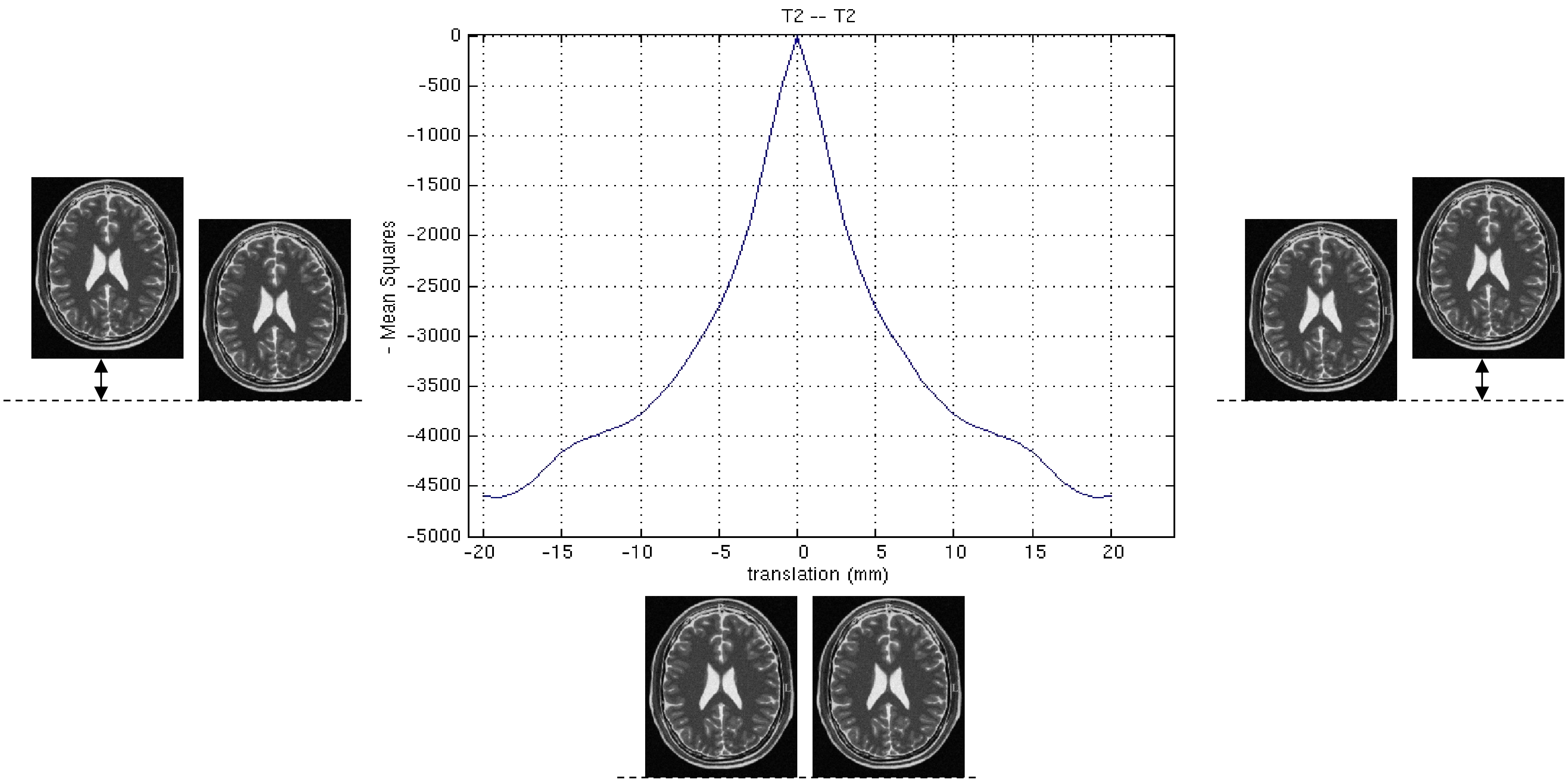
Intensity-based: sum of squared differences

Compare image intensities between template image $\mathcal{T}$ and reference image $\mathcal{R}$:

$$\mathcal{D}(\mathbf{w}) = \frac{1}{2}\sum_{i\in\Omega}\left(\mathcal{T}(\mathbf{y}(\mathbf{x}_i;\mathbf{w})) - \mathcal{R}(\mathbf{x}_i)\right)^2$$

Sum of squared differences



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- Regularization $\mathcal{S}(\mathbf{w})$
- Optimization algorithm $\mathcal{J}(\mathbf{w}) = \mathcal{D}(\mathbf{w}) + \alpha\mathcal{S}(\mathbf{w})$

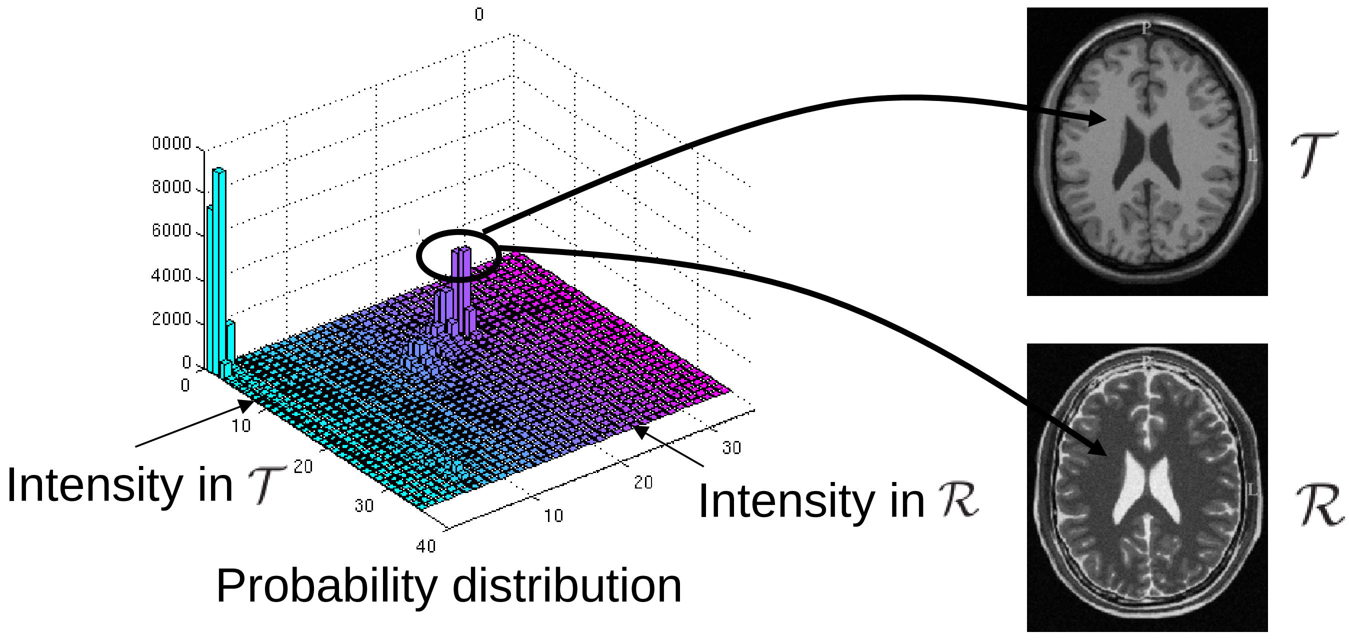
Intensity-based: mutual information

Compare image intensities between template image $\mathcal{T}$ and reference image $\mathcal{R}$:

$$\mathcal{D}(\mathbf{w}) \;=\; H(\mathcal{T}(\mathbf{y}(\mathbf{x}_i;\mathbf{w})),\mathcal{R}(\mathbf{x}_i)) - H(\mathcal{T}(\mathbf{y}(\mathbf{x}_i;\mathbf{w}))) - H(\mathcal{R}(\mathbf{x}_i))$$

