EXTRINSIC ISOPERIMETRIC ANALYSIS ON SUBMANIFOLDS WITH CURVATURES BOUNDED FROM BELOW

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Abstract. We obtain upper bounds for the isoperimetric quotients of extrinsic balls of submanifolds in ambient spaces which have a lower bound on their *radial sectional curvatures*. The submanifolds are themselves only assumed to have lower bounds on the radial part of the mean curvature vector field and on the radial part of the intrinsic unit normals at the boundaries of the extrinsic spheres, respectively. In the same vein we also establish *lower bounds* on the *mean exit time* for Brownian motions in the extrinsic balls, i.e. lower bounds for the time it takes (on average) for Brownian particles to diffuse *within* the extrinsic ball from a given starting point before they hit the boundary of the extrinsic ball. In those cases, where we may extend our analysis to hold all the way to infinity, we apply a *capacity comparison* technique to obtain a sufficient condition for the submanifolds to be *parabolic*, i.e. a condition which will guarantee that any Brownian particle, which is free to move around in the whole submanifold, is bound to eventually revisit any given neighborhood of its starting point with probability 1. The results of this paper are in a rough sense *dual* to similar results obtained previously by the present authors in complementary settings where we assume that the curvatures are bounded from *above*.

1. Introduction

Given a precompact domain Ω in a Riemannian manifold M, the isoperimetric quotient for Ω measures the ratio between the volume of the boundary and the volume of the enclosed domain: $\mathcal{Q}(\Omega) = \operatorname{Vol}(\partial\Omega)/\operatorname{Vol}(\Omega)$. These volume measures and this quotient are descriptors of fundamental importance for obtaining geometric and analytic information about the manifold M. In fact, this holds true on every zoom level, be it global, local, or micro-local.

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1.1. **On the global level.** A classical quest is to find necessary and sufficient conditions for the *type* of a given manifold: Is it hyperbolic or parabolic? As already alluded to in the abstract, parabolicity is a first measure of the relative smallness of the boundary at infinity of the manifold: The Brownian particles are bound to eventually return to any given neighborhood of their starting point - as in $\mathbb{R}^2$; they do not get lost at infinity as they do in $\mathbb{R}^3$ (which is the simplest example of a transient manifold).

In [14] T. Lyons and D. Sullivan collected and proved a number of equivalent conditions (the so-called Kelvin–Nevanlinna–Royden criteria) for non-parabolicity, i.e. hyperbolicity: The Riemannian manifold (M, g) is hyperbolic if one (thence all) of the following equivalent conditions are satisfied: (a) M has *finite resistance* to infinity, (b) M has *positive capacity*, (c) M admits a *Green's function*, (d) There exists a precompact open domain Ω, such that the Brownian motion starting from Ω does *not return* to Ω with probability 1, (e) M admits a *square integrable vector field* with finite, but non-zero, absolute divergence. In particular the capacity condition (b) implies that (M, g) is parabolic if it has vanishing capacity - a condition which we will apply in section 9.

Returning now to the role of isoperimetric information: J. L. Fernandez showed in [3] that M is hyperbolic if the so-called (rooted) isoperimetric profile function $\phi(t)$ has a quare integrable reciprocal, i.e. $\int^{\operatorname{Vol}(M)} \phi^{-2}(t)\, dt \,<\, \infty$. Here the Ω_0-rooted profile function is molded directly from isoperimetric information as follows:

$$\begin{aligned}\phi(t) \,=\, \inf\{\, &\operatorname{Vol}(\partial\Omega) \mid \\ &\Omega \text{ is a smooth relatively compact domain in } M \;, \\ &\Omega \supset \Omega_0 \;, \quad \text{and } \operatorname{Vol}(\Omega) \,\geq\, t \,\} \quad .\end{aligned}$$

The volume of a non-parabolic manifold M is necessarily infinite. In fact, a finite volume manifold is parabolic by the following theorem due to Grigor'yan, Karp, Lyons and Sullivan, and Varopoulos. See [7] section 7.2 for an account of this type of results, which again is stated in terms of the simplest possible 'isoperimetric' information: Let $B_r(q)$ denote the geodesic ball centered at q in M and with radius r. If there exists a point q such that one (or both) of the following conditions is satisfied

$$\begin{aligned}\int^{\infty} \frac{r}{\operatorname{Vol}(B_r(q))}\, dr \,&=\, \infty \\ \int^{\infty} \frac{1}{\operatorname{Vol}(\partial B_r(q))}\, dr \,&=\, \infty \quad ,\end{aligned}$$

then M is parabolic.

In the present paper we obtain generalizations of this parabolicity condition. They are obtained for submanifolds in ambient spaces with a *lower bound* on curvatures by using a capacity comparison technique in combination with the Kelvin–Nevanlinna–Royden condition (b) as stated above, see Proposition 7.5 and Theorem 9.2. These results complement - and are in a rough sense dual to - previous hyperbolicity results that we have obtained using a corresponding *upper bound* on the curvatures of the ambient spaces, see [19, 20].

1.2. **On the local level.** If the boundary of a given domain is relatively small as compared to the domain itself, we also expect the mean exit time for Brownian motion to be correspondingly larger. The main concern of the present paper is to show an upper bound on the isoperimetric quotients of the so-called extrinsic balls of submanifolds under the essential assumption that the ambient spaces have a *lower bound* on their curvatures, see Theorem 6.1. The result for the mean exit time (from such extrinsic balls) then follows, as observed and proved in Theorem 6.3. These results are again dual to results which have been previously obtained under the condition of an upper bound on the curvature of the ambient space, see [25, 18] and [15, 24] , respectively.

1.3. **On the micro-local level.** When considering again the intrinsic geodesic balls $B_\varepsilon(q)$ of M centered at a fixed point q and assuming that the radius ε is approaching 0, then the Taylor series expansion of the volume function of the corresponding metric ball (or metric sphere) contains information about the curvatures of M at q - a classical result (for surfaces) obtained by Gauss and developed by A. Gray and L. Vanhecke in their seminal work [5]. Moreover, the geodesic metric balls have been analyzed by A. Gray and M. Pinsky in [4] in order to extract the geometric information contained in that particular function of ε , which gives the mean exit time from the center point q of the metric ball $B_\varepsilon(q)$.

To motivate even further the *extrinsic geometric* setting under consideration in the present paper, we mention here also yet another nice observation due to L. Karp and M. Pinsky concerning submanifolds in $\mathbb{R}^n$; see [13, 12] , where they show how to extract combinations of the principal curvatures of a submanifold at a point q from suitable power series expansions of the respective functions $\mathrm{Vol}(D_\varepsilon(q))$, $\mathrm{Vol}(\partial D_\varepsilon(q))$, and $E_{D_\varepsilon(q)}$, where $D_\varepsilon(q)$ is the extrinsic ball of (extrinsic) radius ε centered at q, and E is the mean exit time function.

On all levels then, be it global, local, or micro-local, as well as from both viewpoints, intrinsic or extrinsic, we thus encounter a fundamental interplay and inter-dependence between the highly instrumental geometric concepts of measure, shape, and diffusion considered here, namely the notions of volume, curvature, and exit time, respectively.

1.4. **Outline of the paper.** In the first two sections 2, and 3, we first provide intuitive versions, shadows, of our main results, i.e. we present the general results under stronger conditions than actually needed and compare them in particular with previous techniques for obtaining parabolicity (for surfaces of revolution) due to J. Milnor and K. Ichihara. In section 4 we then begin to establish the technical machinery for the paper and give precise definitions of the geometric bounds needed as preparation for our definition of what we call an Isoperimetric Constellation in section 5. This key notion is then applied in section 6 to present and prove our main isoperimetric result, Theorem 6.1. Various consequences of the main result and its proof are shown in sections 7 and 8. In particular we find new inequalities involving the volumes of extrinsic balls and their derivatives as well as intrinsic versions of our main results. In section 9 we establish an inequality for the capacities of extrinsic annular domains, which is then finally applied to prove the parabolicity result, Theorem 9.2, as alluded to above.

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2. A first Glimpse of the Main Results

We first facilitate intuition concerning our main results by considering some of their consequences for submanifolds in constant curvature ambient spaces - in particular for surfaces in $\mathbb{R}^3$. This seems quite relevant and worthwhile, because even in these strongly restricted settings we find results, which we believe are of independent interest. The results presented here are but shadows of the general results. The full versions of the main theorems appear in the sections below as indicated in the Outline, section 1.4.

