Introduction to Differential and Riemannian Geometry

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Outline

1. Motivation
   - Non Linearity
   - Statistics on Non Linear Data
2. Recalls
   - Geometry
   - Topology
   - Calculus on $\mathbb{R}^n$
3. Differentiable Manifolds
   - Definitions
   - Building Manifolds
   - Tangent Space
4. Riemannian Manifolds
   - Metric
   - Gradient Field
   - Length of curves
   - Geodesics
   - Covariant derivatives

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Non Linear data

Many interesting /common objects behave non linearly.

Vector lines in $\mathbb{R}^2$

The projective line as a circle.

Right triangles with fixed hypotenuse

Right rectangles as half-circle (without endpoints).

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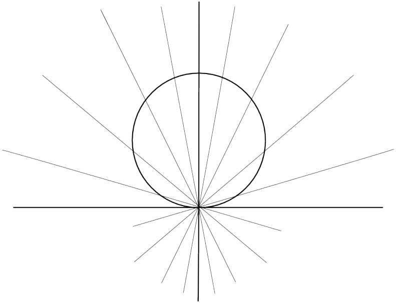

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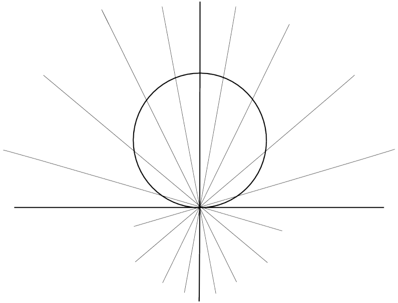

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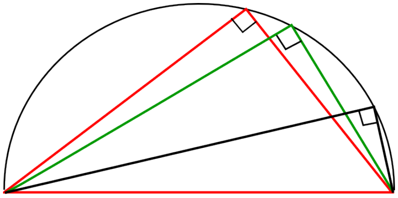

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Averaging

- How to average points on a circle?
- Hm... The linear way does not work!
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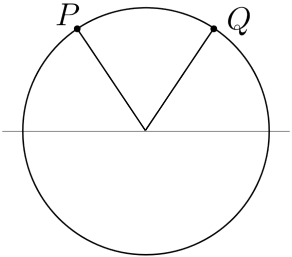

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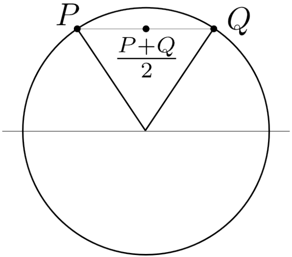

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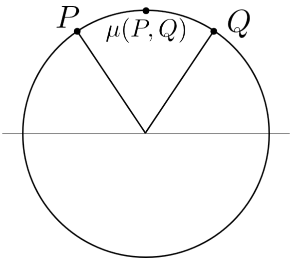

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Inner Products

- Inner Product: product $\langle \mathbf{x}, \mathbf{y} \rangle$ for column vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$
  - linear in $\mathbf{x}$ and $\mathbf{y}$,
  - symmetric: $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
  - positive and definite: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ with equality if $\mathbf{x} = 0$.
- Simplest example: usual dot-product $\mathbf{x} = (x_1, \dots, x_n)^t$, $\mathbf{y} = (y_1, \dots, y_n)^t$,

$$\mathbf{x} \cdot \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i = \mathbf{x}^t \, Id \, \mathbf{y}$$

  Id is the n-order identity matrix.
- Every inner product is of the form $\mathbf{x}^t A \mathbf{y}$, A symmetric positive definite.
  Alternate notation: $\langle \mathbf{x}, \mathbf{y} \rangle_A$.
  Without subscript $\langle -, - \rangle$ will denote standard Euclidean dot-product (i.e. $A = Id$).

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Orthogonality – Norm – Distance

- A-orthogonality: $\mathbf{x} \perp_A \mathbf{y} \Leftrightarrow \langle \mathbf{x}, \mathbf{y} \rangle_A = 0$.
- A-norm of $\mathbf{x}$: $\|x\|_A = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_A}$.
- A-distance on $\mathbb{R}^n$: $d_A(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_A$.

