Shape spaces and metrics in an application perspective

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Overview

The application perspective:

Data and task

Which structure is needed on the models to subserve the task?

Disclaimer

Slides have been stolen from Mumford, Dam, Chennai, and many more

Osteoarthritis (OA)

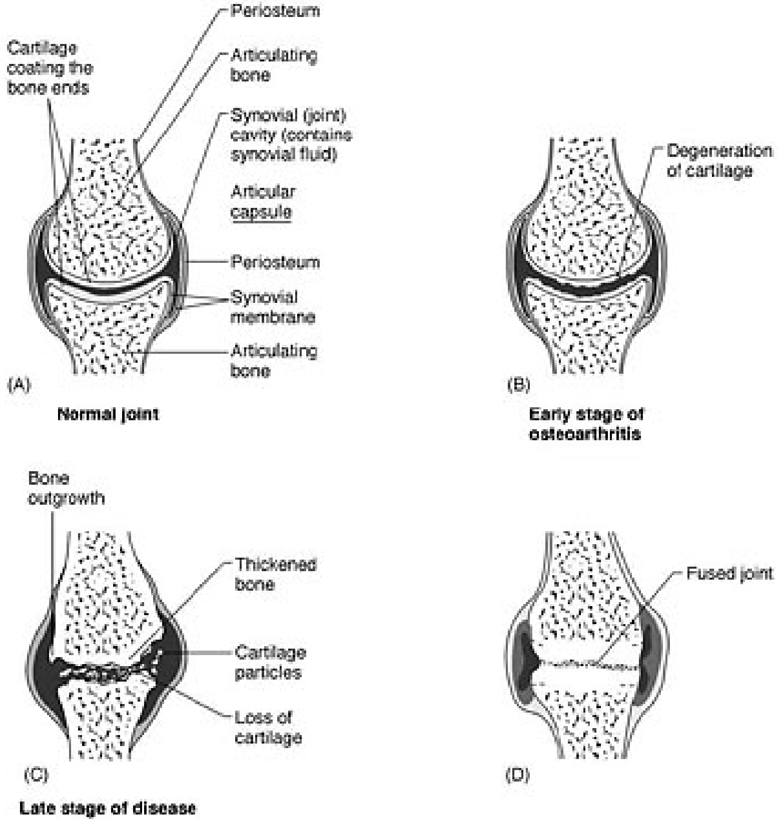
OA is a degenerative joint disease in knees, hips, …

Effect:
Pain, Reduced range of motion

Rule of thumb:
Age in years gives % chance of OA

Treatment:
Symptom control

Current golden standard:
-Kellgren & Lawrence Index
-Joint space width



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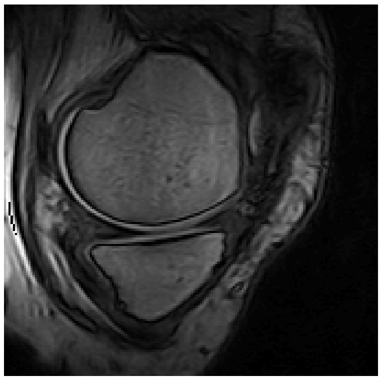
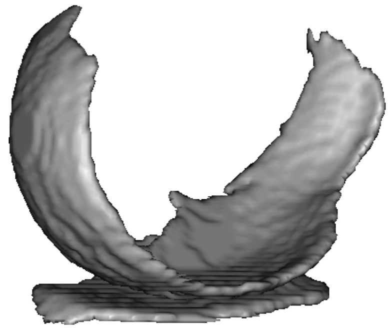
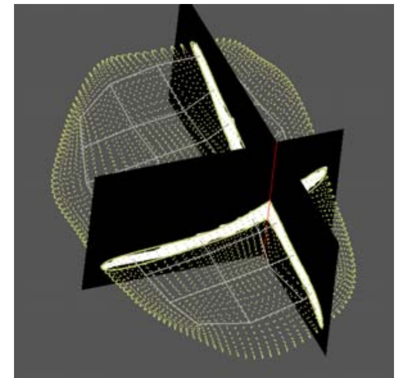
Quantification Framework