Mutual information

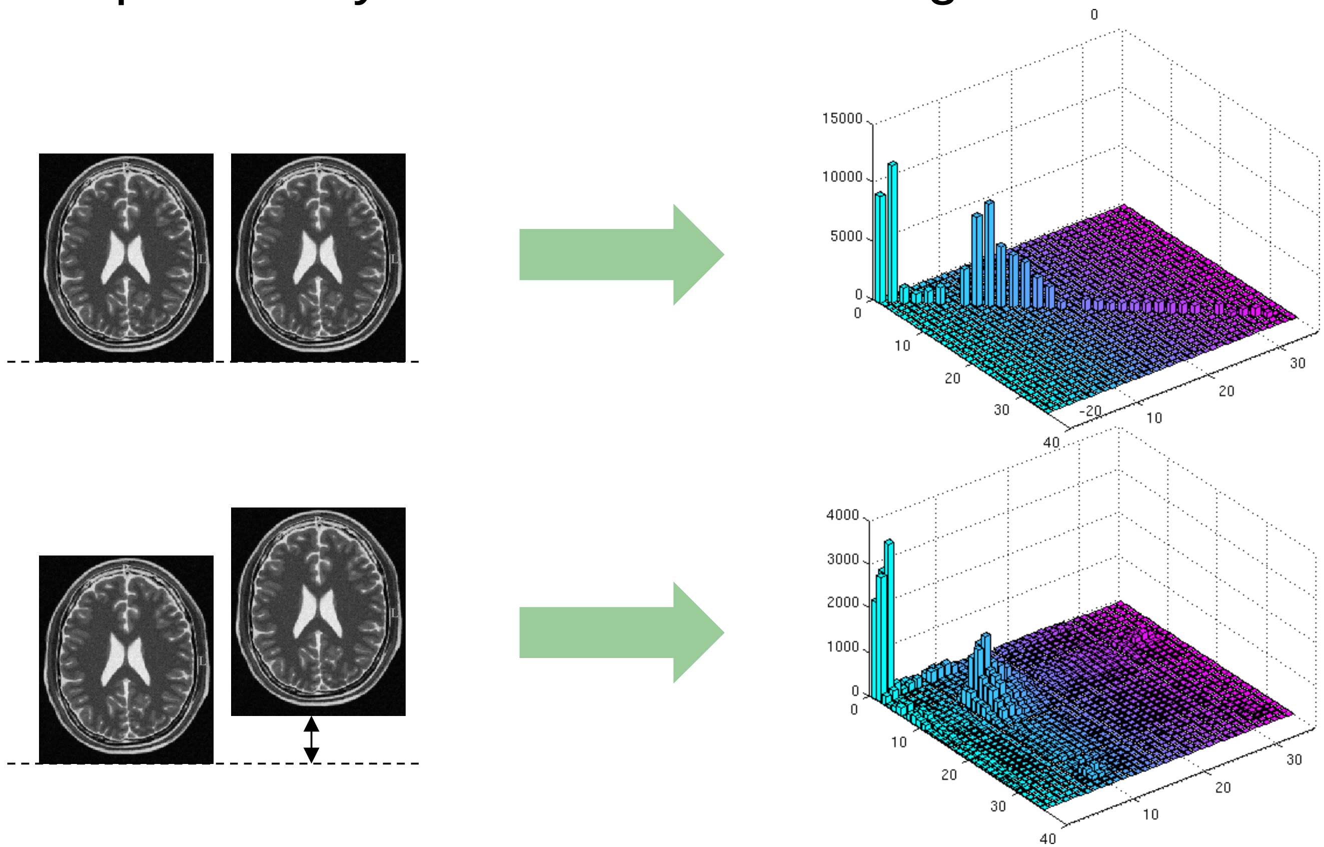
Consider two images as a realization of N draws from a joint distribution



Probability distribution

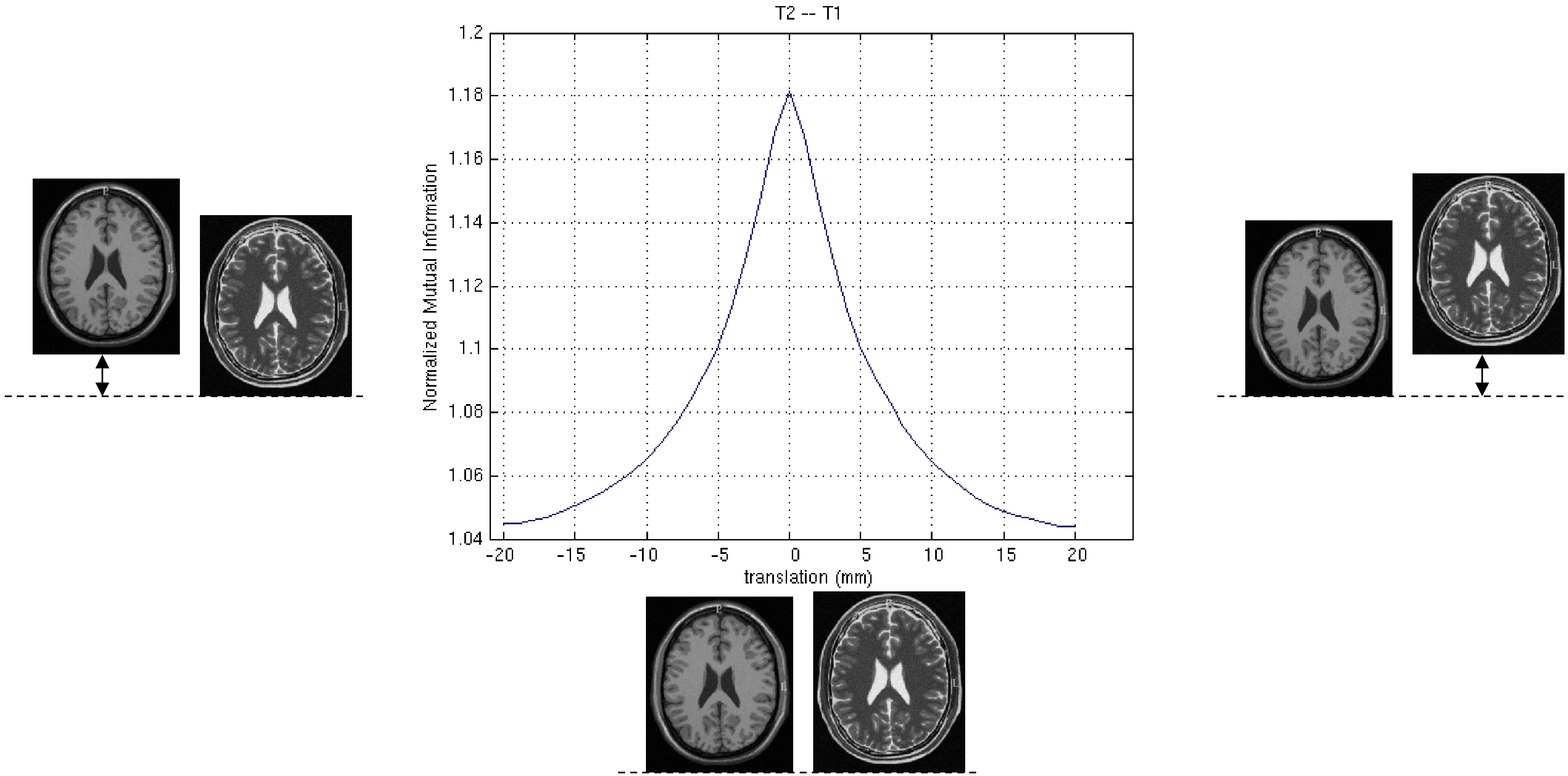
Mutual information

Estimate probability distribution from images



Compute joint and marginal entropies from estimated distribution

Mutual information



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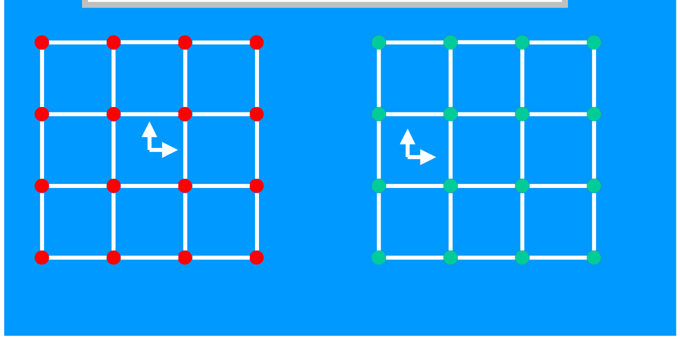
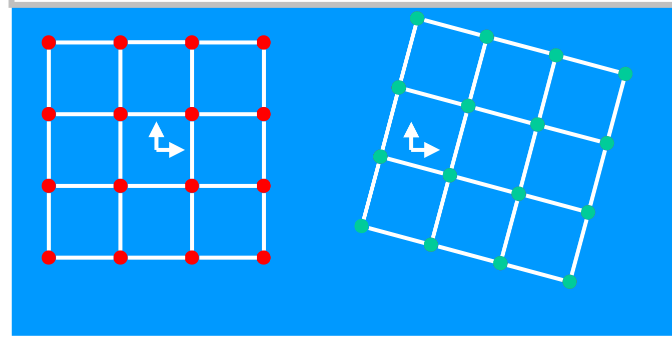
Spatial mappings (linear)

translation



$$\mathbf{y}(\mathbf{x};\mathbf{t}) = \mathbf{x} + \mathbf{t}$$

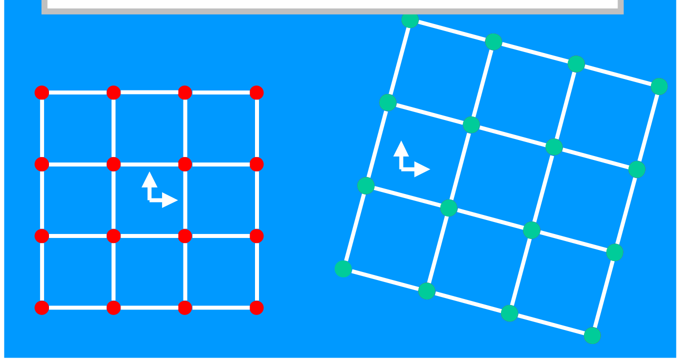
rigid: translation + rotation



$$\mathbf{y}(\mathbf{x};\mathbf{R},\mathbf{t}) = \mathbf{R}\mathbf{x} + \mathbf{t}$$