- Standard orthogonal transform on $\mathbb{R}^n$: $n \times n$ matrix R satisfying $R^t R = Id$. They form the orthogonal group $O(n)$. Matrices R with det $= 1$ form the special orthogonal group $SO(n)$.
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Duality

- Linear form $h : \mathbb{R}^n \to \mathbb{R}$: $h(\mathbf{x}) = \sum_{i=1}^n h_i x_i$.
- Given an inner product $\langle -, - \rangle_A$ on $\mathbb{R}^n$, h represented by a unique vector $\mathbf{h}_A$ s.t

$$h(\mathbf{x}) = \langle \mathbf{h}_A, \mathbf{x} \rangle_A$$

  $\mathbf{h}_A$ is the dual of h (w.r.t $\langle -, - \rangle_A$).
- for standard dot product:

$$\mathbf{h} = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} !$$

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Open sets, Continuity...

- Topology on $\mathbb{R}^n$. A open set of $\mathbb{R}^n$ is a union (not necessarily finite) of open balls. $\mathbb{R}^N$ and the empty set $\emptyset$ are open.
- A map $f : X \to Y$ between topological spaces is continuous if

$$V \subset Y \text{ open} \Rightarrow f^{-1}(V) \subset X \text{ open}.$$

- A map $h : X \to Y$ between topological spaces is a homeomorphism is it is continuous, one-to-one and h^{-1} is continuous.

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Differentiable and smooth functions

- $f : U$ open $\subset \mathbb{R}^n \to \mathbb{R}^q$ continuous: write

$$(y_1, \dots, y_1) = f(x_1, \dots, x_n)$$

- f is of class $\mathcal{C}^r$ if f has continuous partial derivatives

$$\frac{\partial^{r_1+\cdots+r_n} y_k}{\partial x_1^{r_1} \dots \partial x_n^{r_n}}$$

$k = 1 \dots q$, $r_1 + \dots r_n \leq r$.

- When $r = \infty$, I say that f is smooth. This is the main situation of interest.

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Differential, Jacobian Matrix

- Differential of f in $\mathbf{x}$: unique linear map (if exists) $d_x f : \mathbb{R}^n \to \mathbb{R}^q$ s.t.

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + d_x f(\mathbf{h}) + o(\mathbf{h}).$$

- Jacobian matrix of f: matrix $q \times n$ of partial derivatives of f:

$$J_{\mathbf{x}} f = \begin{pmatrix} \frac{\partial y_1}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial y_1}{\partial x_n}(\mathbf{x}) \\ \vdots & & \vdots \\ \frac{\partial y_q}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial y_q}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

- if $n \geq q$ and $\mathrm{rank}(J_{\mathbf{x}} f) = q$, f is a submersion at $\mathbf{x}$.
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Differential, Jacobian Matrix

- Differential of f in $\mathbf{x}$: unique linear map (if exists) $d_x f : \mathbb{R}^n \to \mathbb{R}^q$ s.t.

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + d_x f(\mathbf{h}) + o(\mathbf{h}).$$

- Jacobian matrix of f: matrix $q \times n$ of partial derivatives of f:

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Diffeomorphism

- when $n = q$: if f is 1-1 $\mathcal{C}^r$ and its inverse is also $\mathcal{C}^r$, f is a $\mathcal{C}^r$-diffeomorphism. A smooth diffeomorphism is simply referred to as a diffeomorphism.
- If f is a diffeomorphism, $\det(J_{\mathbf{x}} f) \neq 0$. Conversely, if $\det(J_{\mathbf{x}} f) \neq 0$, by the Inverse Function Theorem, f is a local diffeomorphism in a neighborhood of $\mathbf{x}$.
- f may be a local diffeomorphism everywhere but fail to be a global diffeomorphism. Example:

$$f : \mathbb{R}^2 \backslash 0 \to \mathbb{R}^2, \quad (x, y) \to (e^x \cos(y), e^x \sin(y)).$$

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Gradient of a function

- $f : U \subset \mathbb{R}^n \to \mathbb{R}$, $d_\mathbf{x} f$ it differential at $\mathbf{x} \in U$.
- $d_\mathbf{x} f$ is represented by a unique vector, the gradient of f for the standard inner product:

$$d_\mathbf{x} f(\mathbf{h}) = \nabla f_\mathbf{x} \cdot \mathbf{h}$$

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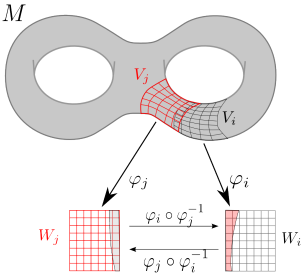
Outline