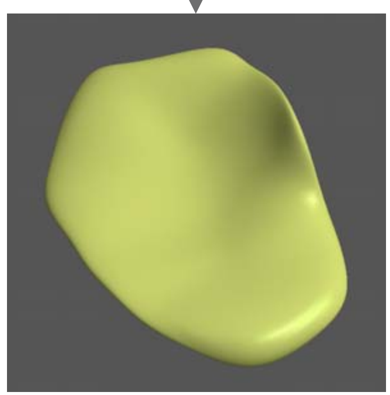
Smoothness,
Homogeneity

Folkesson, Dam et al. 2007
Trans Medical Imaging

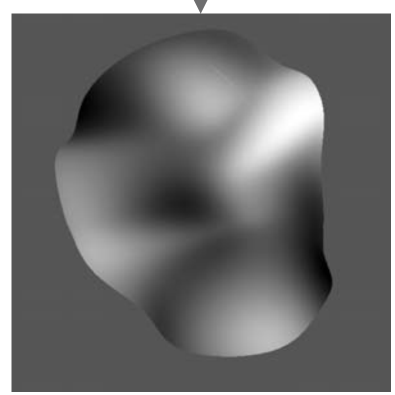
Dam et al. 2008
Medical Image Analysis

Dam, Folkesson et al. 2007
Osteoarthritis & Cartilage

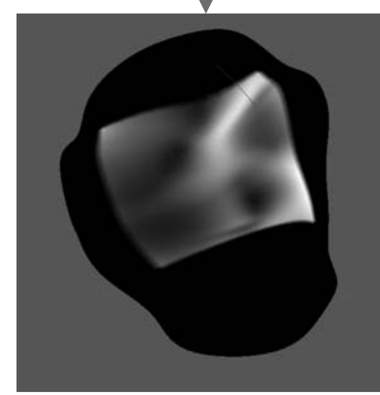
Qazi, Dam, Karsdal et al. 2007
Osteoarthritis & Cartilage