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}, \quad \det(\mathbf{R}) = 1$$

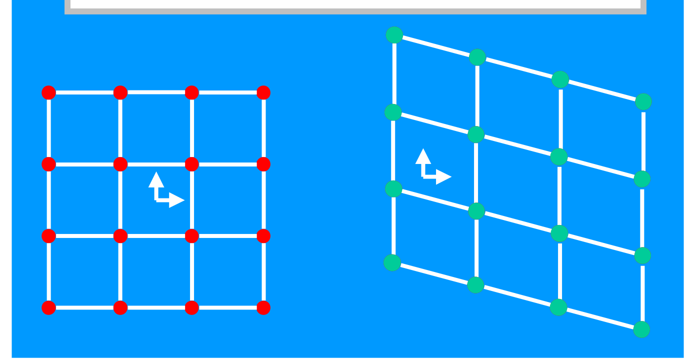
similarity transformation



$$\mathbf{y}(\mathbf{x};s,\mathbf{R},\mathbf{t}) = s\mathbf{R}\mathbf{x} + \mathbf{t}$$

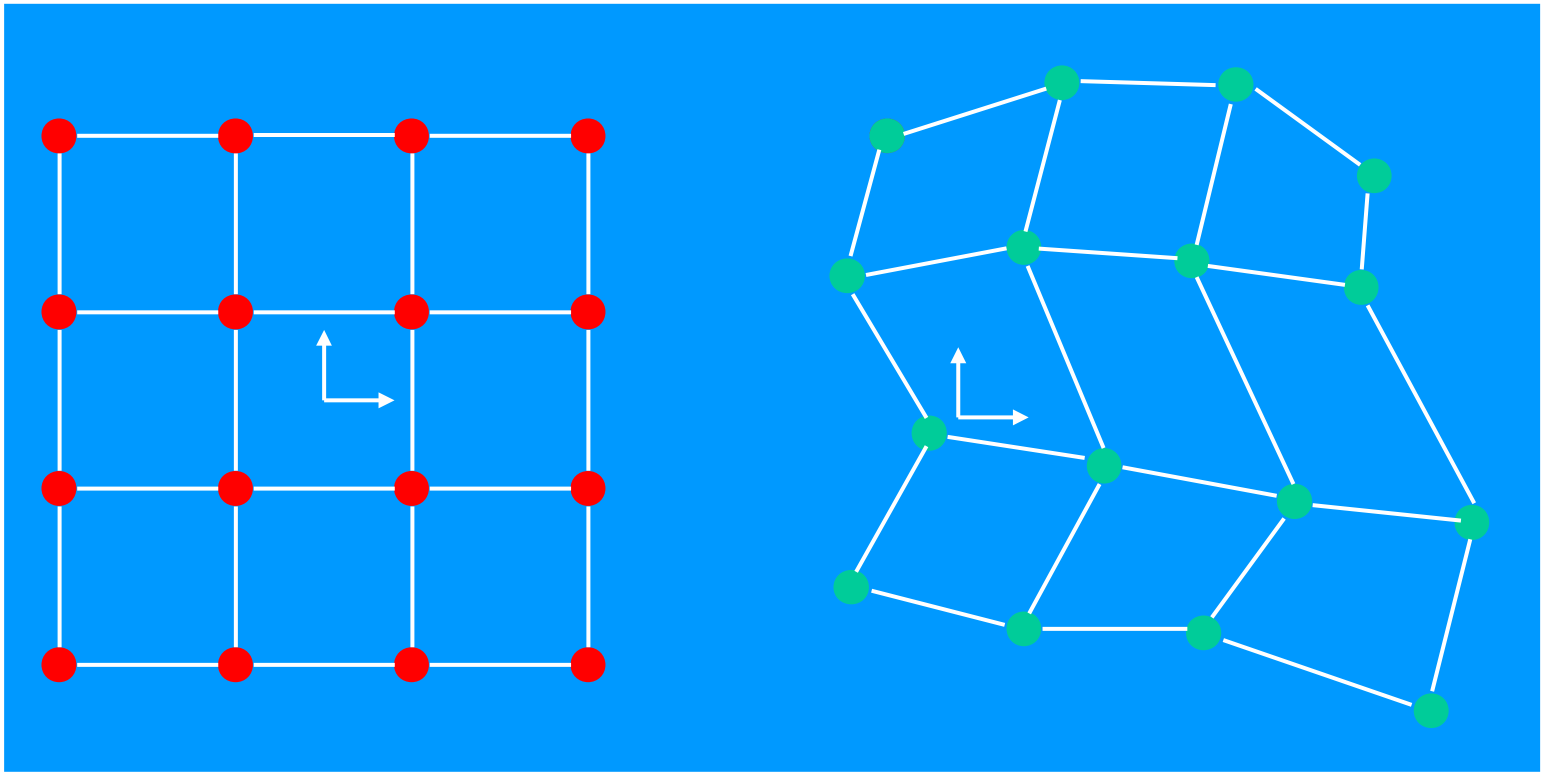
$$s > 0$$

affine transformation



$$\mathbf{y}(\mathbf{x};\mathbf{A},\mathbf{t}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$

Spatial mappings (non-linear)



- Tissue motion (cardiac cycle/respiratory motion)
- Deformation compensation (intra-operative, soft tissue)
- Longitudinal tissue changes (e.g., tumor growth)
- Inter-subject registration

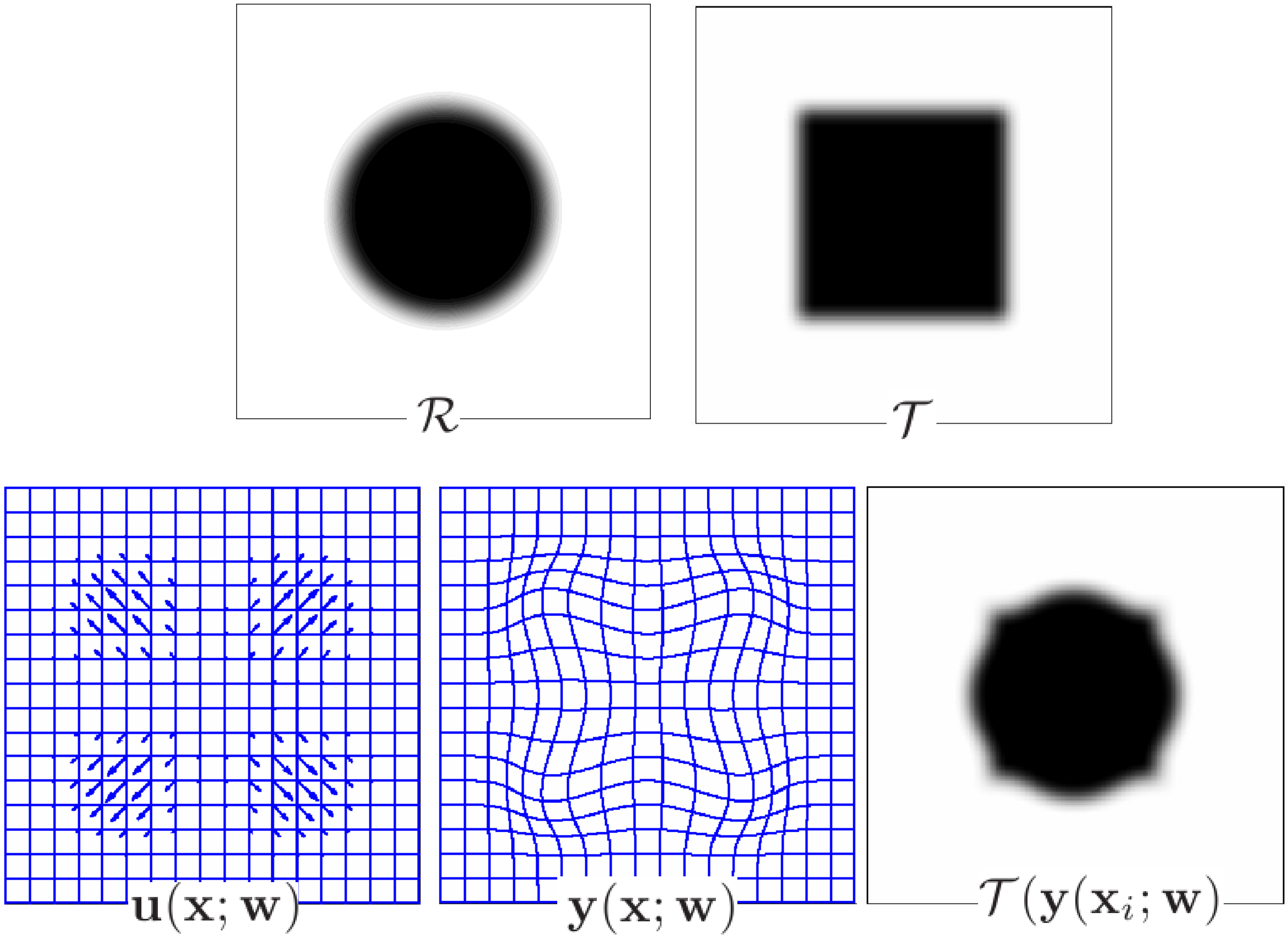
Canonical form

deformation part

$$\mathbf{y}(\mathbf{x};\mathbf{w}) = \mathbf{x} + \mathbf{u}(\mathbf{x};\mathbf{w})$$

identity part

Canonical form: $\mathbf{y}(\mathbf{x};\mathbf{w}) = \mathbf{x} + \mathbf{u}(\mathbf{x};\mathbf{w})$