Differentiable Manifold



- Differential manifold M of dim n:
- smoothly glued open pieces of Euclidean space $\mathbb{R}^n$ via $M = \cup_i V_i$, homeomorphisms $\varphi_i : V_i \to W_i \subset \mathbb{R}^n$,
- $\varphi_i(P) = (x_1(P), \dots, x_n(P))$ Charts or local coordinates
- Smoothness in gluing: the changes of coordinates

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(V_i \cap V_j) \to \varphi_j(V_i \cap V_j)$$

are smooth.
- Set

$$\varphi_j(P) = (y^1(P), \dots, y^n(P)),$$

then

$$\varphi_j \circ \varphi_i^{-1}(x_1, \dots, x_n) = (y_1, \dots, y_n)$$

and the $n \times n$ Jacobian matrices $\left(\frac{\partial y^k}{\partial x^h}\right)_{k,h}$ are invertible.
- Maps $\varphi_i^{-1} : W_i \to V_i$ are local parametrization of M.

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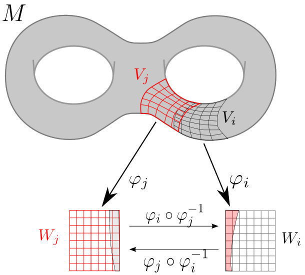

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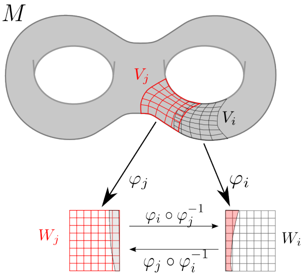

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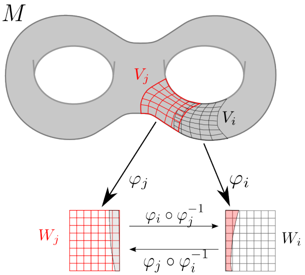

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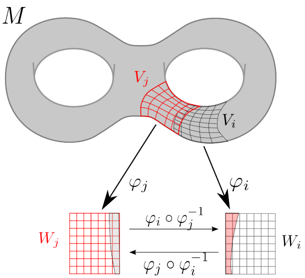

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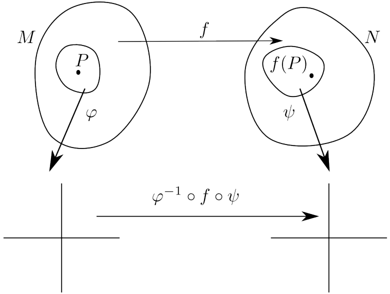

- $f : M \rightarrow N$ is differentiable if its expression in any coordinates for M and N is.
- φ local coordinates at $P \in M$, ψ local coordinates at $f(P) \in N$

$$\varphi^{-1} \circ f \circ \psi \text{ differentiable.}$$

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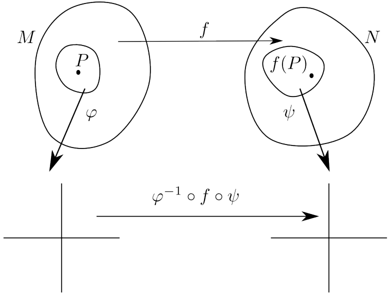

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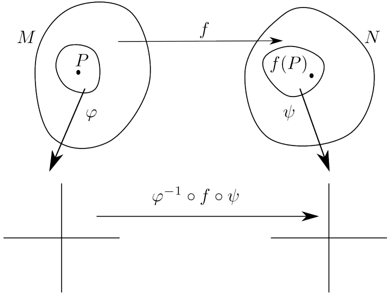

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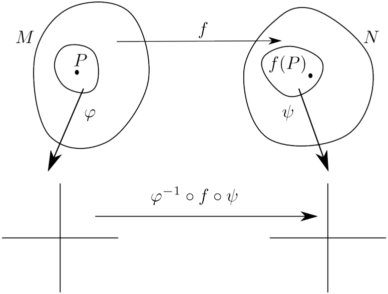

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- The Euclidean space $\mathbb{R}^n$ is a manifold: take $\varphi = Id$ as global coordinate system!
- The sphere $\mathbb{S}^2 = \{(x, y, z), x^2 + y^2 + z^2 = 1\}$