Volume



Thickness

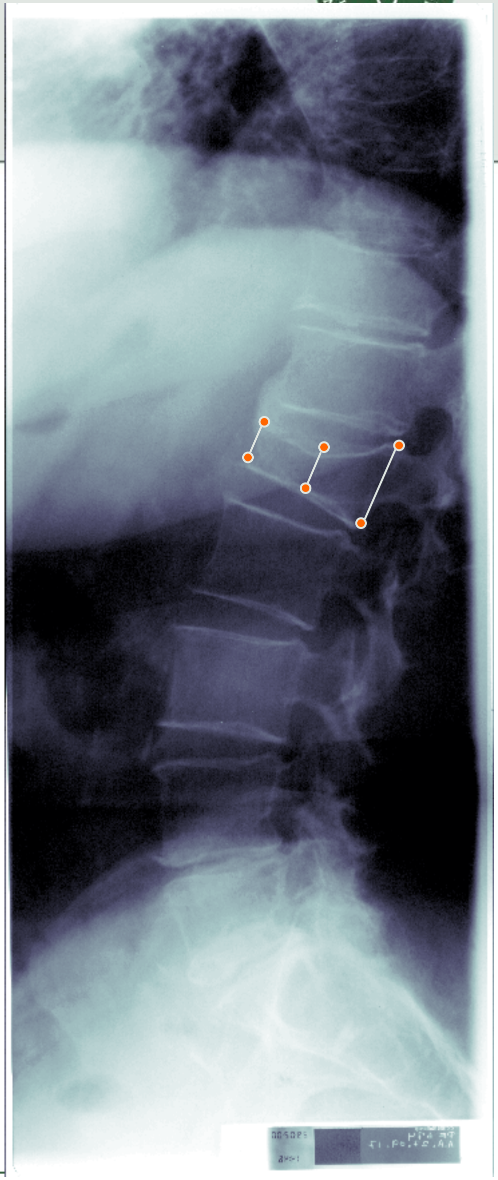


Curvature

Risk of vertebral fractures

Current standard of fracture grading:
Bone Mineral Density based on dual x-ray

Our approach:
Statistical shape analysis

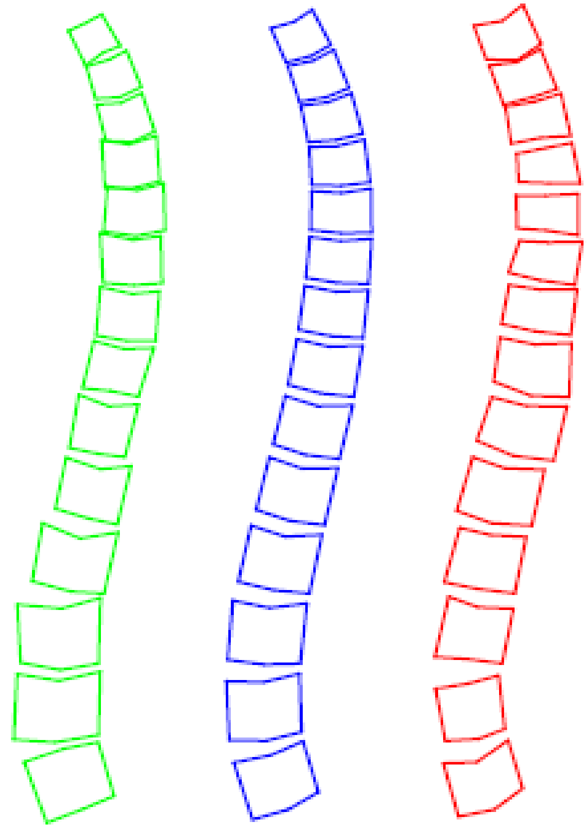


Visualization

Green: Likely to stay intact

Blue: Mean spine shape

Red: Likely to fracture



Tasks

Classification	S —> $[L_1, L_2, \ldots L_n]$
Shape regression	S(t): R^n-> ***S***
Marker regression	t(S): ***S*** -> R
Prior for segmentation	p(S): dist. on ***S***

In all cases, a metric on the space ***S*** of shapes S is essential

Finding usefull metrics is non-trivial

Shapes

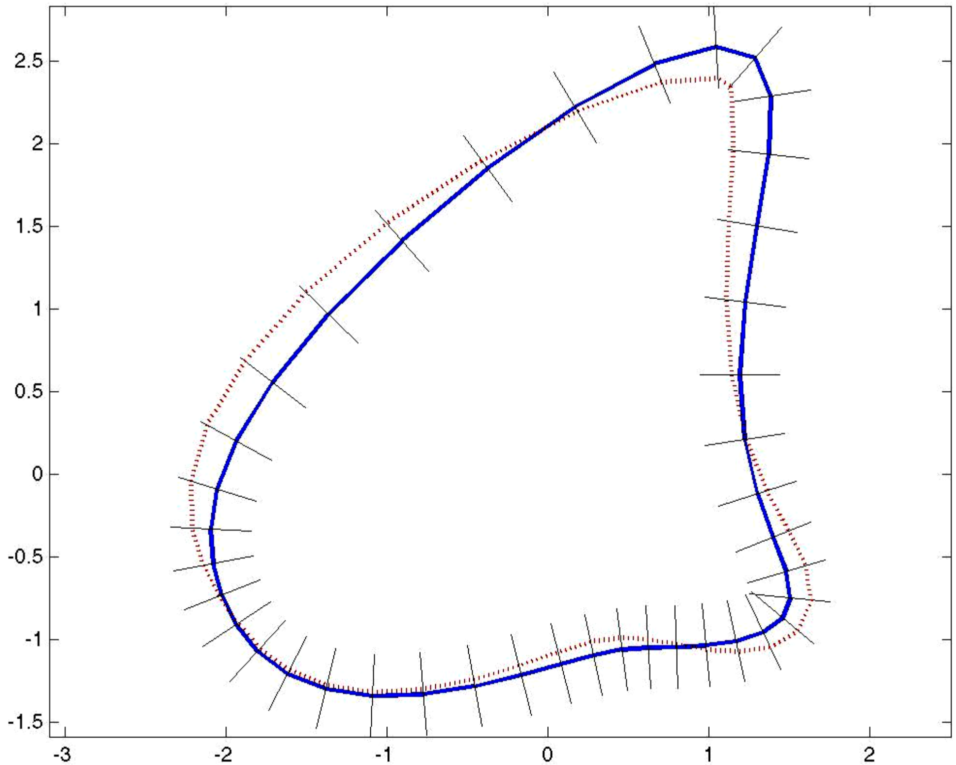
Shape = Geometry \ Position

Shape is a qouotient manifold (mayby embedded in Geometry space)

Metric on the geometry space, may be inherited (projected) to the shape space

Kendall : Points in R^{2n} \ Similarity

The set Σ of all smooth plane curves forms a manifold!