Affine transformation

$$\begin{aligned}\mathbf{y}(\mathbf{x}_i;\mathbf{A},\mathbf{t}) &= \mathbf{A}\mathbf{x}_i + \mathbf{t} \\ &= \mathbf{x}_i + \underbrace{(\mathbf{A}-\mathbf{I}_3)\mathbf{x}_i + \mathbf{t}}_{\mathbf{u}(\mathbf{x}_i;\mathbf{A},\mathbf{t})}\end{aligned}$$

with $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $\mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$

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All-important trick: look at each coordinate k separately!!!

$$y^k(\mathbf{x}_i;\mathbf{A},\mathbf{t}) = x_i^k + \mathbf{q}(\mathbf{x}_i)^{\mathrm{T}}\mathbf{w}^k$$

$$\mathbf{q}(\mathbf{x}_i) = (1,\ x_i^1,\ x_i^2,\ x_i^3)^{\mathrm{T}}$$

$$\mathbf{w}^k = (t_k,\ a_{k1} - 1,\ a_{k2},\ a_{k3})^{\mathrm{T}} \quad \boxed{(\text{for } k=1)}$$

Affine transformation

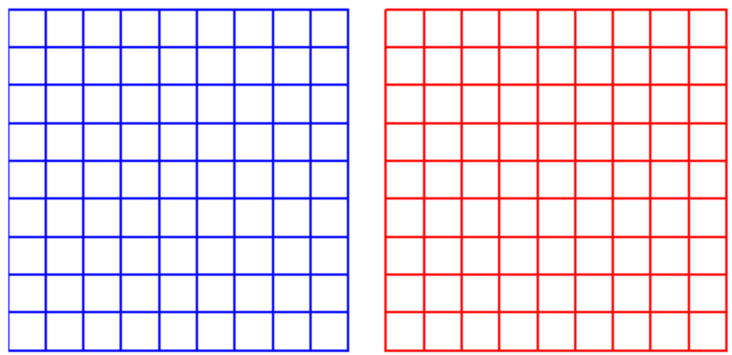
In matrix form: (all N input points simultaneously)

Define

$$\mathbf{x}^k = \begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_N^k \end{pmatrix} \quad \mathbf{y}^k = \begin{pmatrix} y_1^k \\ y_2^k \\ \vdots \\ y_N^k \end{pmatrix} \quad \mathbf{Q}' = \begin{pmatrix} 1 & x_1^1 & x_1^2 & x_1^3 \\ 1 & x_2^1 & x_2^2 & x_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_N^1 & x_N^2 & x_N^3 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{y}^k \quad = \quad \mathbf{x}^k + \mathbf{Q}'\mathbf{w}_k$$

Affine transformation

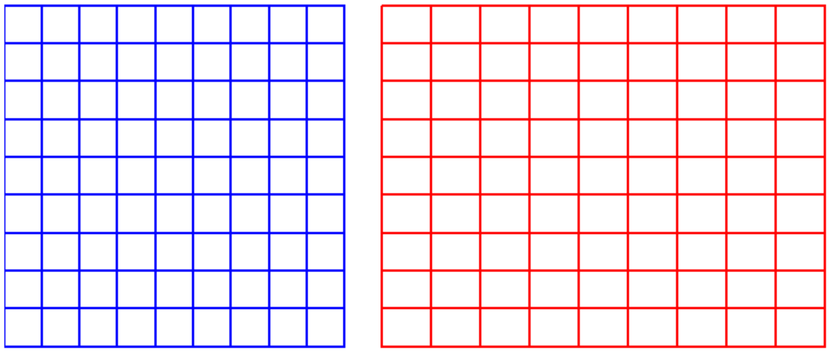


$$\mathbf{y}^1 = \mathbf{x}^1 + \mathbf{Q}' \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{y}^2 = \mathbf{x}^2 + \mathbf{Q}' \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{t} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$$

Affine transformation



$$\mathbf{y}^1 = \mathbf{x}^1 + \mathbf{Q}' \begin{pmatrix} 10 \\ 0.3 \\ 0 \end{pmatrix}$$

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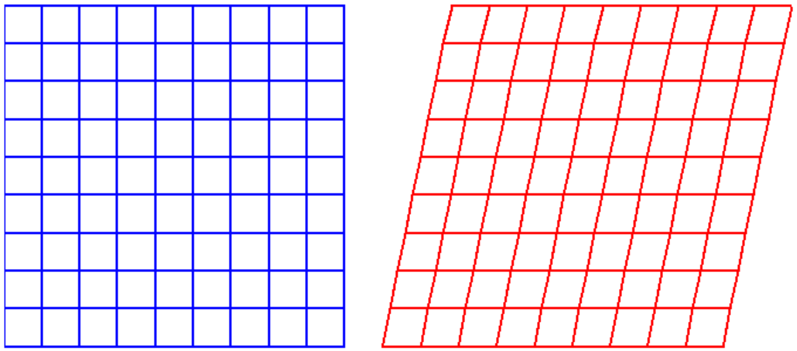
Affine transformation



$$\mathbf{y}^1 = \mathbf{x}^1 + \mathbf{Q}' \begin{pmatrix} 10 \\ 0 \\ 0.2 \end{pmatrix}$$

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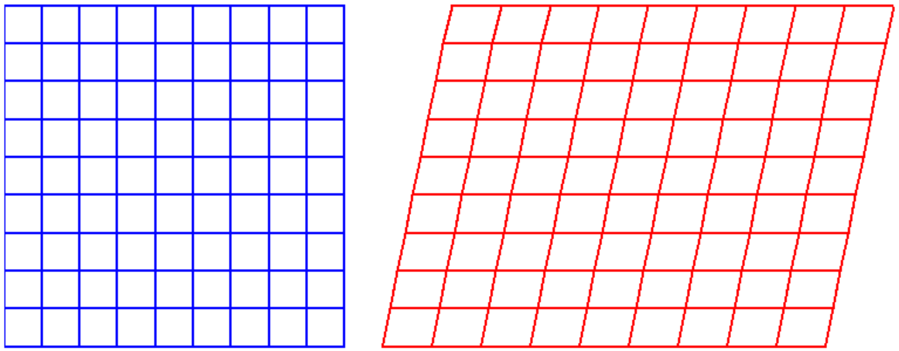
$$\mathbf{A} = \begin{pmatrix} 1 & 0.2 \\ 0 & 1 \end{pmatrix}, \mathbf{t} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$$

Affine transformation



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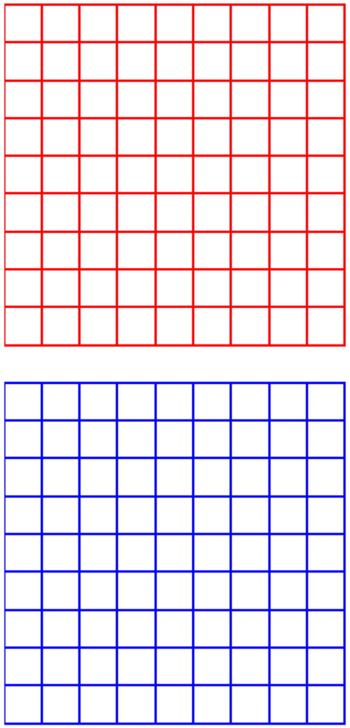

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$$\mathbf{y}^2 = \mathbf{x}^2 + \mathbf{Q}' \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

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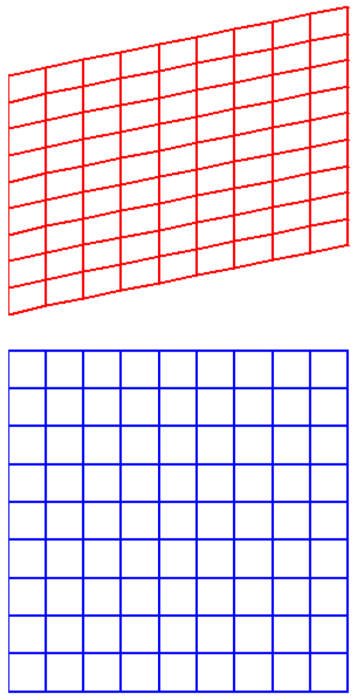
Affine transformation