For instance the projection from North Pole, given, for a point $P = (x, y, z) \neq N$ of the sphere, by

$$\varphi_N(P) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

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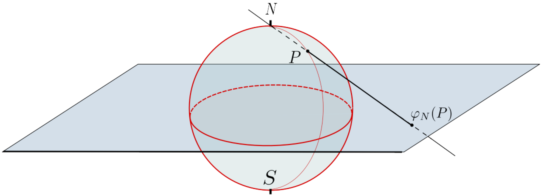

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The Moebius strip

$$u \in [0, 2\pi], v \in [\frac{1}{2}, \frac{1}{2}]$$

$$\begin{pmatrix} \cos(u)\left(1 + \frac{1}{2} v \cos(\frac{u}{2})\right) \\ \sin(u)\left(1 + \frac{1}{2} v \cos(\frac{u}{2})\right) \\ \frac{1}{2} v \sin(\frac{u}{2}) \end{pmatrix}$$

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$$(u, v) \in [0, 2\pi]^2, R \gg r > 0$$

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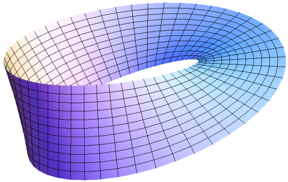

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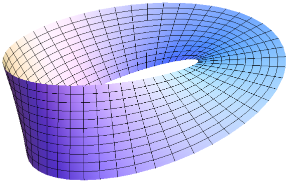

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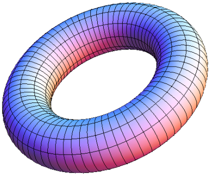

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Outline

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- Take $f : U \in \mathbb{R}^n \to \mathbb{R}^q$, $q \leq n$ smooth.
- Set $M = f^{-1}(0)$.
- If for all $\mathbf{x} \in M$, f is a submersion at $\mathbf{x}$, M is a manifold of dimension $n - q$.
- Example:

$$f(x_1, \dots, x_n) = 1 - \sum_{i=1}^{n} x_i^2 :$$

  $f^{-1}(0)$ is the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$.
- Many common examples of manifolds in practice are of that type.

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- Take $f : U \in \mathbb{R}^n \longrightarrow \mathbb{R}^q$, $q \leq n$ smooth.
- Set $M = f^{-1}(0)$.
- If for all $\mathbf{x} \in M$, f is a submersion at $\mathbf{x}$, M is a manifold of dimension $n - q$.
- Example:

$$f(x_1, \dots, x_n) = 1 - \sum_{i=1}^{n} x_i^2 :$$

  $f^{-1}(0)$ is the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$.
- Many common examples of manifolds in practice are of that type.

Product Manifolds

- M and N manifolds, so is $M \times N$.
- Just consider the products of charts of M and N!
- Example: $M = \mathbb{S}^1$, $N = \mathbb{R}$: cylinder.
- Example: $M = N = \mathbb{S}^1$: the torus!

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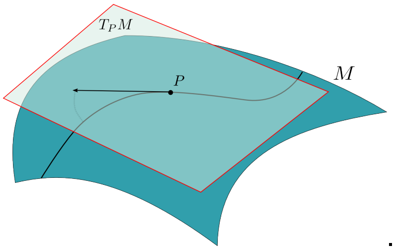
Outline

Tangent vectors informally



- Informally: a tangent vector at $P \in M$: draw a curve $c : (-\epsilon, \epsilon) \to M$, $c(0) = P$, then $\dot{c}(0)$ is a tangent vector.

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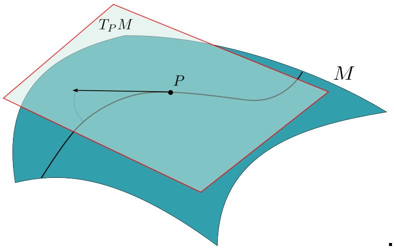

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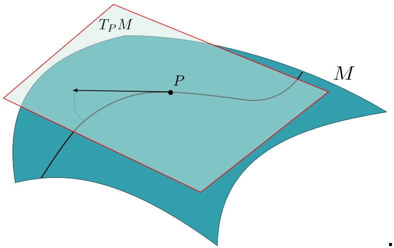