2.1. **Strong Assumptions and Constant Curvature.** The general strong conditions applied for these initial statements are as follows: We let P^m denote a complete immersed submanifold in an ambient space form $N^n = \mathbb{K}^n_b$ with constant sectional curvature $b \leq 0$. Suppose further that P is *radially mean* $C-$*convex on the interval* $[\rho, \infty[$ for some $\rho \geq 0$ in N as viewed from a point $p \in P$ in the following sense: The unique oriented, arc length parametrized geodesic $\gamma_{p\to x}$ from p to $x \in P$ in the ambient space $\mathbb{K}^n_b$ has length $r(x)$ and its tangent $\gamma'_{p\to x}$ at x has an inner product with the mean curvature vector $H_P(x)$ of P in N at x, which we assume is bounded as follows:

$$\mathcal{C}(x) \,=\, -\langle\, \gamma'_{p\to x} \,,\, H_P(x)\,\rangle \,\geq\, C \tag{2.1}$$

for some constant C and for $r(x) \geq \rho$. We note that, by construction, $\mathcal{C}(p) = 0$ so for $C > 0$ the condition in (2.1) can only be satisfied away from p, see Theorem A below.

Condition (2.1) with $C = 0$ is e.g. satisfied for all r by *convex* hypersurfaces (cf. [2, 24]), as well as by all *minimal* submanifolds, where $H_P(x) = 0$ for all $x \in P$.

We consider a special type of compact subsets of P, the so-called extrinsic balls D_R, which for any given $R > 0$ consists of those points x in P, which have extrinsic distance $r(x)$ to p less than or equal to R.

At each point x on its boundary ∂D_R, the extrinsic ball D_R has a unique outward pointing unit vector ν_x, which has the following inner product with the unit tangent vector of $\gamma_{p\to x}$:

$$\mathcal{T}(x) \,=\, \langle\, \gamma'_{p\to x} \,,\, \nu_x\,\rangle \quad . \tag{2.2}$$

We let $g(R)$ denote the minimum tangency value $\mathcal{T}$ along the boundary of the extrinsic ball D_R - in general:

$$g(r) \,=\, \min_{x\in\partial D_r} \mathcal{T}(x) \leq 1 \quad . \tag{2.3}$$

We note that, by construction, $g(r) \to 1$ for $r \to 0$. An instrumental assumption to be satisfied throughout this paper, is that $g(r) > 0$ for all $r \in\,]0, R]$ and for each D_R under consideration.

With these initial ingredients and concepts we can now state the following first instances of our parabolicity criterion. Theorem A and Theorem B will eventually follow from Theorem 9.2 to be stated and proved in section 9.

Theorem A. *We assume the strong general conditions stated above in section 2.1 i.e. that* P^m *is radially mean* $C-$*convex on* $[\rho, \infty[$ *in* $\mathbb{K}^n_b$ *for* $b < 0$ *and for some* $\rho > 0$*. Suppose that* $C < \sqrt{-b}$ *and that*

$$\int_\rho^\infty Q_b(r)\exp\left(-\int_\rho^r \frac{m}{g^2(t)}\left(\frac{Q'_b(t)}{Q_b(t)} - C\right)dt\right)dr \,=\, \infty \quad , \tag{2.4}$$

where $g(r)$ *is the minimum tangency value at distance* r *from the pole (as in 2.3) and* $Q_b(r) = \frac{1}{\sqrt{-b}}\sinh(\sqrt{-b}\,r)$*. Then* P^m *is parabolic.*

Remark 2.1. We note that $Q_b'(t)/Q_b(t) > \sqrt{-b} > C$ for all t so that the condition (2.4) can only be met if $mC \geq (m-1)\sqrt{-b}$, cf. Corollary 2.4 below, where this is an explicit assumption. In the limiting case $b = 0$ the bound $C = 0$ is thence particularly interesting. For $b = 0$ we have $Q_0(r) = r$, and the condition (2.4) with $C = 0$ then reads:

$$\int_{\rho}^{\infty} r \exp\left(-\int_{\rho}^{r} \frac{m}{g^2(t)}\left(\frac{1}{t}\right) dt\right) dr = \infty \quad . \tag{2.5}$$

This condition can only be satisfied for $m = 2$. Hence, we are left with the following 'squeezed' version of the above theorem, which again will be proved as a consequence of Theorem 9.2. (Note also that we need here to assume $0-$convexity on the full interval $[0, \infty[$.)

Theorem B. *Suppose that P^2 is radially mean $0-$convex on $[0, \infty[$ in $\mathbb{R}^n$ and that*

$$\int_{\rho}^{\infty} r \exp\left(-\int_{\rho}^{r} \frac{2}{t\, g^2(t)} dt\right) dr = \infty \quad , \tag{2.6}$$

Then P^2 is parabolic.

One restriction on $g(r)$ which will satisfy (2.6) is explicitly displayed in the following:

Corollary 2.2. *Let P^2 denote a two-dimensional surface which is radially mean $0-$convex on $[0, \infty]$ in $\mathbb{R}^n$ and which has a radial tangency bounding function $g(r)$ which satisfies the following inequality for all sufficiently large r (so that also $\tilde{g}(r)$ is well defined and less than 1):*

$$g(r) \geq \tilde{g}(r) = \sqrt{\frac{2\log(r)}{1 + 2\log(r)}} \quad . \tag{2.7}$$

Then P^2 is parabolic.

Remark 2.3. This particular result should be viewed in the light of the fact that there are well known minimal (hence $0-$convex) surfaces in $\mathbb{R}^3$ - like Scherk's doubly periodic minimal surface - which are non-parabolic (i.e. hyperbolic), but which nevertheless also - in partial contrast to what could be expected from the above Corollary - support radial tangency functions $\mathcal{T}(x)$ which are 'mostly' close to 1 at infinity. The Scherk surface alluded to is the graph surface of the function $f(x, y) = \log(\cos(y)/\cos(x))$, which is smooth and well-defined on a checkerboard pattern in the $(x, y)-$plane. It was proved in [17] - via methods quite different from those considered in the present paper - that Scherk's surface is hyperbolic. Roughly speaking, the Scherk's surface may be considered as the most 'slim' known hyperbolic surface in $\mathbb{R}^3$. If that surface is viewed from a point far away from the $(x, y)-$plane, the surface looks like two sets of parallel half-planes, both orthogonal to the $(x, y)-$plane, one set below and the other above, and the two sets being rotated $\pi/2$ with respect to each other, see e.g. [16, pp. 46–49].

The radial tangency (from any fixed point p in the $(x, y)-$plane) is 'mostly' close to 1 at infinity except for the points in the $(x, y)-$plane itself, where the tangency function is 'wiggling' sufficiently close to 0, so that the condition 2.7 cannot be satisfied. In fact, if we accept for a moment the rough description of the surface as two sets of parallel half planes, then the integral over an extrinsic disc D_R of the tangency function $\mathcal{T}(x)$ from $p = (0, 0, 0)$ is roughly $0.8\,\mathrm{Area}(D_R)$ for large values of R. Note that for this simplified calculation the extrinsic disc consists of a finite number of flat half-discs, each one of which has radius $\sqrt{R^2 - \rho_0^2}$, where $\rho_0 \leq R$ denotes the orthogonal distance to p from the plane containing the half-disc.

We have discussed this particular example at some length, because it seems to be a good example for displaying in purely geometric terms what goes on at or close to the otherwise still quite unknown borderline between hyperbolic and parabolic surfaces in $\mathbb{R}^3$. In other words, the tangency function $\mathcal{T}$ introduced here seems to have an interesting and instrumental rôle to play concerning the quest of finding a necessary and sufficient condition for a surface to be hyperbolic, resp. parabolic.

For $b < 0$ we have a similar explicit bound on $g(r)$ which will satisfy equation (2.4):

Corollary 2.4. *Let P^m denote an $m-$dimensional submanifold which is radially mean $C-$convex on $[\,\rho, \infty[$, $\rho > 0$, in the space form $\mathbb{K}^n_b$ of constant curvature $b < 0$. Suppose (in accordance with Remark 2.1) that*

$$0 < m\left(\sqrt{-b} - C\right) \leq \sqrt{-b} \quad , \tag{2.8}$$

and suppose that P^m admits a radial tangency bounding function $g(r)$ which satisfies $g(r) \geq \tilde{g}(r)$ for all sufficiently large r where now - using shorthand notation $\tau = r\sqrt{-b}$ - we define

$$\tilde{g}(r) = \sqrt{\frac{m\,r\left(\sqrt{-b} - C\tanh(\tau)\right)}{\tanh(\tau)\left(1 + \frac{1}{\log(r)}\right) + \tau}} \quad . \tag{2.9}$$

(The conditions (2.8) imply that this function is well defined and less than 1 for sufficiently large r.) Then P^m is parabolic.