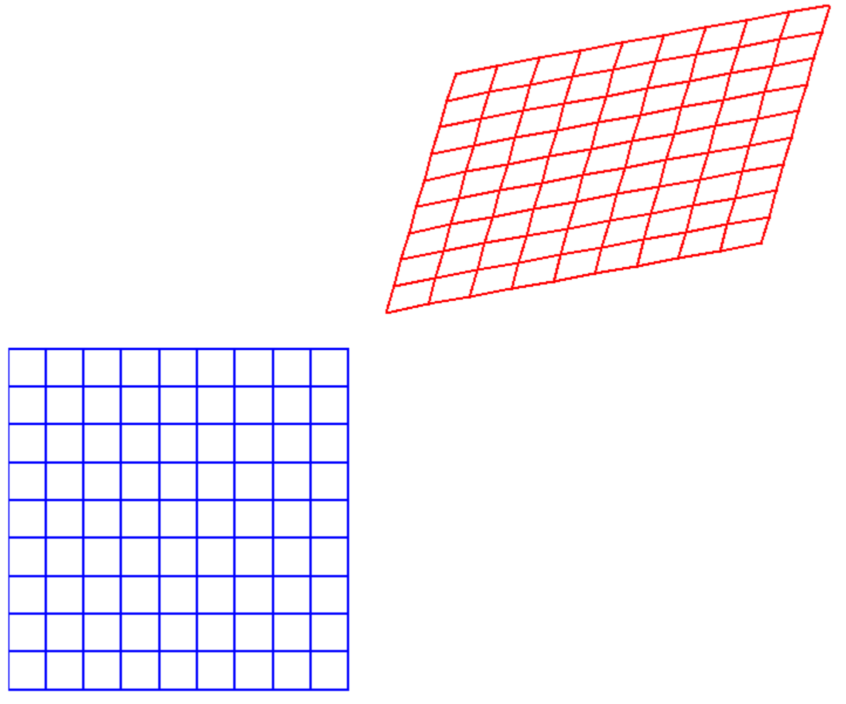


$$\mathbf{y}^1 = \mathbf{x}^1 + \mathbf{Q}' \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{y}^2 = \mathbf{x}^2 + \mathbf{Q}' \begin{pmatrix} 10 \\ 0.2 \\ 0.3 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0.2 & 0.7 \end{pmatrix}, \mathbf{t} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$$

Affine transformation



$$\mathbf{y}^1 = \mathbf{x}^1 + \mathbf{Q}' \begin{pmatrix} 10 \\ 0.1 \\ 0.2 \end{pmatrix}$$

$$\mathbf{y}^2 = \mathbf{x}^2 + \mathbf{Q}' \begin{pmatrix} 10 \\ 0.2 \\ 0.3 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1.1 & 0.2 \\ 0.2 & 0.7 \end{pmatrix}, \mathbf{t} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

Non-linear transformations

Just use basis functions that depend ***non-linearly*** on $\mathbf{x}$

$$\mathbf{y}^k \quad = \quad \mathbf{x}^k + \mathbf{Q}'\mathbf{w}_k$$

$$\mathbf{Q}' = \begin{pmatrix} 1 & x_1^1 & x_1^2 & x_1^3 \\ 1 & x_2^1 & x_2^2 & x_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_N^1 & x_N^2 & x_N^3 \end{pmatrix}$$

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Non-linear transformations

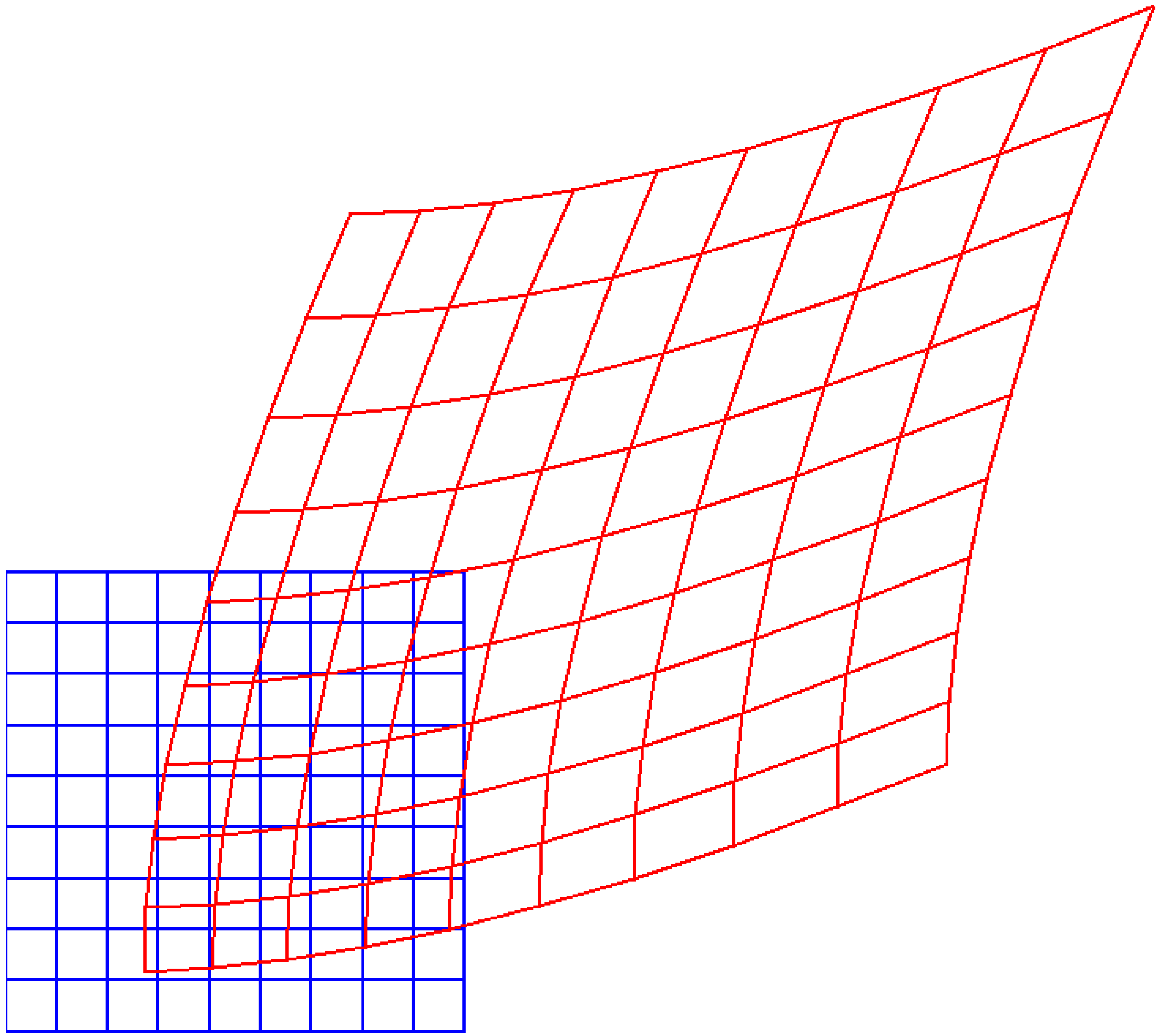
Just use basis functions that depend ***non-linearly*** on $\mathbf{x}$

$$\mathbf{y}^k = \mathbf{x}^k + \mathbf{Q}'\mathbf{w}_k$$

$$\mathbf{Q}' = \begin{pmatrix} \phi_1(\mathbf{x}_1) & \phi_2(\mathbf{x}_1) & \cdots & \phi_M(\mathbf{x}_1) \\ \phi_1(\mathbf{x}_2) & \phi_2(\mathbf{x}_2) & \cdots & \phi_M(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(\mathbf{x}_N) & \phi_2(\mathbf{x}_N) & \cdots & \phi_M(\mathbf{x}_N) \end{pmatrix}$$