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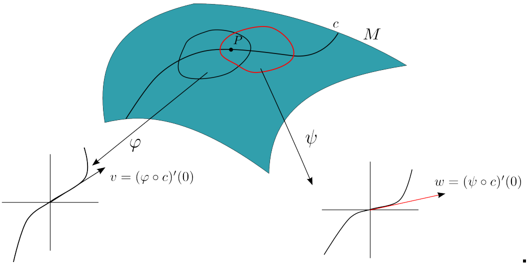
A bit more formally



- c : $c : (-\epsilon, \epsilon) \to M$, $c(0) = P$. In chart φ, the map $t \mapsto \varphi \circ c(t)$ is a curve in Euclidean space, and so is $t \mapsto \psi \circ c(t)$.
- set $v = \frac{d}{dt}(\varphi \circ c)|_0$, $w = \frac{d}{dt}(\psi \circ c)|_0$ then

$$w = J_0\left(\varphi^{-1} \circ \psi\right) v.$$

- Use this relation to identify vectors in different coordinate systems!

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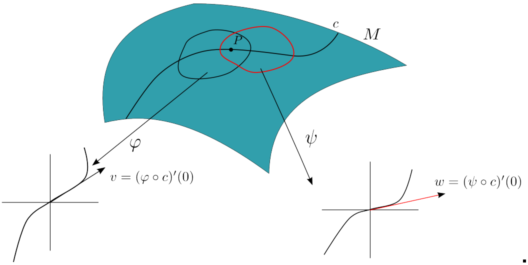

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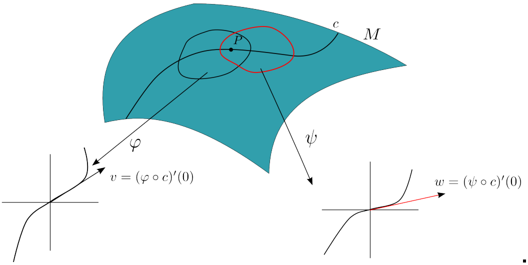

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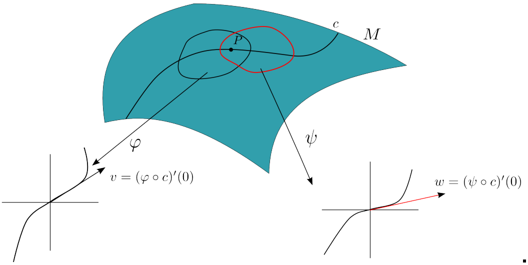

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Tangent space

- The set of tangent vectors to the n-dimensional manifold M at point P is the tangent space of M at P denoted $T_P M$.
- It is a vector space of dimension n: let θ a local parametrization of M, $\theta(x_1, \dots, x_n) \in M$ with $\theta(0) = P$. Define curves

$$x_i : t \mapsto \theta(0, \dots, 0, t, 0, 0)$$

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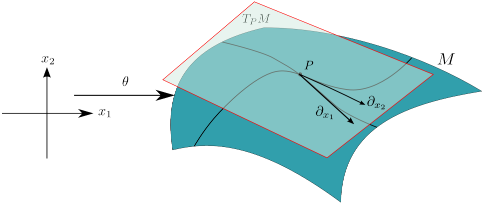
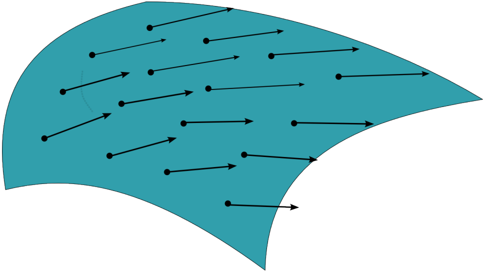


Vector fields

- A vector field is a smooth map that send $P \in M$ to a vector $v(P) \in T_P M$.



Differential of a differentiable map

- $f : M \to N$ differentiable, $P \in M$, $f(P) \in N$
- $d_P f : T_P M \to T_{f(P)} N$ linear map corresponding to the Jacobian matrix of f in local coordinates.
- When $N = \mathbb{R}$, $d_P f$ is a linear form $T_P M \to \mathbb{R}$.