Remark 2.5. This result should likewise be compared with the fact established in [20], that every minimal submanifold of any co-dimension in a negatively curved space form is transient. Such submanifolds are *not* $C-$convex, of course, for any positive C (note that (2.8) implies $C > 0$). In relation to the discussion in the previous remark 2.3, what 'induces' parabolicity in negatively curved ambient spaces as in

Corollary 2.2 above, is to a large extent the radial mean C −convexity assumption - not just the tangency condition $\mathcal{T}(x) \geq \tilde{g}(r(x))$.

Proof of Corollary 2.2. With $b = 0$ we have $Q_b(r) = r$ and since $\tilde{g}(r)$ is designed to satisfy

$$\frac{2}{r\,\tilde{g}^2(r)} = \frac{2r\log(r) + r}{r^2\log(r)} = \frac{d}{dr}\log(r^2\log(r)) \tag{2.10}$$

for sufficiently large values of r, say $r \geq A$, we get for some positive constant c_1:

$$-\int_A^r \frac{2}{t\,\tilde{g}^2(t)}\,dt = -\log(r^2\log(r)) + c_1 \quad , \tag{2.11}$$

so that, for some other positive constant c_2:

$$\begin{aligned}
&\int^\infty Q_b(r)\exp\left(-\int_A^r \frac{m\,\eta_{Q_b}(t)}{g^2(t)}\,dt\right)\,dr \\
&\geq \int^\infty r\,c_2\exp\left(-\int_A^r \frac{2}{t\,\tilde{g}^2(t)}\,dt\right)\,dr \\
&= \int^\infty \frac{c_2}{r\log(r)}\,dr \\
&= \infty \quad ,
\end{aligned} \tag{2.12}$$

which then implies parabolicity according to Theorem B. □

The proof of Corollary 2.4 follows essentially verbatim from Theorem A except for handling the allowed C−interval for given b and m. The condition (2.8) simply stems from the two obvious conditions, that the square root defining $\tilde{g}(r)$ in (2.9) must be well-defined and less than 1. Indeed it follows directly from the definition (2.9) that

$$\lim_{r\to\infty}\tilde{g}(r) = \sqrt{\frac{m\left(\sqrt{-b} - C\right)}{\sqrt{-b}}} \quad . \tag{2.13}$$

It is of independent interest to note as well, that when b approaches 0 then C must go to 0, i.e. we are then back in the case of Corollary 2.2.

3. Examples and Benchmarking Surfaces of Revolution

We show in this section, that the catenoid and the hyperboloid of one sheet, are parabolic using the condition established in Corollary 2.2. Parabolicity of those surfaces is known already from criteria due to by Milnor and Ichihara, (see [22] and [9]).

The examples we have in mind are classical but suitably modified to provide well defined (and simple) extrinsic balls (discs) to exemplify our analysis. We consider piecewise smooth radially mean 0−convex surfaces of revolution in $\mathbb{R}^3$ constructed as follows. In the $(x, z)-$plane we consider the profile generating curve consisting of a (possibly empty) line segment along the $x-$axis: from $(0, 0)$ to $(a, 0)$ for some $a \in [\,0, \infty\,[$ together with a smooth curve $\Gamma(u) = (x(u), z(u))$, $u \in [\,0, c\,]$ for some $c \in]\,0, \infty\,[$ with $x(u) > 0$ for all u and $x(0) = a$. The corresponding surfaces of revolution then (possibly) have a flat (bottom) disc of radius a.

The center of this disc, i.e. the origin $p = (0, 0, 0)$, will serve as the point from which the surfaces under consideration will be $p-$radially mean $0-$convex as well as radially symmetric via the specific choices of generating functions $x(u)$ and $z(u)$.

The extrinsic radius of the defining cutting sphere centered at p is chosen to be $R(c) > a$, so that the corresponding extrinsic disc containing p consists of the flat bottom disc of radius a together with the following non-vanishing part of the surface of revolution:

$$\Omega_c \,:\, r(u, v) = (x(u)\cos(v), x(u)\sin(v), z(u))\,, \;\; u \in [\,0, c\,]\,, \;\; v \in [-\pi, \pi[\,.$$

The area of this extrinsic disc is then a function of a and c as follows:

$$\text{Area}(D_{R(c)}) = A(a, c) = \pi\, a^2 + 2\,\pi \int_0^c x(t)\sqrt{x'^2(t) + z'^2(t)}\; dt \quad ,$$

and the length of its boundary is simply

$$\text{Length}(\partial D_{R(c)}) = L(a, c) = 2\pi\, x(c) \quad .$$

The exact isoperimetric quotient of the extrinsic disc of the resulting surface of revolution is thus

$$\mathcal{Q} = \mathcal{Q}(a, c) = \frac{x(c)}{(a^2/2) + \int_0^c x(t)\sqrt{x'^2(t) + z'^2(t)}\; dt} \quad . \tag{3.1}$$

Under the assumed condition, that the surface of revolution is radially mean $0-$convex from p, Theorem 6.1 asserts that the quotient $\mathcal{Q}(a, c)$ is bounded from above by

$$\frac{2}{R(c)\, g(R(c))} = \frac{2}{g(c)\,\sqrt{x^2(c) + z^2(c)}} \quad , \tag{3.2}$$

where we use $g(u_0)$ as shorthand for $g(R(u_0))$ which is the exact common radial tangency value for the surface of revolution at each point of the circle $(x(u_0)\cos(v), x(u_0)\sin(v), z(u_0))$. It is by definition the inner product:

$$\begin{aligned} g(u_0) &= \langle\, \frac{\Gamma'(u_0)}{\|\Gamma'(u_0)\|}\,,\, \frac{\Gamma(u_0)}{\|\Gamma(u_0)\|}\,\rangle \\ &= \frac{x(u_0)\, x'(u_0) + z(u_0)\, z'(u_0)}{\sqrt{x'^2(u_0) + z'^2(u_0)}\;\sqrt{x^2(u_0) + z^2(u_0)}} \quad . \end{aligned} \tag{3.3}$$

Therefore the comparison upper bound reduces to:

$$\frac{2}{R(c)\,g(R(c))} \;=\; \frac{2\sqrt{x'^2(c) + z'^2(c)}}{x(c)\,x'(c) + z(c)\,z'(c)} \quad . \tag{3.4}$$

The examples below then serve as illustrations of our main theorem within the category of surfaces of revolution. Given the functions $x(u)$ and $z(u)$ we simply verify the following inequality in each case:

$$\mathcal{Q}(a,c) \;=\; \frac{A(a,c)}{L(a,c)} \leq \frac{2}{R(c)\,g(c)} \quad , \tag{3.5}$$

which (as shown above) is equivalent to:

$$\frac{x(c)}{(a^2/2) + \int_0^c x(t)\sqrt{x'^2(t) + z'^2(t)}\;dt} \leq \frac{2\sqrt{x'^2(c) + z'^2(c)}}{x(c)\,x'(c) + z(c)\,z'(c)} \quad . \tag{3.6}$$

In our short presentations below we also illustrate how the conditions of Corollary 2.2 and the criteria due to Milnor and Ichihara apply as well to prove parabolicity of each one of the surfaces in question.