Start with a fixed curve C Î S

parametrized by $s \mapsto f(s)$

Define a local chart near f :

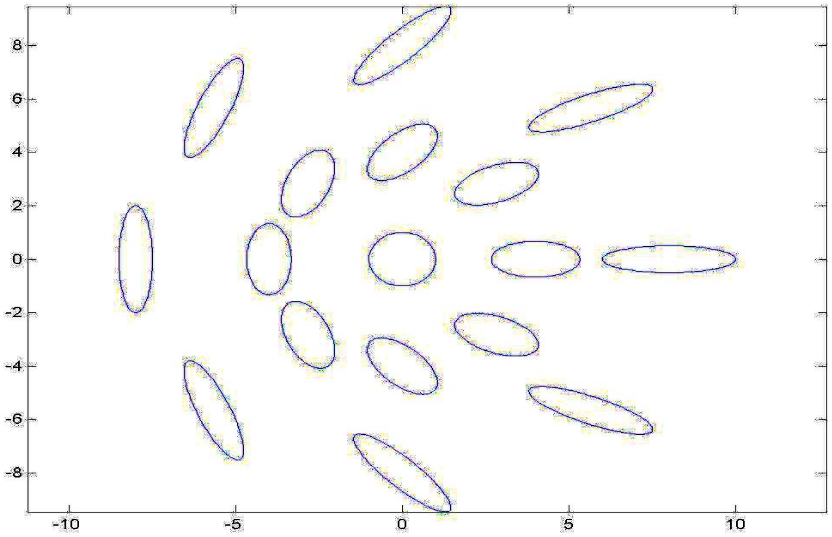
$y_a(s) = f(s) + a(s).\vec{n}(s),$

$\vec{n}(s) =$ unit normal to C,

$C_a =$ image of y_a

Think of Σ geometrically

- A curve on Σ is a warping of one shape to another.
- On Σ, the set of ellipses forms a surface:



- The geometric heat equation:

$$\frac{\partial C_t}{\partial t} = \kappa_{C_t} \vec{n}_{C_t},$$

is a <u>vector field</u> on Σ

Advantages of L^2 metrics

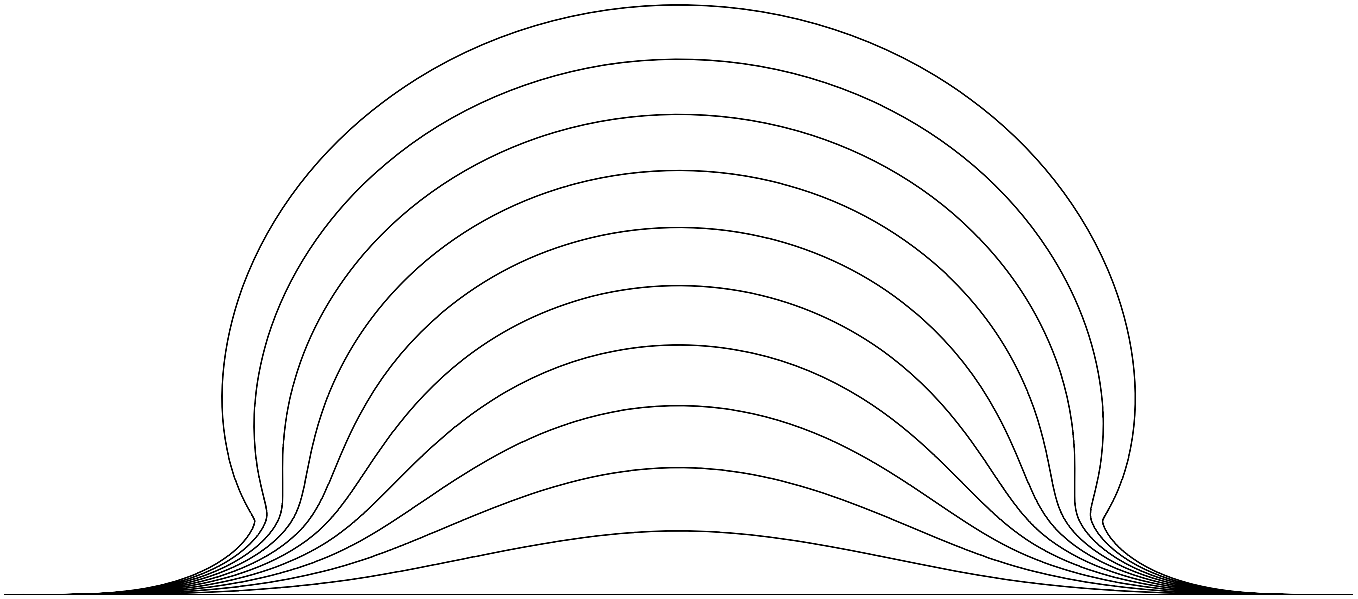
Have simple notion of a gradient to form flows