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   - Non Linearity
   - Statistics on Non Linear Data
2. Recalls
   - Geometry
   - Topology
   - Calculus on $\mathbb{R}^n$
3. Differentiable Manifolds
   - Definitions
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   - Tangent Space
4. Riemannian Manifolds
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   - Gradient Field
   - Length of curves
   - Geodesics
   - Covariant derivatives

Riemannian Metric

- A Riemannian metric on a $n-$dimensional manifold is a smooth family g_P of inner products on the tangent spaces $T_P M$ of M, $u, v \in T_P M \mapsto g_p(u,v) := \langle u, v\rangle_P \in \mathbb{R}$. With it, one can compute length of vectors in tangent spaces, check orthogonality of them...
- With a local parametrization $\theta(\mathbf{x}) = (x_1, \dots, x_n) \to M$, it corresponds to a smooth family of positive definite matrices:

$$g_{\mathbf{x}} = \begin{pmatrix} g_{\mathbf{x}11} & \cdots & g_{\mathbf{x}1n} \\ \vdots & & \vdots \\ g_{\mathbf{x}n1} & \cdots & g_{\mathbf{x}nn} \end{pmatrix}$$

- $u = \sum_{i=1}^n u_i \partial_{x_i}$, $v = \sum_{i=1}^n v_i \partial_{x_i}$ $\langle u, v\rangle_{\mathbf{x}} = (u_1, \dots, u_n) g_{\mathbf{x}} (v_1, \dots, v_n)^t$

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Riemannian Manifold

A differential manifold with a Riemannian metric is a Riemannian manifold.

Outline

Gradient, Gradient vector field.

- M Riemannian, $f : M \to \mathbb{R}$ differentiable. Then

$$d_P f(h) = \langle v, h \rangle_P, \quad \text{for a unique } v.$$

- $v := \nabla f_P$ is the gradient of f at P.
- $P \mapsto \nabla f_P$ is the gradient vector field of f.
- One can thus make gradient descent/ascent... Not possible without Riemannian structure.

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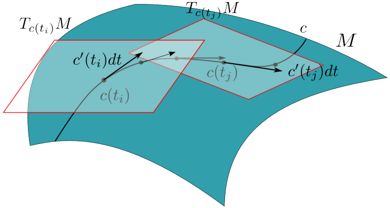

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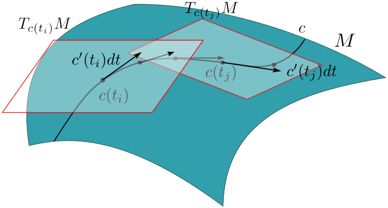
- It has length $\|\dot{c}(t)\| = \sqrt{\langle c'(t), c'(t)\rangle_{c(t)}}$
- Define the length of c as

$$\ell(c) = \int_a^b \|\dot{c}(t)\|\, dt$$

as in the Euclidean case, by now with variable inner products.



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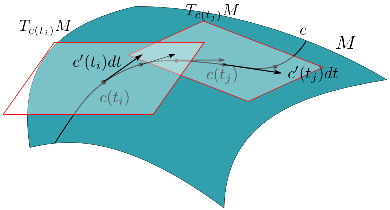

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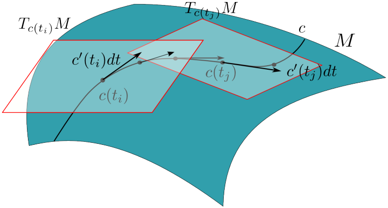

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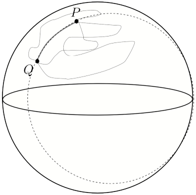
Geodesics



- Restricted definition: Riemannian Geodesics are curves of (locally) minimal length among curves with fixed endpoints say P and Q.
- They are also minimizers of the curve energy:

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- The shortest length of a curve joining P and Q is the geodesic distance $d(P, Q)$.

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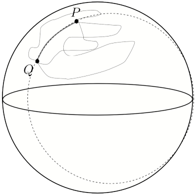

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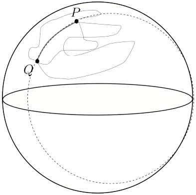

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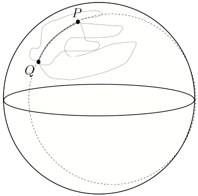

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How to characterize geodesics?