Example 3.1. The *Catenoid* is a minimal surface and hence radially mean $0-$convex from any point on the surface; It is also clearly radially mean $0-$convex from p in its truncated version considered here - completed with a "flat disc bottom" of radius $a = 1$. We have in this particular case:

$$\begin{aligned} x(u) \;&=\; \cosh(u) \\ z(u) \;&=\; u \end{aligned} \tag{3.7}$$

for $u \in [\,0, c\,]$. Our isoperimetric inequality (in the form of (3.6)) is easily verified for all c. The tangency function follows from equation (3.3) and is given by:

$$\mathcal{T}(u,v) \;=\; g(u) \;=\; \frac{\sinh(u)\cosh(u) + u}{\cosh(u)\sqrt{\cosh(u)^2 + u^2}} \quad , \tag{3.8}$$

and the corresponding distance function to p, the origin, is:

$$r(u,v) \;=\; \sqrt{\cosh(u)^2 + u^2} \quad , \tag{3.9}$$

It is a simple calculation to see that indeed the lower tangency bound of Corollary 2.2, (2.7), holds true, so that we can conclude parabolicity of the truncated catenoid. Moreover, the Gaussian curvature of the catenoid is:

$$K(u,v) \;=\; -\frac{1}{\cosh^4(u)} \tag{3.10}$$

The surface integral of the absolute curvature is clearly finite. A direct calculation gives:

$$\int_{M^2} |K|\,d\mu \;=\; 2\pi \int_{-\infty}^{\infty} \frac{1 + \cosh(2u)}{2\cosh^4(u)}\,du \;=\; 4\pi \quad . \tag{3.11}$$

Icihara's condition in [9] then also gives parabolicity. To apply Milnor's condition, (see [22]), we calculate the arc length presentation:

$$s(u) \;=\; \sinh(u) \quad , \tag{3.12}$$

and parabolicity then follows from the inequality:

$$-\frac{1}{\cosh^4(u)} \;\geq\; -\,\frac{1}{\sinh^2(u)\log(\sinh(u))} \quad . \tag{3.13}$$

Example 3.2. The truncated and completed *Hyperboloid of one sheet* is also radially mean $0-$convex from the center point p of its flat bottom, although this is not clear from its shape. Indeed, from the generating functions

$$\begin{aligned} x(u) \;&=\; \sqrt{1+u^2} \\ z(u) \;&=\; u \end{aligned} \tag{3.14}$$

a short calculation reveals the following non-negative radial mean convexity function:

$$\mathcal{C}(x) \;=\; h(u(x)) \;=\; \frac{u^2}{(2u^2+1)^{5/2}} \geq 0 \quad . \tag{3.15}$$

The inequality (3.6) holds true for all $c \geq 0$ and our isoperimetric inequality is thus verified in this case. The curvature is

$$K(u,v) \;=\; -\,\frac{1}{(1+2u^2)^2} \tag{3.16}$$

and the intrinsic distance is essentially:

$$s(u) \;=\; i\,E(-u\,i, \sqrt{2}) \quad , \tag{3.17}$$

where $E(z,k)$ is the incomplete elliptic integral of the second kind. Parabolicity of the standard hyperboloid of one sheet then follows from Milnor's condition:

$$-\,\frac{1}{(1+2u^2)^2} \;\geq\; -\frac{1}{s^2(u)\log(s(u))} \quad . \tag{3.18}$$

The surface integral of the absolute value of the Gauss curvature is again finite. A direct calculation gives:

$$\int_{M^2} |K|\, d\mu \;=\; 2\pi \int_{-\infty}^{\infty} \frac{1}{(1+2\,u^2)^{3/2}}\, du \;=\; 2\pi\sqrt{2} \quad . \tag{3.19}$$

The tangency function is

$$\mathcal{T}(u,v) \;=\; \frac{2u\sqrt{1+u^2}}{1+2u^2} \quad , \tag{3.20}$$

and the corresponding extrinsic distance function to the origin is:

$$r(u,v) \;=\; \sqrt{1+2u^2} \quad . \tag{3.21}$$

It is a simple calculation to see that indeed the lower tangency bound of (2.7) holds true, so that parabolicity also stems from that condition.

4. Geometric Bounds from Below and Model Space Carriers

4.1. **Lower Mean Convexity Bounds.** Given an immersed, complete $m-$dimensional submanifold P^m in a complete Riemannian manifold N^n with a pole p, we denote the distance function from p in the ambient space N^n by $r(x) = \operatorname{dist}_N(p, x)$ for all $x \in N$. Since p is a pole there is - by definition - a unique geodesic from x to p which realizes the distance $r(x)$. We also denote by r the restriction $r|_P : P \longrightarrow \mathbb{R}_+ \cup \{0\}$. This restriction is then called the extrinsic distance function from p in P^m. The corresponding extrinsic metric balls of (sufficiently large) radius R and center p are denoted by $D_R(p) \subseteq P$ and defined as follows:

$$D_R(p) = B_R(p) \cap P = \{x \in P \,|\, r(x) < R\} \quad ,$$

where $B_R(p)$ denotes the geodesic R-ball around the pole p in N^n. The extrinsic ball $D_R(p)$ is assumed throughout to be a connected, pre-compact domain in P^m which contains the pole p of the ambient space. Since P^m is (unless the contrary is clearly stated) assumed to be unbounded in N we have for every sufficiently large R that $B_R(p) \cap P \neq P$.

In order to control the mean curvatures $H_P(x)$ of P^m at distance r from p in N^n we introduce the following definition:

Definition 4.1. The p-radial mean curvature function for P in N is defined in terms of the inner product of H_P with the N-gradient of the distance function $r(x)$ as follows:

$$\mathcal{C}(x) = -\langle \nabla^N r(x), H_P(x) \rangle \quad \text{for all} \quad x \in P \; .$$

We say that the submanifold P satisfies a *radial mean convexity condition* from $p \in P$ when we have a smooth function $h : P \mapsto \mathbb{R}$, such that

$$\mathcal{C}(x) \,\geq\, h(r(x)) \quad \text{for all} \quad x \in P \; . \tag{4.1}$$

Remark 4.2. For technical reasons pertaining to the proper construction of the *Isoperimetric Comparison Space* below in Definition 4.14 we may (and do) assume that $h(r)$ for $r < \varepsilon$ (for some sufficiently small ε) is represented by a Taylor series polynomial which only contains *odd* powers of r. Such a choice of bounding function h can always be done without lack of generality. Indeed, since $\mathcal{C}(x)$ is a smooth function of $x \in P$ with value 0 attained at $x = p$, we even have for a sufficiently large positive value of a that $h(r(x)) = -a\, r(x) \leq \mathcal{C}(x)$ for all $x \in D_\varepsilon(p)$.

4.2. **Lower Tangency Bounds.** The final notion needed to describe our comparison setting is the idea of *radial tangency.* If we denote by $\nabla^N r$ and $\nabla^P r$ the gradients of r in N and P respectively, then let us first remark that $\nabla^P r(q)$ is just the tangential component in P of $\nabla^N r(q)$, for all $q \in P$. Hence we have the following basic relation:

$$\nabla^N r = \nabla^P r + (\nabla^P r)^\perp \quad , \tag{4.2}$$

where $(\nabla^P r)^\perp(q)$ is perpendicular to T_qP for all $q \in P$.

Considering the extrinsic disc $D_{r(x)} \subset P$ and the outward pointing unit normal vector $\nu(x)$ to the extrinsic sphere $\partial D_{r(x)}$, then

$$\nabla^P r(x) \,=\, \langle \nabla^N r(x), \nu(x)\rangle\, \nu(x) \quad , \tag{4.3}$$

so that $\|\nabla^P r(x)\|$ measures the local *tangency* to P at x of the geodesics issuing from p. Full tangency means $\|\nabla^P r(x)\| \,=\, \langle \nabla^N r(x), \nu(x)\rangle \,=\, 1$, i.e. $\nabla^N r(x) \,=\, \nu(x)$ and minimal tangency means orthogonality, i.e. $\|\nabla^P r(x)\| \,=\, \langle \nabla^N r(x), \nu(x)\rangle \,=\, 0$.

In order to control this tangency of geodesics to the submanifold P we introduce the following

Definition 4.3. We say that the submanifold P satisfies a *radial tangency condition* from $p \in P$ when we have a smooth positive function $g : P \mapsto \mathbb{R}_+$, such that

$$\mathcal{T}(x) \,=\, \|\nabla^P r(x)\| \geq g(r(x)) \,>\, 0 \quad \text{for all} \quad x \in P\ . \tag{4.4}$$

Remark 4.4. We may (and do) assume that $g(r)$ for $r < \varepsilon$ (for some sufficiently small ε) is represented by a Taylor series polynomial which only contains *even* powers of r. This can always be assumed without lack of generality since $\mathcal{T}(x)$ is a smooth function of $x \in P$ with maximum value 1 attained at $x = p$, so that we even have for a sufficiently large positive value of b that $g(r(x)) = 1 - b\, r^2(x) \leq \mathcal{T}(x)$ for all $x \in D_\varepsilon(p)$.