Have a beautiful theory of locally unique geodesics, thus a warping of one shape to another.

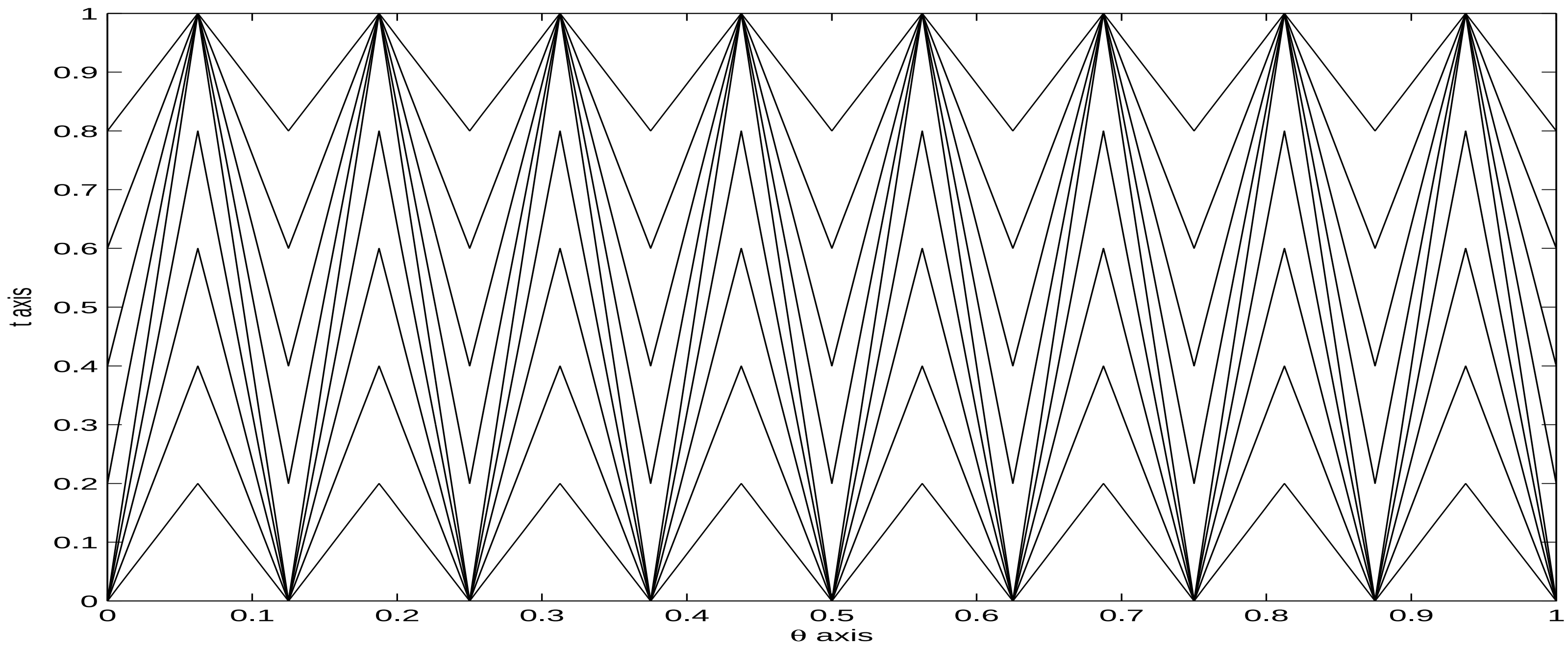
Can define the Riemannian curvature tensor. If non-positive, have a good theory of *means.*