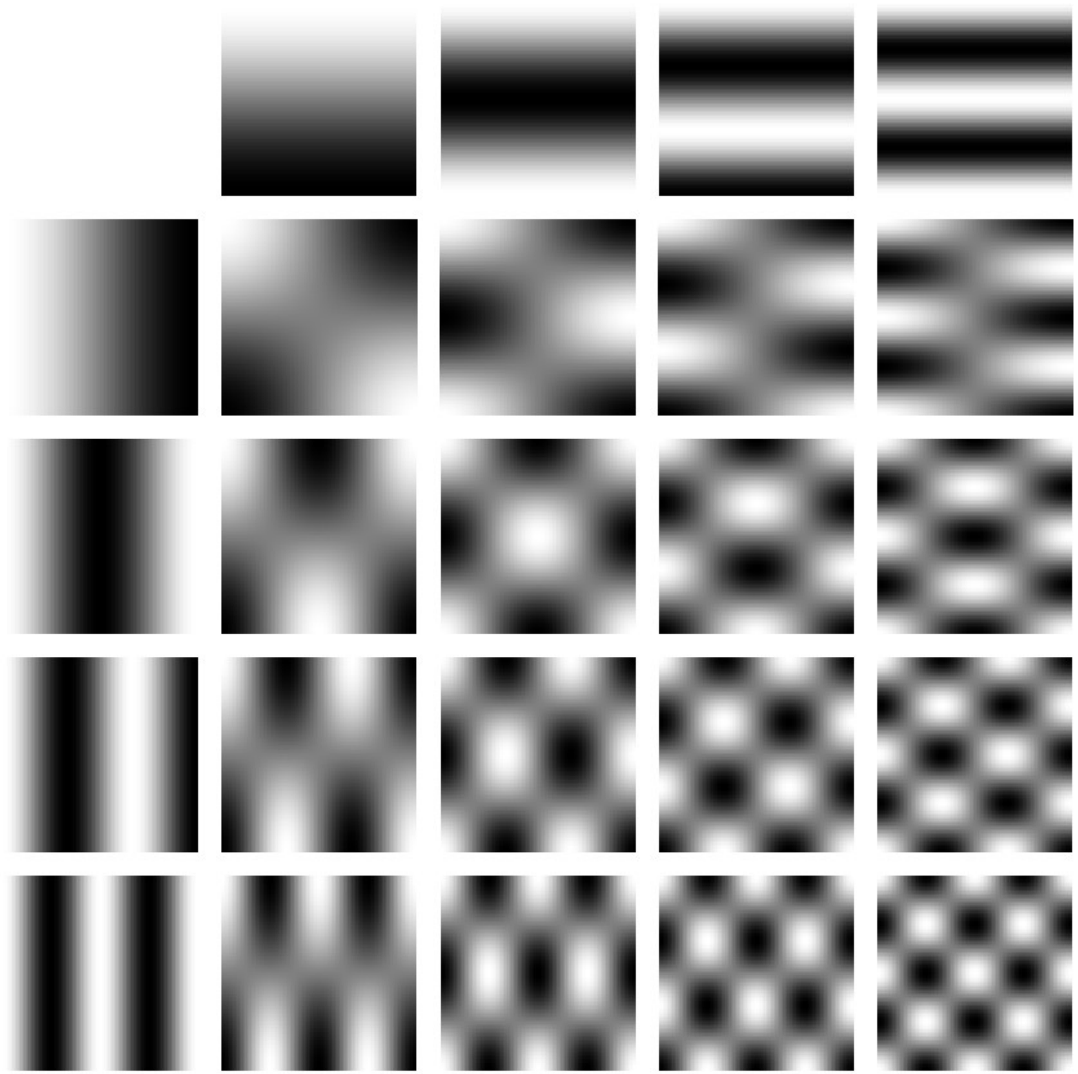
Non-linear in input variables, but still linear in parameters!