4.3. **Auxiliary Model Spaces.** The concept of a model space is of instrumental importance for the precise statements of our comparison results. We therefore consider the definition and some first well-known properties in some detail:

Definition 4.5 (See [6], [7], [26])**.** A $w-$model space M^m_w is a smooth warped product with base $B^1 = [\,0,\, R[\, \subset \mathbb{R}$ (where $0 < R \leq \infty$), fiber $F^{m-1} = S^{m-1}_1$ (i.e. the unit $(m-1)-$sphere with standard metric), and warping function $w :\ [\,0,\, R[\to \mathbb{R}_+ \cup \{0\}$ with $w(0) = 0$, $w'(0) = 1$, $w^{(k)}(0) \,=\, 0$ for all even derivation orders k, and $w(r) \,>\, 0$ for all $r > 0$. The point $p_w = \pi^{-1}(0)$, where π denotes the projection onto B^1, is called the *center point* of the model space. If $R = \infty$, then p_w is a pole of M^m_w.

Remark 4.6. The simply connected space forms $\mathbb{K}^m_b$ of constant curvature b can be constructed as $w-$models with any given point as center point using the warping functions $w(r) = Q_b(r)$, (see Theorems A and B). Note that for $b > 0$ the function $Q_b(r)$ admits a smooth extension to $r = \pi/\sqrt{b}$. For $b \leq 0$ any center point is a pole.

In the papers [23], [6], [7], [20] and [21], we have a complete description of these model spaces, which we can summarize with the following results.

Proposition 4.7 (See [23] p. 206)**.** *Let M^m_w be a $w-$model with warping function $w(r)$ and center p_w. The distance sphere of radius r and center p_w in M^m_w, denoted as S^w_r, is the fiber $\pi^{-1}(r)$. This distance sphere has the following constant mean curvature vector in M^m_w*

$$H_{\pi^{-1}(r)} = -\eta_w(r)\,\nabla^M \pi = -\eta_w(r)\,\nabla^M r \quad , \tag{4.5}$$

where the mean curvature function $\eta_w(r)$ is defined by

$$\eta_w(r) = \frac{w'(r)}{w(r)} = \frac{d}{dr}\ln(w(r)) \quad . \tag{4.6}$$

In particular we have for the constant curvature space forms $\mathbb{K}^m_b$:

$$\eta_{Q_b}(r) = \begin{cases} \sqrt{b}\cot(\sqrt{b}\,r) & \text{if } b > 0 \\ \quad 1/r & \text{if } b = 0 \\ \sqrt{-b}\coth(\sqrt{-b}\,r) & \text{if } b < 0 \end{cases} \quad . \tag{4.7}$$

Definition 4.8. Let p be a point in a Riemannian manifold M and let $x \in M - \{p\}$. The sectional curvature $K_M(\sigma_x)$ of the two-plane $\sigma_x \in T_xM$ is then called a *p-radial sectional curvature* of M at x if σ_x contains the tangent vector to a minimal geodesic from p to x. We denote these curvatures by $K_{p,M}(\sigma_x)$.

Proposition 4.9 (See [6] and [7])**.** *Let M^m_w be a $w-$model with center point p_w. Then the p_w-radial sectional curvatures of M^m_w at every $x \in \pi^{-1}(r)$ (for $r > 0$) are all identical and determined by the radial function $K_w(r)$ defined as follows:*

$$K_{p_w, M^m_w}(\sigma_x) \;=\; K_w(r) \;=\; -\frac{w''(r)}{w(r)} \quad . \tag{4.8}$$

For any given warping function $w(r)$ we introduce the isoperimetric quotient function $q_w(r)$ for the corresponding $w-$model space M^m_w as follows:

$$q_w(r) \;=\; \frac{\mathrm{Vol}(B^w_r)}{\mathrm{Vol}(S^w_r)} \;=\; \frac{\int_0^r \, w^{m-1}(t)\,dt}{w^{m-1}(r)} \quad , \tag{4.9}$$

where B^w_r denotes the polar centered geodesic $r-$ball of radius r in M^m_w with boundary sphere S^w_r .

4.4. **The Laplacian Comparison Space.** The 2.nd order analysis of the restricted distance function $r_{|_P}$ defined on manifolds with a pole is firstly and foremost governed by the Hessian comparison Theorem A in [6]. As a consequence of this result, we have the following Laplacian inequality, (see too [20] and [8]):

Proposition 4.10. *Let N^n be a manifold with a pole p, let M_w^m denote a $w-$model with center p_w. Suppose that every p-radial sectional curvature at $x \in N - \{p\}$ is bounded from below by the p_w-radial sectional curvatures in M_w^m as follows:*

$$\mathcal{K}(\sigma(x)) \,=\, K_{p,N}(\sigma_x) \geq -\frac{w''(r)}{w(r)} \tag{4.10}$$

for every radial two-plane $\sigma_x \in T_xN$ at distance $r = r(x) = \operatorname{dist}_N(p, x)$ from p in N. Then we have for every smooth function $f(r)$ with $f'(r) \leq 0$ for all r, (respectively $f'(r) \geq 0$ for all r):

$$\begin{aligned}\Delta^P(f\circ r) \,\geq (\leq)\, &\left(\, f''(r) - f'(r)\eta_w(r)\,\right) \|\nabla^P r\|^2 \\ &+ mf'(r)\left(\,\eta_w(r) + \langle\, \nabla^N r,\, H_P\,\rangle\,\right) \quad ,\end{aligned} \tag{4.11}$$

where H_P denotes the mean curvature vector of P in N.

4.5. **The Isoperimetric Comparison Space.** Given the tangency and convexity bounding functions $g(r)$, $h(r)$ and the ambient curvature controller function $w(r)$ we construct a new model space $C^m_{w,g,h}$, which eventually will serve as the precise comparison space for the isoperimetric quotients of extrinsic balls in P.

Proposition 4.11. *For the given smooth functions $w(r)$, $g(r)$, and $h(r)$ defined above on the closed interval $[\,0, R\,]$, we consider the auxiliary function $\Lambda(r)$ for $r \in \,]\,0, R\,]$, which is independent of R and defined via the following equation. (We assume without lack of generality that $R > 1$ which, if needed, can be obtained by scaling the metric of N and thence P by a constant.)*

$$\Lambda(r)w(r)g(r) \,=\, T\,\exp\left(-\int_r^1 \frac{m}{g^2(t)}\,\left(\eta_w(t) - h(t)\right)\,dt\right), \tag{4.12}$$

where T is a positive constant - to be fixed shortly. Then there exists a unique smooth extension of $\Lambda(r)$ to the closed interval $[\,0, R\,]$ with $\Lambda(0) = 0$. The constant T can be (and is) chosen so that

$$\frac{d}{dr}_{|_{r=0}}\left(\Lambda^{\frac{1}{m-1}}(r)\right) = 1 \quad . \tag{4.13}$$

Moreover, from the technical assumptions on the bounding functions $h(r)$ and $g(r)$ alluded to in Remarks 4.2 and 4.4, we get that the following derivatives at $r = 0$ vanish:

$$\left(\Lambda^{\frac{1}{m-1}}(r)\right)^{(k)}_{|_{r=0}} \,=\, 0 \text{ for all even } k \quad . \tag{4.14}$$

Proof. Concerning the definition of Λ, the only problem is with the smooth extension to $r = 0$. We first observe that the integral of the right hand side of (4.12) can be expressed as follows:

$$\mathcal{I} \,=\, T_1\,\exp\left(-\int_r^{\varepsilon}\frac{m}{g^2(t)}\,\left(\eta_w(t)-h(t)\right)\,dt\right)\quad , \tag{4.15}$$

where T_1 is a well defined constant proportional to T. When $r \leq \varepsilon$ the Taylor series assumptions on $h(r)$ and $g(r)$ together with the model space warping function properties for $w(r)$ give for suitable coefficients as indicated:

$$g(r) = 1 + \sum_{i=1} a_i\, r^{2i}\ ,\ g^{-1}(r) = 1 + \sum_{i=1} \widehat{a}_i\, r^{2i}\ ,$$

$$g^{-2}(r) = 1 + \sum_{i=1} \widetilde{a}_i\, r^{2i}\ ,\ h(r) = \sum_{j=1} b_j\, r^{2j-1}$$

$$w(r) = 1 + \sum_{l=2} c_l\, r^{2l-1}\ ,\ \frac{w'(r)}{w(r)} = \eta_w(r) = \frac{1}{r} + \sum_{i=1} \widetilde{c}_l\, r^{2l-1}\quad ,$$

which, when inserted into (4.15) gives:

$$\begin{aligned}
\mathcal{I} \,&=\, T_1\,\exp\left(-m\int_r^{\varepsilon}\left(1+\sum_{i=1}\widetilde{a}_i\,t^{2i}\right)\left(\eta_w(t)-\sum_{j=1}b_j\,t^{2j-1}\right)\,dt\right)\\
&= T_1\,\exp\left(-m\int_r^{\varepsilon}\left(\eta_w(t)+\sum_{k=1}d_k\,t^{2k-1}\right)\,dt\right)\\
&= T_2\,w^m(r)\,\exp\left(-m\int_r^{\varepsilon}\left(\sum_{k=1}d_k\,t^{2k-1}\right)\,dt\right)\\
&= T_3\,w^m(r)\,\exp\left(\sum_{k=1}\widetilde{d}_k\,r^{2k}\right)\\
&= T_3\,w^m(r)\,\left(1+\sum_{k=1}\widehat{d}_k\,r^{2k}\right)
\end{aligned}$$

for suitable constants T_2, T_3, $\widetilde{d}_k$, $\widehat{d}_k$.