Can expect a theory of diffusion, of Brownian motion, hence Gaussian-type measures and their mixtures.

A geodesic in the simple L^2 metric



But distances collapse in this metric:



The line on the bottom is moved to the line on the top by growing "teeth" upwards and then shrinking them again.

Fixes of this bug:

Michor + Mumford: L_2 + curvature

Yezzi: Sobolev metric

Charpiat: Bounded curvature

Sommer: Finite bandwith by resampling

Trouve, Younes: Diffeomorphic

Others use Hausdorff metric

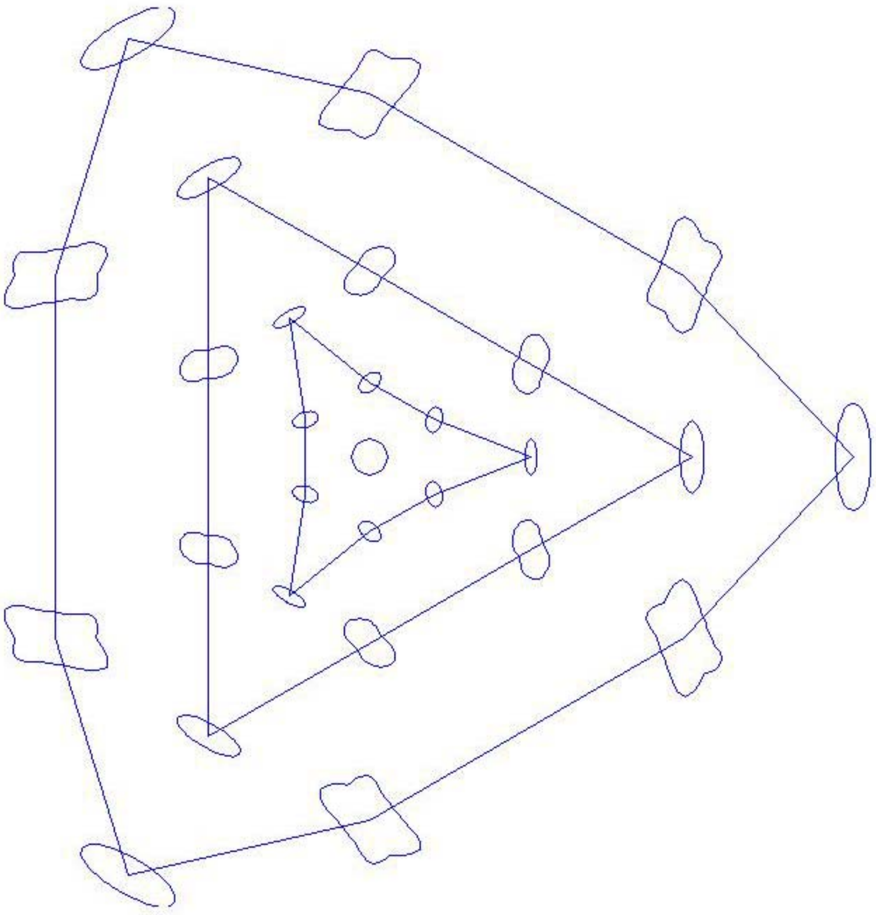
Michor+Mumford

$$\|a\|^2 = \int (1 + A\kappa^2)|a|^2 \, ds$$

For small shapes, curvature is negative and the path nearly goes back to the circle (= the ‘origin’). Angle sum = 102 degrees.