Non-linear transformations

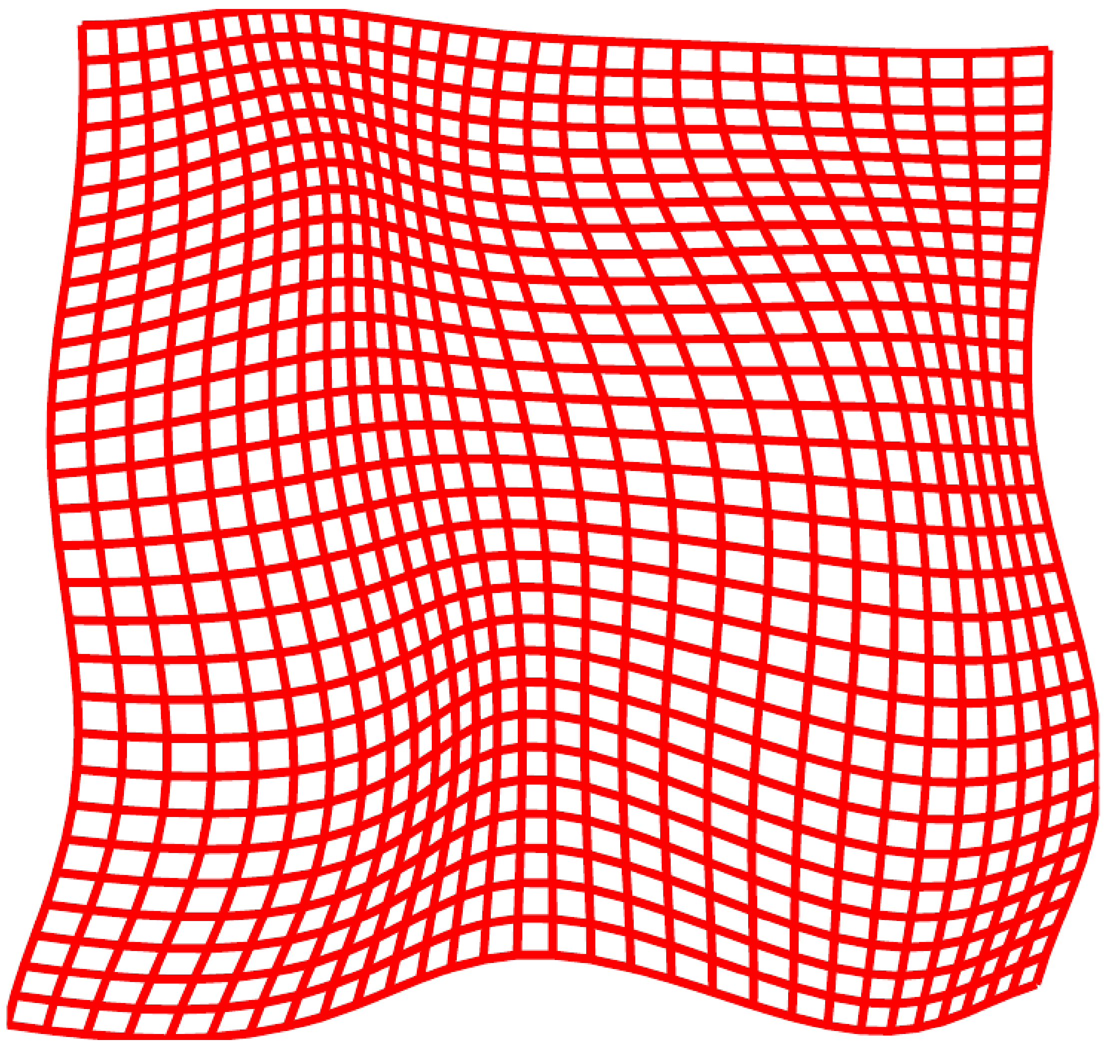


Non-linear transformations

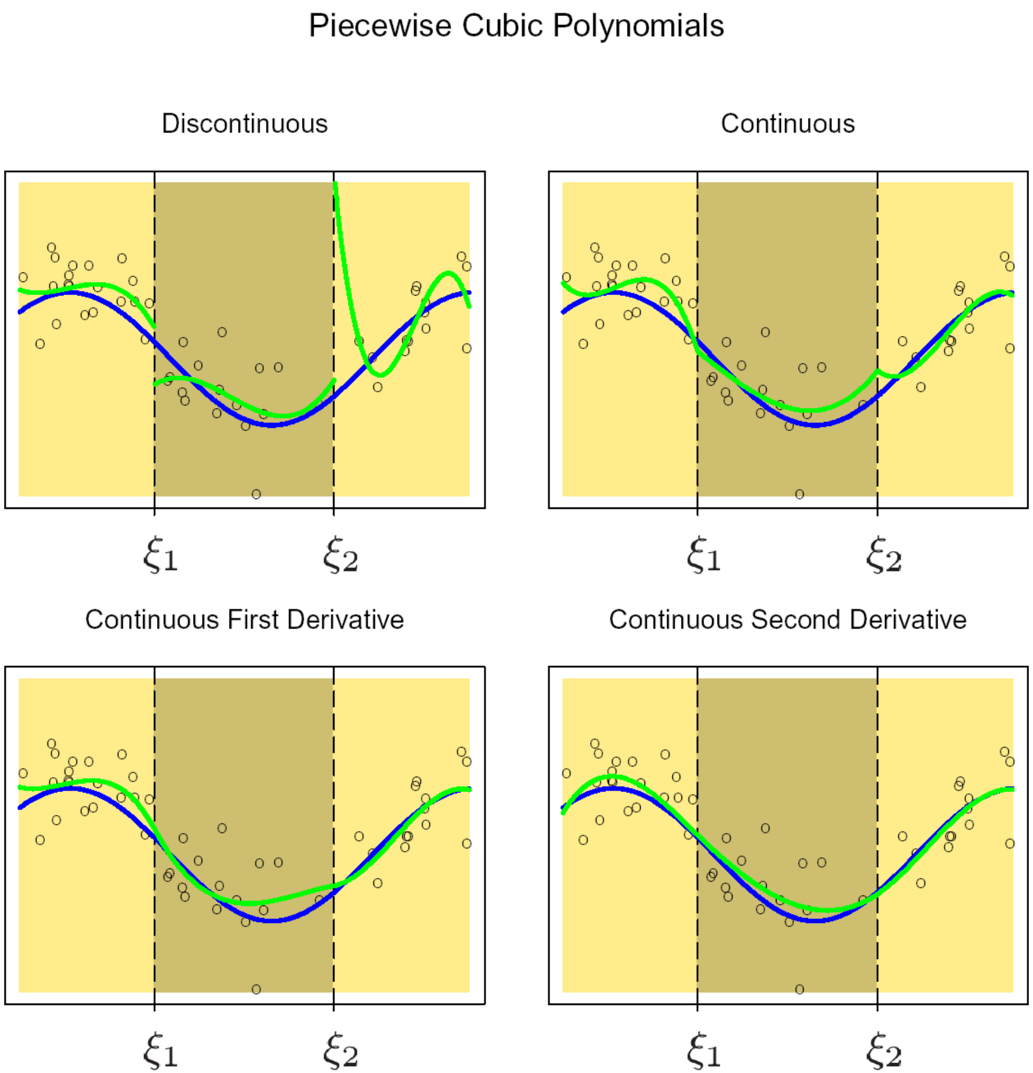
$\phi_m(\mathbf{x})$



Non-linear deformations

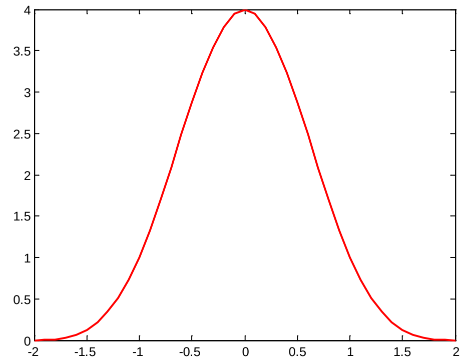


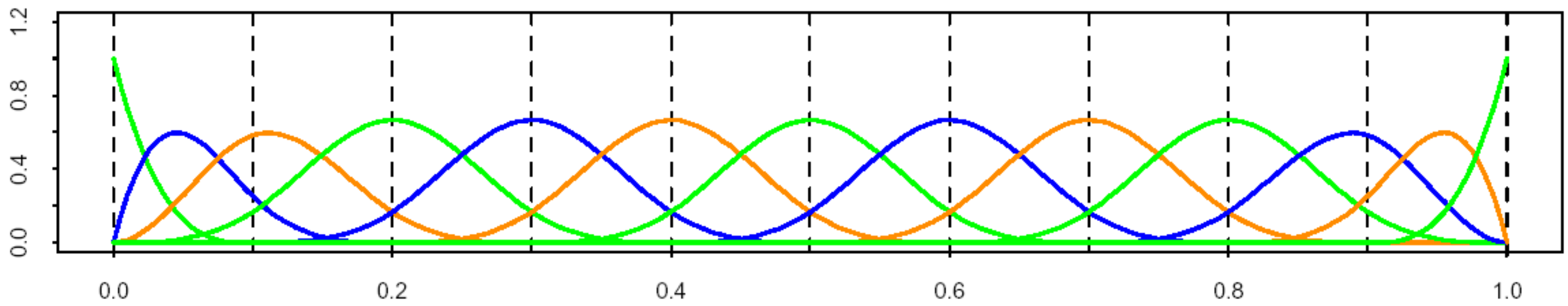
Cubic b-spline I



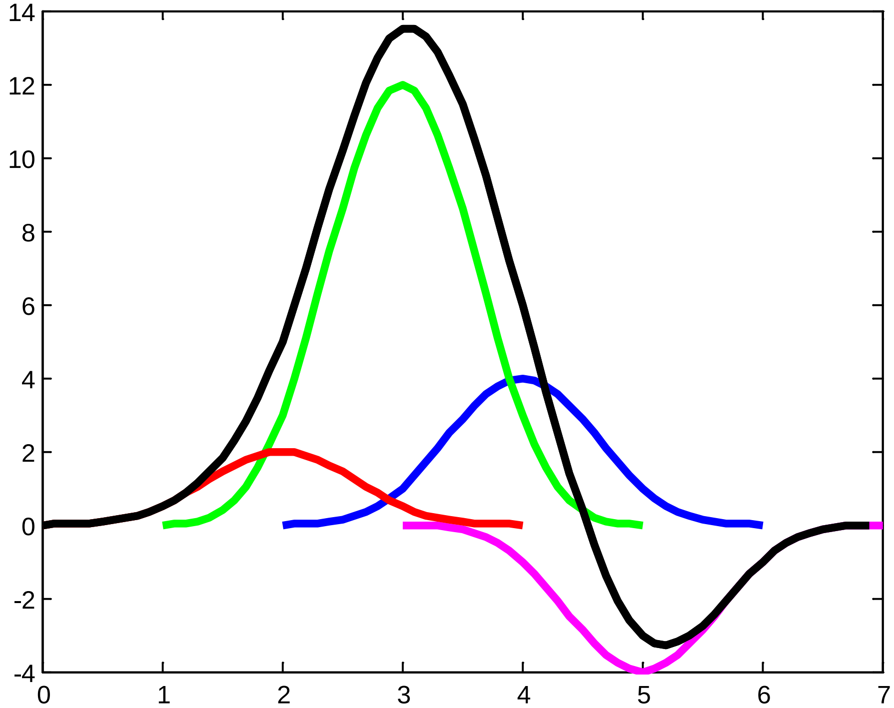
Cubic b-spline II

$$b_j(x) = \begin{cases} (2 + x_j)^3 & \text{for } -2 \le x_j < -1 \\ -(3x_j + 6)x_j^2 + 4 & \text{for } -1 \le x_j < 0 \\ (3 * x_j - 6)x_j^2 + 4 & \text{for } 0 \le x_j < 1 \\ (2 - x_j)^3 & \text{for } 1 \le x_j < 2 \end{cases}$$