Thus we have:

$$\Lambda(r)w(r)g(r) \,=\, T_3\,w^m(r)\,\left(1+\sum_{k=1}\widehat{d}_k\,r^{2k}\right)\quad , \tag{4.16}$$

so that

$$\begin{aligned}
\Lambda(r)g(r) \,&=\, T_3\,w^{m-1}(r)\,\left(1+\sum_{k=1}\widehat{d}_k\,r^{2k}\right)\\
\Lambda(r) \,&=\, T_3\,w^{m-1}(r)\,\left(1+\sum_{k=1}\bar{d}_k\,r^{2k}\right)
\end{aligned} \tag{4.17}$$

for suitable constants $\bar{d}_k$. It follows that

$$
\begin{aligned}
\Lambda(0) &= 0 \quad , \\
\frac{d}{dr}_{|r=0} \left(\Lambda^{\frac{1}{m-1}}(r)\right) &= 1 \quad \text{for a suitable choice of } T_3 \text{, i.e. of } T, \\
\left(\Lambda^{\frac{1}{m-1}}(r)\right)^{(k)}_{|r=0} &= 0 \quad \text{for all even } k \quad .
\end{aligned}
\tag{4.18}
$$

Note that this follows essentially because of the structure of the Taylor series involved and not least because the warping function $w(r)$ itself, by assumption, satisfies these identities, i.e.

$$
\begin{aligned}
w(0) &= 0 \quad , \\
\frac{d}{dr}_{|r=0} w(r) &= 1 \quad , \\
w^{(k)}(0) &= 0 \quad \text{for all even } k \quad .
\end{aligned}
\tag{4.19}
$$

□

The following observation is a direct consequence of the construction of $\Lambda(r)$:

Lemma 4.12. *The function $\Lambda(r)$ satisfies the following differential equation:*

$$
\begin{aligned}
\frac{d}{dr}\,\Lambda(r)w(r)g(r) &= \Lambda(r)w(r)g(r)\left(\frac{m}{g^2(r)}\left(\eta_w(r) - h(r)\right)\right) \\
&= m\,\frac{\Lambda(r)}{g(r)}\left(w'(r) - h(r)w(r)\right)
\end{aligned}
\tag{4.20}
$$

Remark 4.13. In passing we also note that the function $\Lambda(r)$ defined as above is the essential ingredient in the solution to a Dirichlet–Poisson problem (6.4) to be considered below and is in this sense of instrumental importance for obtaining our main isoperimetric inequality in Theorem 6.1.

A 'stretching' function s is defined as follows

$$
s(r) = \int_0^r \frac{1}{g(t)}\,dt \quad . \tag{4.21}
$$

It has a well-defined inverse $r(s)$ for $s \in [\,0, s(R)\,]$ with derivative $r'(s) = g(r(s))$. In particular $r'(0) = g(0) = 1$ and by specific assumption on $g(r)$ we have close to $r = 0$ coefficients q_i and $\widehat{q}_i$ which give the stretching function and its inverse as follows:

$$
\begin{aligned}
s(r) &= r + \sum_{i=1} q_i r^{2i+1} \\
r(s) &= s + \sum_{i=1} \widehat{q}_i s^{2i+1} \quad .
\end{aligned}
\tag{4.22}
$$

With these concepts and key features of the auxiliary function $\Lambda(r)$ we are now able to define the carrier of the isoperimetric quotients to be compared with the isoperimetric quotients of the extrinsic balls D_R:

Definition 4.14. The *isoperimetric comparison space* $C^m_{w,g,h}$ is the $W-$model space with base interval $B = [\,0, s(R)\,]$ and warping function $W(s)$ defined by

$$W(s) \,=\, \Lambda^{\frac{1}{m-1}}(r(s)) \quad . \tag{4.23}$$

We observe firstly, that in spite of its relatively complicated construction, $C^m_{w,g,h}$ is indeed a model space with a well defined pole p_W at $s = 0$: $W(s) \geq 0$ for all s and $W(s)$ is only 0 at $s = 0$, where also, because of the explicit construction in Proposition 4.11 and equation (4.13): $W'(0) \,=\, 1$. Moreover, the second expansion (of $r(s)$) in (4.22) together with equation (4.14) gives as well: $W^{(k)}(0) \,=\, 0$ for every even k.

Secondly it should not be forgotten, that the spaces $C^m_{w,g,h}$ are specially tailor made to facilitate the proofs of the isoperimetric inequalities, that we are about to develop in section 6 as well as the explicit capacity comparison result in section 9.

In order for this to work out we need one further particular property to be satisfied by these comparison spaces:

4.6. **A Balance Condition.** Any given $C^m_{w,g,h}$ inherits all its properties from the bounding functions w, g, and h from which it is molded in the first place. Concerning the associated volume growth properties we note the following expressions for the isoperimetric quotient function:

Proposition 4.15. *Let $B^W_s(p_W)$ denote the metric ball of radius s centered at p_W in $C^m_{w,g,h}$. Then the corresponding isoperimetric quotient function is*

$$\begin{aligned} q_W(s) \,&=\, \frac{\mathrm{Vol}(B^W_s(p_W))}{\mathrm{Vol}(\partial B^W_s(p_W))} \,= \\ &\frac{\int_0^s W^{m-1}(t)\,dt}{W^{m-1}(s)} \,=\, \frac{\int_0^{r(s)} \frac{\Lambda(u)}{g(u)}\,du}{\Lambda(r(s))} \quad . \end{aligned} \tag{4.24}$$

The extra balance condition alluded to above is the following:

Definition 4.16. The model space $M^m_W \,=\, C^m_{w,g,h}$ is $w-$*balanced from below* on $[\,\rho, R\,]$ (with respect to the intermediary model space M^m_w) if the following holds for all $r \in [\,\rho, R\,]$, resp. all $s \in [\,s(\rho), s(R)\,]$:

$$\frac{d}{ds}\left(\frac{q_W(s)}{g(r(s))w(r(s))}\right) \,\leq\, 0 \quad . \tag{4.25}$$

We shall need the following observations concerning this notion of balance:

Lemma 4.17. *The balance condition (4.25) is equivalent to the following inequality:*

$$m\, q_W(s)\, (\eta_w(r(s)) - h(r(s))) \geq g(r(s)) \quad , \tag{4.26}$$

which, in terms of the auxiliary function $\Lambda(r)$, directly reads:

$$m \left(\int_0^r \frac{\Lambda(t)}{g(t)}\, dt \right) (w'(r) - h(r)w(r)) \,\geq\, \Lambda(r)w(r)g(r) \quad . \tag{4.27}$$

Moreover, if the balance condition (4.25), hence also (4.26) and (4.27), is satisfied for some value of r, say $r = \rho$, then the balance condition is satisfied for all $r \in [\rho, R]$ if the following stronger inequality (which does not involve the tangency bounding function $g(r)$), holds for all $r \in [\rho, R]$:

$$w''(r) - w'(r)h(r) - w(r)h'(r) \,\geq\, 0 \quad . \tag{4.28}$$

Remark 4.18. In particular the w-balance condition implies that

$$\eta_w(r) \,-\, h(r) \,>\, 0 \quad . \tag{4.29}$$

The above Definition 4.16 is an extension of the balance condition from below as applied in [21]. The original condition there reads as follows and is obtained from (4.26) precisely when $g(r) \,=\, 1$ and $h(r) \,=\, 0$ for all $r \in [\,0, R\,]$ so that $r(s) \,=\, s$, $W(s) \,=\, w(r)$:

$$m\, q_w(r)\eta_w(r) \,\geq 1 \quad . \tag{4.30}$$

Proof of Lemma 4.17. A direct differentiation using (4.24) but with respect to r amounts to the following via Lemma 4.12:

$$\begin{aligned} \frac{d}{dr}\left(\frac{q_W(s(r))}{g(r)w(r)}\right) \,&=\, \frac{d}{dr}\left(\frac{\int_0^r \frac{\Lambda(u)}{g(u)}\, du}{\Lambda(r)g(r)w(r)}\right) \\ &= \frac{1}{\Lambda(r)g^3(r)w^2(r)}\left(\Lambda(r)w(r)g(r) - m\left(\int_0^r \frac{\Lambda(t)}{g(t)}\, dt\right)(w'(r) - h(r)w(r))\right) \quad , \end{aligned}$$

which shows that (4.25), (4.26), and (4.27) are all equivalent for each choice of $r \in [\,\rho, R\,]$.