- In $\mathbb{R}^n$, The calculus of variations for curve energy gives : $\ddot{c} = 0$.
- In a general manifold: problem to define $\ddot{c}$:

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$\dot{c}(t) \in T_{c(t)}M$ and $\dot{c}(0) \in T_{c(0)}M$: these tangent spaces are distinct!

- Need for a "device" that "connects" tangent spaces of close enough points. Such a device is called an affine connection.

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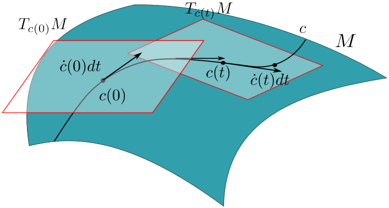

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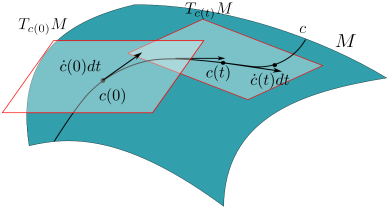

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Outline

Covariant Derivative

- Allows to differentiate a vector field along a curve: given a curve $\gamma(t) \in M$, X a vector field,

  $$\frac{D}{dt}X(t) = \dot{X}(t) \in T_{\gamma(t)}$$

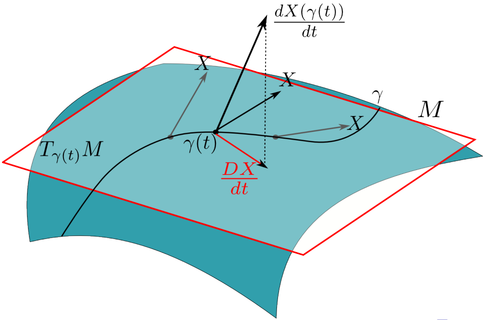
  We ask that $\frac{D}{dt}$ depends only on the value $\dot{\gamma}(t)$ and not on the behaviour of γ around $\gamma(t)$. The computation $\frac{D}{dt}X(t)$ depends on values of X around $\gamma(t)$.
- Many choices are possible, but exactly one is compatible with the Riemannian structure in the sense that

  $$\frac{d}{dt}\langle X, Y\rangle = \langle \frac{DX}{dt}, Y\rangle + \langle X, \frac{DY}{dt}\rangle$$

  plus another property. Levi-Cività connexion.

Concrete construction

- Assume $M \subset \mathbb{R}^n$. A vector field X on M can be seen as a vector field on $\mathbb{R}^n$ and a curve γ on M can be seen as a curve in $\mathbb{R}^n$. Then
  1. Compute the usual derivative
  $$\tilde{X(t)} = \frac{d}{dt}X(\gamma(t))$$
  its a vector field on $\mathbb{R}^n$ but not a tangent vector field on M in general.
  2. Project $\tilde{X(t)}$ orthogonally on $T_{\gamma(t)}M \subset \mathbb{R}^3$. The result is DX/dt!



Characterization of Geodesics

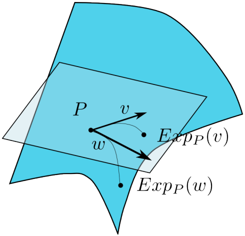
A curve γ is geodesic if its covariant is acceleration 0.

$$\ddot{\gamma}(t) = \frac{D\dot{\gamma}(t)}{dt} = 0!$$

This is in fact a second order ODE: given initial position $\gamma(0)$ and velocity $\dot{\gamma}(0)$ there is a unique solution.

Exponential map

- The uniqueness above leads to the following definition: given $P \in M$, $v \in T_P M$, the exponential map $Exp_P(v)$ is the solution at time 1 of the previous ODE. For small enough v: diffeomorphism.



- The curve $t \to Exp_P(tv)$, $t \in [0, 1]$ is geodesic, its length is $\|v\|$.

Log map

- The inverse map of the exponential map is called the Log map! For $Q \in M$ “not too far from P”, $Log_P(Q)$ is the vector v of $T_P M$ s.t. $Exp_P(v) = Q$.
- The exponential map is relatively easy to compute. The Log map is generally much more complicated, but badly needed in many optimization problems!

Bibliography

- Boothby: Introduction to Differential Manifolds and Riemannian Geometry, Wiley
- do Carmo: Riemannian Geometry, Birkhäuser.
- Hulin-Lafontaine: Riemannian Geometry, Springer
- Small: The statistical theory of shapes, Springer.