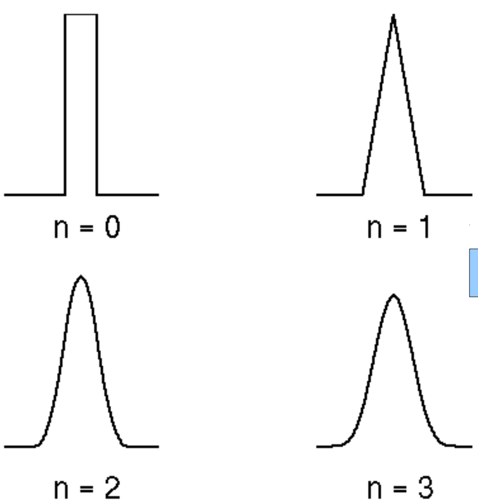


Cubic b-spline III

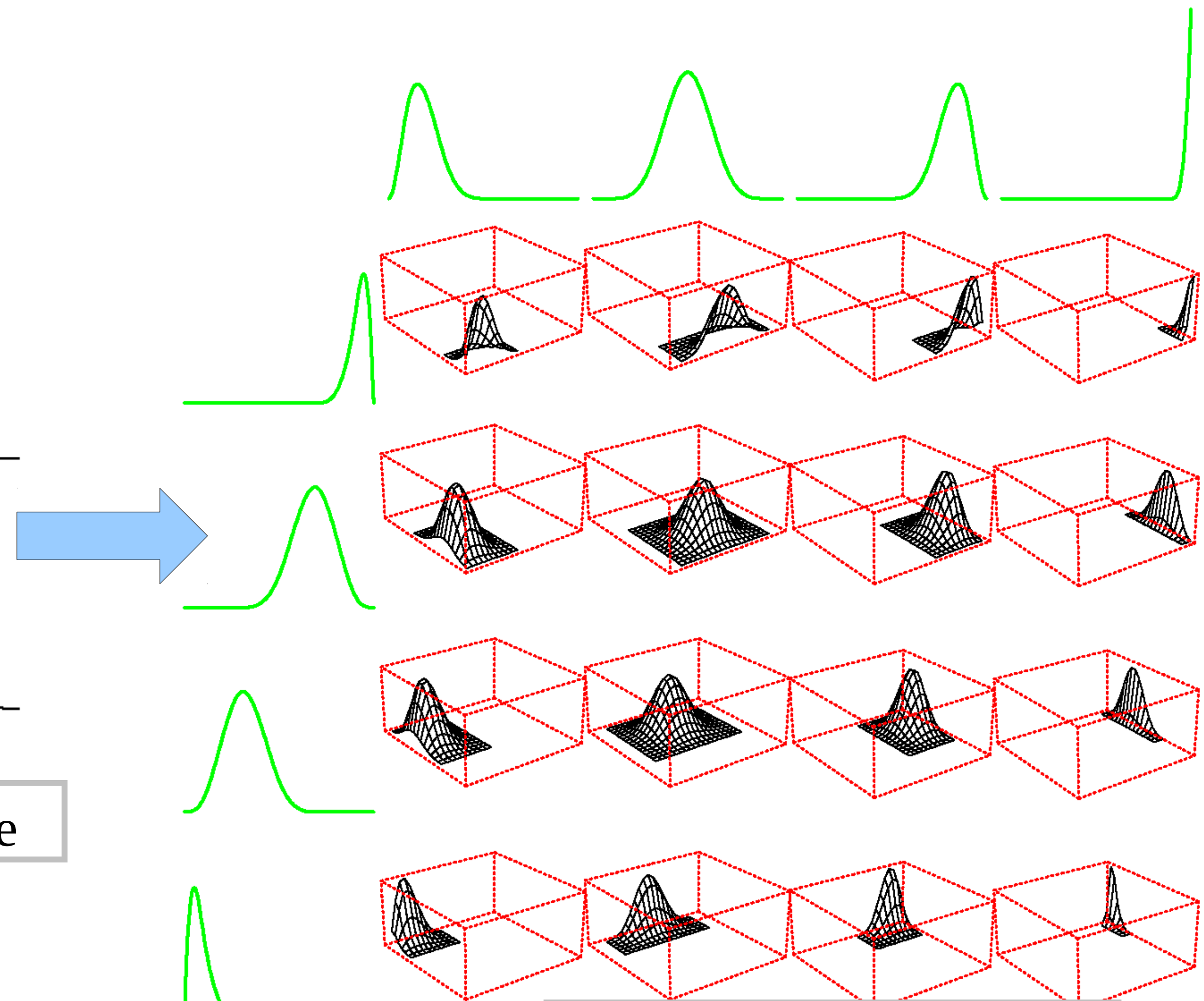


0.5 –
+ 3.0 –
+ 1.0 –
- 1.0 –
= ------

Non-linear basis functions used in exercise next week



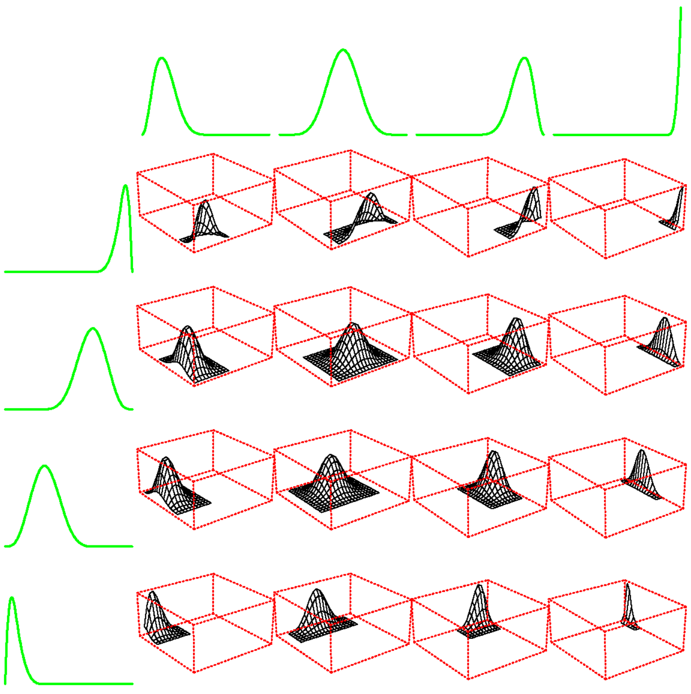
1-D: B-spline



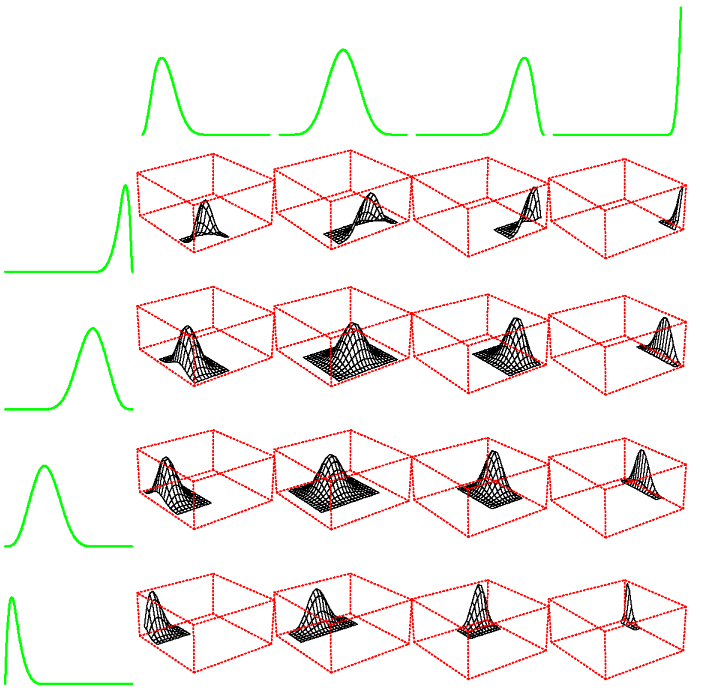
2-D: “tensor B-spline”

Non-linear basis functions used in exercise next week

Warp field: need one for each coordinate!



$$\mathbf{y}^1 = \mathbf{x}^1 + \mathbf{Q}'\mathbf{w}_1$$



$$\mathbf{y}^2 = \mathbf{x}^2 + \mathbf{Q}'\mathbf{w}_2$$

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- Similarity measure $\mathcal{D}(\mathbf{w})$
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Optimization of image registration

Optimize sum of squared differences

$$\mathcal{D}_{\mathrm{SSD}}(\mathbf{w}) \quad = \quad \frac{1}{2}\sum_{i\in\Omega}\left(\mathcal{T}(\mathbf{y}(\mathbf{x}_i;\mathbf{w})) - \mathcal{R}(\mathbf{x}_i)\right)^2$$

New notation: all coordinates simultaneously by stacking them

$$\tilde{\mathbf{x}} = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{pmatrix} \qquad \tilde{\mathbf{y}} = \begin{pmatrix} \mathbf{y}^1 \\ \mathbf{y}^2 \\ \mathbf{y}^3 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} \mathbf{w}^1 \\ \mathbf{w}^2 \\ \mathbf{w}^3 \end{pmatrix}$$

$$\tilde{\mathbf{y}} \quad = \quad \tilde{\mathbf{x}} + \begin{pmatrix} \mathbf{Q}' & 0 & 0 \\ 0 & \mathbf{Q}' & 0 \\ 0 & 0 & \mathbf{Q}' \end{pmatrix}\mathbf{w} \quad = \tilde{\mathbf{x}} + \mathbf{Q}\mathbf{w}$$

Optimization of image registration

More new notation:

$\mathcal{R}(\tilde{\mathbf{x}})$: vector of all reference image values

$\mathcal{T}(\tilde{\mathbf{y}})$: vector of corresponding template image values

$$\begin{aligned}\mathcal{D}_{\mathrm{SSD}}(\mathbf{w}) &= \frac{1}{2}\sum_{i\in\Omega}\left(\mathcal{T}(\mathbf{y}(\mathbf{x}_i;\mathbf{w})) - \mathcal{R}(\mathbf{x}_i)\right)^2 \\ &= \frac{1}{2}\left\|\mathcal{T}(\tilde{\mathbf{y}}) - \mathcal{R}(\tilde{\mathbf{x}})\right\|^2\end{aligned}$$

Optimization of image registration

For a small change $\mathbf{s}$ to the current parameter estimate $\mathbf{w}$ we have (linearization):

$$\begin{aligned} \mathcal{D}_{\mathrm{SSD}}(\mathbf{w}+\mathbf{s}) &= \frac{1}{2}\left\|\mathcal{T}(\tilde{\mathbf{x}}+\mathbf{Q}(\mathbf{w}+\mathbf{s}))-\mathcal{R}(\tilde{\mathbf{x}})\right\|^2 \\ &\approx \frac{1}{2}\left\|\mathcal{T}(\tilde{\mathbf{x}}+\mathbf{Q}\mathbf{w})+\nabla\mathcal{T}(\tilde{\mathbf{x}}+\mathbf{Q}\mathbf{w})\mathbf{Q}\mathbf{s}-\mathcal{R}(\tilde{\mathbf{x}})\right\|^2 \end{aligned}$$

with $\nabla\mathcal{T}(\tilde{\mathbf{x}}+\mathbf{Q}\mathbf{w}) = \begin{pmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \mathbf{G}_3 \end{pmatrix}$

and $\mathbf{G}_k = \mathrm{diag}\left(\left.\frac{\partial\mathcal{T}}{\partial y^k}\right|_{\mathbf{y}_1}, \ldots \left.\frac{\partial\mathcal{T}}{\partial y^k}\right|_{\mathbf{y}_N}\right)$

Optimization of image registration

Gauss-Newton optimization: search for change $\mathbf{s}$ that minimizes

$$\mathcal{D}_{\mathrm{SSD}}(\mathbf{w}+\mathbf{s}) \approx \frac{1}{2}\left\|\mathcal{T}(\tilde{\mathbf{x}}+\mathbf{Q}\mathbf{w}) + \nabla\mathcal{T}(\tilde{\mathbf{x}}+\mathbf{Q}\mathbf{w})\mathbf{Q}\mathbf{s} - \mathcal{R}(\tilde{\mathbf{x}})\right\|^2$$

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Is standard least-squares fit!

solve $(\mathbf{J}^{\mathrm{T}}\mathbf{J})\mathbf{s} = \mathbf{J}^{\mathrm{T}}(\mathcal{R}(\tilde{\mathbf{x}})-\mathcal{T}(\tilde{\mathbf{x}}+\mathbf{Q}\mathbf{w}))$

where $\mathbf{J} = \nabla\mathcal{T}(\tilde{\mathbf{x}}+\mathbf{Q}\mathbf{w})\mathbf{Q}$

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When done, update current parameter estimate $\mathbf{w} := \mathbf{w}+\mathbf{s}$

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