Now let $G(r)$ denote the left hand side of the balance condition inequality (4.27), and let $F(r)$ denote the right hand side:

$$G(r) \,=\, m\left(\int_0^r \frac{\Lambda(t)}{g(t)}\, dt\right)(w'(r) - h(r)w(r)) \,\geq\, \Lambda(r)w(r)g(r) \,=\, F(r) \quad .$$

Suppose $G(\rho) \geq F(\rho)$ for some ρ and assume that (4.28) is satisfied for all $r \in [\rho, R]$. Then $G'(r) \geq F'(r)$ for all r in that interval. In fact

$$\begin{aligned}
G'(r) &= m\frac{\Lambda(r)}{g(r)}\left(w'(r) - h(r)w(r)\right) \\
&\quad + m\left(\int_0^r \frac{\Lambda(t)}{g(t)}\,dt\right)\left(w''(r) - w'(r)h(r) - w(r)h'(r)\right) \\
&\geq m\frac{\Lambda(r)}{g(r)}\left(w'(r) - h(r)w(r)\right) \\
&= F'(r) \quad ,
\end{aligned} \tag{4.31}$$

where we have used Lemma 4.12 (for $F'(r)$) and the condition (4.28). It follows that $G(r) \geq F(r)$ for all $r \in [\rho, R]$, and this proves the rest of the Lemma. □

For the proof of Theorem A via Theorem 9.2 below we shall need a special but simple extension principle for the balance condition. It is based directly upon (4.28):

Lemma 4.19. *Let $C^m_{w,g,h}$ denote an isoperimetric comparison space for some P^m in $\mathbb{K}^n_b$, $b < 0$. Specifically we then use $w(r) = Q_b(r) = \frac{1}{\sqrt{-b}}\sinh(\sqrt{-b}r)$ for all $r \in [0, \infty]$. Suppose further that $h(r) = C < \sqrt{-b}$ for all r in the interval $r \in [\rho_1, \infty]$ where $\rho_1 > 0$. Note that (4.28) is then satisfied on this interval. Then there exists a lower support function $\hat{h}(r)$ for $h(r)$ defined on $[0, \infty]$ so that $C^m_{w,g,\hat{h}}$ is an isoperimetric comparison space for P^m in $\mathbb{K}^n_b$ with $\hat{h}(r) \leq h(r)$ for all $r \in [0, \rho_1[$, $\hat{h}(r) = h(r) = C$ for $r \in [\rho_2, \infty[$ for some $\rho_2 \geq \rho_1$ and such that (4.28) is satisfied for the pair $w(r)$, $\hat{h}(r)$ on $[0, \infty[$, i.e. $C^m_{w,g,\hat{h}}$ is balanced from below on $[0, \infty[$ - independent of the tangency bounding function $g(r)$.*

Proof. First construct a smooth function $\hat{h}(r)$ on $[0, \rho_1]$ which supports $h(r)$ from below with $\hat{h}(0) = 0$ and $\hat{h}'(r) \leq 0$ on that interval and with only odd non-zero derivatives at $r = 0$ as per Remark 4.2. Then $\hat{h}(r)$ is non-positive and satisfies the differential inequality (4.28) on $[0, \rho_1]$:

$$\tanh(r\sqrt{-b})(-b - \hat{h}'(r)) \geq \hat{h}(r)\sqrt{-b} \quad . \tag{4.32}$$

From the value $\hat{h}(\rho_1) \leq 0$ at $r = \rho_1$ we may now extend the function smoothly to the interval $[\rho_1, \rho_2]$ such that $\hat{h}(\rho_2) = C$, $\hat{h}'(\rho_2) = 0$, and such that

$$\tanh(r\sqrt{-b})(-b - \hat{h}'(r)) \geq C\sqrt{-b} = \hat{h}(r)\sqrt{-b} \tag{4.33}$$

is satisfied on $[\rho_1, \rho_2]$ for a sufficiently large ρ_2. This is possible precisely because of the assumption $C < \sqrt{-b}$. Indeed, the only problem is when $\hat{h}(r)$ is close to C. But for sufficiently large r we have

$\tanh(r\sqrt{-b}) > 1 - \varepsilon$ for some small $\varepsilon > 0$. Under the condition of equation (4.33) we may thence choose $\hat{h}'(r)$ as large as follows: $\hat{h}'(r) \leq -b - \frac{C\sqrt{-b}}{1-\varepsilon}$, where the right hand side is positive when ε is small enough. In particular $\hat{h}(r)$ can increase smoothly from any non-positive value $\hat{h}(\rho_1)$ to attain the value $\hat{h}(\rho_2) = C$ if only the allowed interval $[\,\rho_1, \rho_2\,]$ is large enough. This proves the Lemma. □

4.7. **Balance on the Edge.**

Lemma 4.20. *Equality in the balance condition (4.26) is equivalent to equality in the stronger condition (4.28) and equivalent to each one of the following identities:*

$$\eta_w(r) - h(r) = \frac{1}{w(r)} \tag{4.34}$$

$$\Lambda(r)w(r)g(r) = m\int_0^r \frac{\Lambda(t)}{g(t)}\,dt \quad . \tag{4.35}$$

$$q_W(s) = \frac{1}{m}w(r(s))g(r(s)) \quad . \tag{4.36}$$

Proof. From equality in (4.28) we get

$$w'(r) = w(r)h(r) + c \tag{4.37}$$

for some constant c which must then be $c = 1$ since $w'(0) = 1$. Then $w'(r) - w(r)h(r) = 1$ gives identity in equation (4.31) as well and vice versa. □

Remark 4.21. Special cases of equality in the balance condition are obtained by $h(r) = 0$ and $w(r) = r$. This corresponds to the situation considered in section 3 - where we analyzed radially mean $0-$convex surfaces in Euclidean $3-$space.

5. The Isoperimetric Comparison Constellation

The intermediate observations considered above together with the previously introduced bounds on radial curvature and tangency now constitute the notion of a comparison constellation (for the isoperimetric inequality) as follows.

Definition 5.1. Let N^n denote a Riemannian manifold with a pole p and distance function $r = r(x) = \mathrm{dist}_N(p, x)$. Let P^m denote an unbounded complete and closed submanifold in N^n. Suppose $p \in P^m$, and suppose that the following conditions are satisfied for all $x \in P^m$ with $r(x) \in [\,0, R\,]$:

(a) The p-radial sectional curvatures of N are bounded from below by the p_w-radial sectional curvatures of the $w-$model space M_w^m:

$$\mathcal{K}(\sigma_x) \geq -\frac{w''(r(x))}{w(r(x))} \quad .$$

(b) The p-radial mean curvature of P is bounded from below by a smooth radial function $h(r)$ which, for r sufficiently close to 0, is represented by a Taylor series polynomial with only odd powers of r :

$$\mathcal{C}(x) \geq h(r(x)) \quad .$$

(c) The p-radial tangency of P is bounded from below by a smooth radial function $g(r)$ which, for r sufficiently close to 0, is represented by a Taylor series polynomial with only even powers of r :

$$\mathcal{T}(x) \geq g(r(x)) > 0 \quad .$$

Let $C_{w,g,h}^m$ denote the W-model with the specific warping function $W : \pi(C_{w,g,h}^m) \to \mathbb{R}_+$ which is constructed above in Definition 4.14 via w, g, and h. Then the triple $\{N^n, P^m, C_{w,g,h}^m\}$ is called an *isoperimetric comparison constellation* on the interval $[\,0, R\,]$.