For large shapes, curvature is positive, 2 protrusions grow while 2 shrink. Angle sum = 207 degrees.

Hence construction depends on scale



Charpiat, Faugeras, Keriven

The shape space ***S*** is limited to shapes S where the curvature (the extrinsic curvature of the curve in R^2) is limited to $k < k_0$

This is scale dependent and the ordinary L_2 geodesic depends on k_0

Yezzi:

Geometric Sobolev-type norms;

We define

$$|h|^2_{\text{Sobolev}} := |h|^2 + \lambda L^2 |Ds\ h|_2^{\,2}$$

where $h : S_1 \rightarrow R^2$ is a perturbation of the curve c,

L is the length of c,

Ds is the arclength derivative,

This is a negatively curved space.

This construction is independent of scale

Later today

Trouve, Younes

Define a sobolev type metric on flows on the embedding plane.

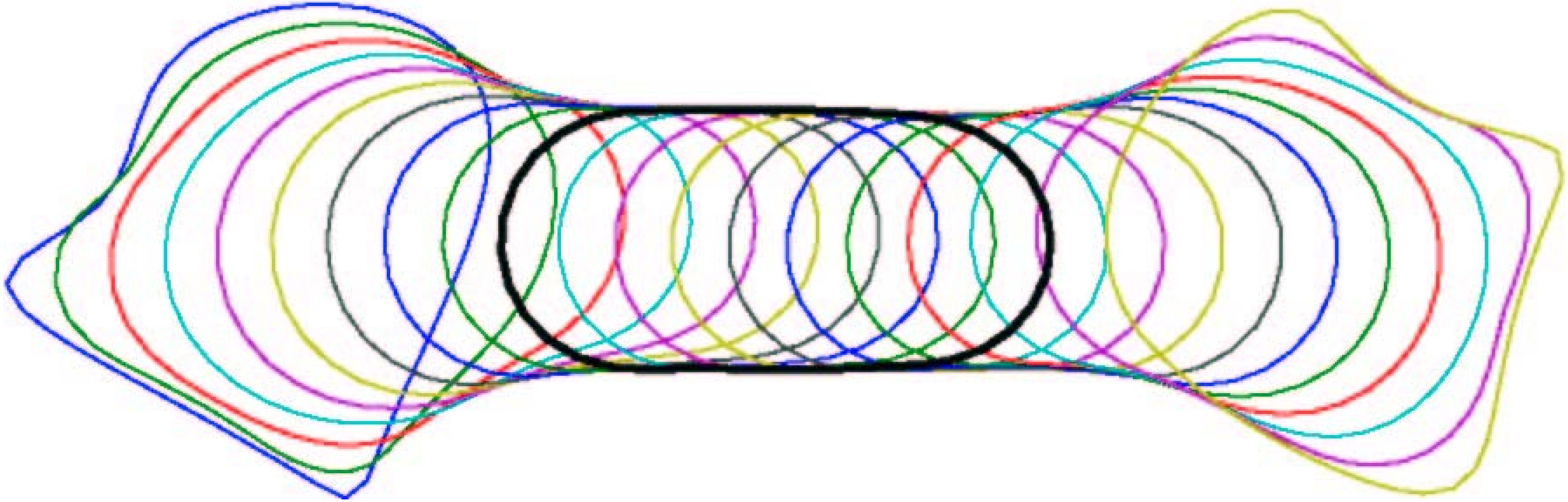
This introduces also a flow on embedded curves

This was introduced here by Tom Fletcher

It is negatively curved

Negatively curved spaces

<u>Same metric</u>: a reflection of its negative curvature for small shapes: to get from any shape to any other *which is far* away, go via 'cigars' (in neg. curved space, to get from one city to another, everyone takes the same highway)



Hausdorff approaches

The distance depends on the largest smallest distance to the other curve

Geodesics seems not very informative?

Implementations via level sets and distance transform

Funny solutions

Conclusion

At best, we still have things to understand:

Informative statistics on negatively curved spaces?

Better L_2-like metrics?

Problem dependent?

Questions?