6. Main isoperimetric results

In this section we find upper bounds for the isoperimetric quotient defined as the volume of the extrinsic sphere divided by the volume of the extrinsic ball, in the setting given by the comparison constellations defined in Definition 5.1:

Theorem 6.1. *We consider an isoperimetric comparison constellation $\{N^n, P^m, C_{w,g,h}^m\}$ on the interval $[\,0, R\,]$. Suppose further that the comparison space $C_{w,g,h}^m$ is $w-$balanced from below on $[\,0, R\,]$ in the sense of Definition 4.16. Then*

$$\frac{\operatorname{Vol}(\partial D_R)}{\operatorname{Vol}(D_R)} \leq \frac{\operatorname{Vol}(\partial B_{s(R)}^W)}{\operatorname{Vol}(B_{s(R)}^W)} \leq \frac{m}{g(R)}\left(\eta_w(R) - h(R)\right) \quad . \tag{6.1}$$

If the comparison space $C_{w,g,h}^m$ satisfies the balance condition with equality in (4.26) (or equivalently in (4.28)) for all $r \in [\,0, R\,]$, then

$$\frac{\operatorname{Vol}(\partial D_R)}{\operatorname{Vol}(D_R)} \leq \frac{\operatorname{Vol}(\partial B_{s(R)}^W)}{\operatorname{Vol}(B_{s(R)}^W)} = \frac{m}{w(R)g(R)} \quad . \tag{6.2}$$

Proof. We define a second order differential operator L on functions f of one real variable as follows:

$$\operatorname{L} f(r) = f''(r)\, g^2(r) + f'(r)\left((m - g^2(r))\, \eta_w(r) - m\, h(r)\right) \quad , \tag{6.3}$$

and consider the smooth solution $\psi(r)$ to the following Dirichlet–Poisson problem:

$$\begin{aligned} \mathrm{L}\,\psi(r) &= -1 \quad \text{on} \quad [\,0, R\,] \\ \psi(R) &= 0 \quad . \end{aligned} \tag{6.4}$$

The ODE problem is equivalent to the following:

$$\psi''(r) + \psi'(r)\left(-\eta_w(r) + \frac{m}{g^2(r)}\left(\eta_w(r) - \,h(r)\right)\right) \;=\; -\frac{1}{g^2(r)} \quad . \tag{6.5}$$

The solution is constructed via the auxiliary function $\Lambda(r)$ from (4.12) as follows:

$$\begin{aligned} \psi'(r) \;&=\; \Gamma(r) \\ &=\; \exp(-\mathcal{P}(r))\,\int_0^r \exp(\mathcal{P}(t))\left(-\frac{1}{g^2(t)}\right)\,dt \quad , \end{aligned} \tag{6.6}$$

where the auxiliary function $\mathcal{P}$ is defined as follows, assuming again without lack of generality, that $R\,>\,1$, so that the domain of definition of all involved functions contains $[\,0,1\,]$:

$$\begin{aligned} \mathcal{P}(r) \;&=\; \int_r^1 \left(\eta_w(t) - \frac{m}{g^2(t)}\left(\eta_w(t) - \,h(t)\right)\right)\,dt \\ &=\; \log\left(\frac{w(1)}{w(r)}\right) - \int_r^1 \frac{m}{g^2(t)}\left(\eta_w(t) - \,h(t)\right)\,dt \quad . \end{aligned} \tag{6.7}$$

Hence

$$\begin{aligned} &\exp(\mathcal{P}(t))\left(\frac{1}{g^2(t)}\right) \;= \\ &\frac{w(1)}{w(t)\,g^2(t)}\exp\left(-\int_t^1 \frac{m}{g^2(u)}\,\left(\eta_w(u) - h(u)\right)\,du\right) \\ &=\; \frac{w(1)}{w(t)\,g^2(t)\,T}\Lambda(t)w(t)g(t) \;=\; \left(\frac{w(1)}{T}\right)\left(\frac{\Lambda(t)}{g(t)}\right) \quad . \end{aligned} \tag{6.8}$$

From this latter expression we then have as well

$$\exp(-\mathcal{P}(r)) \;=\; \left(\frac{T}{w(1)}\right)\left(\frac{1}{g(r)\Lambda(r)}\right) \quad , \tag{6.9}$$

so that

$$\begin{aligned} \psi'(r) \;&=\; \frac{-1}{g(r)\,\Lambda(r)}\int_0^r \frac{\Lambda(t)}{g(t)}\,dt \\ &=\; -\frac{\mathrm{Vol}(B^W_{s(r)})}{g(r)\,\mathrm{Vol}(\partial B^W_{s(r)})} \\ &=\; -\frac{q_W(s(r))}{g(r)} \quad . \end{aligned} \tag{6.10}$$

At this instance we must observe, that it follows clearly from this latter expression for $\psi'(r)$ that the above formal, yet standard, method of solving (6.4) indeed does produce a smooth solution on $[0, R]$ with $\psi'(0) = 0$. Then we have:

$$\begin{aligned}\psi(r) &= \int_r^R \frac{1}{g(u)\,\Lambda(u)}\left(\int_0^u \frac{\Lambda(t)}{g(t)}\,dt\right)du \\ &= \int_r^R \frac{q_W(s(u))}{g(u)}\,du \\ &= \int_{s(r)}^{s(R)} q_W(t)\,dt \quad .\end{aligned} \tag{6.11}$$

We now show that - because of the balance condition (4.28) - the function $\psi(r)$ enjoys the following inequality:

Lemma 6.2.

$$\psi''(r) - \psi'(r)\,\eta_w(r) \geq 0 \quad . \tag{6.12}$$

Proof of Lemma. We must show that

$$\Gamma'(r) - \Gamma(r)\eta_w(r) \geq 0 \quad . \tag{6.13}$$

Equation (6.5) implies

$$\Gamma'(r) - \Gamma(r)\eta_w(r) = -\frac{1}{g^2(r)}\left(1 + m\Gamma(r)\left(\eta_w(r) - h(r)\right)\right) \quad . \tag{6.14}$$

Therefore equation (6.13) is equivalent to the following inequality, observing that $\Gamma(r) < 0$ for all $r \in]0, R]$:

$$m\left(\int_0^r \frac{\Lambda(t)}{g(t)}\,dt\right)\left(w'(r) - h(r)w(r)\right) \geq \Lambda(r)w(r)g(r) \quad , \tag{6.15}$$

which in its turn is equivalent to (4.26) via Lemma 4.17. □

With Lemma 6.2 in hand we continue the proof of Theorem 6.1. Applying the Laplace inequality (4.11) for the function $\psi(r)$ transplanted into P^m in N^n now gives the following comparison

$$\begin{aligned}\Delta^P\psi(r(x)) &\geq \left(\psi''(r(x)) - \psi'(r(x))\eta_w(r(x))\right) g^2(r(x)) \\ &\quad + m\psi'(r(x))\left(\eta_w(r(x)) - h(r(x))\right) \\ &= \mathrm{L}\,\psi(r(x)) \\ &= -1 \\ &= \Delta^P E(x) \quad ,\end{aligned} \tag{6.16}$$

where $E(x)$ is the mean exit time function for the extrinsic ball D_R, with $E_{|\partial D_R} = 0$.

Applying the divergence theorem, taking the unit normal to ∂D_R as $\frac{\nabla^P r}{\|\nabla^P r\|}$, we get

$$
\begin{aligned}
\operatorname{Vol}(D_R) &= \int_{D_R} -\Delta^P E(x)\, d\mu \\
&\geq \int_{D_R} -\Delta^P \psi(r(x))\, d\mu \\
&= -\int_{D_R} \operatorname{div}(\nabla^P \psi(r(x)))\, d\mu \\
&= -\int_{\partial D_R} < \nabla^P \psi(r(x)), \frac{\nabla^P r(x)}{\|\nabla^P r\|} > d\nu \\
&= -\Gamma(R) \int_{\partial D_R} \|\nabla^P r\|\, d\nu \\
&\geq -\Gamma(R)\, g(R)\, \operatorname{Vol}(\partial D_R) \quad ,
\end{aligned}
\tag{6.17}
$$

which shows the first part of the isoperimetric inequality (6.1). The second inequality in (6.1) follows from (4.26).

We should remark that we can apply the maximum principle to inequality (6.16), to conclude

$$
E(x) \,\geq\, \psi(r(x)) \ , \ \text{for all } x \in D_R \quad . \tag{6.18}
$$

We finally consider the case of equality in the balance condition, which implies in particular that (4.35) holds true. Using this we have immediately, as claimed:

$$
\frac{\operatorname{Vol}(\partial B^W_{s(R)})}{\operatorname{Vol}(B^W_{s(R)})} \,=\, \frac{\Lambda(R)}{\int_0^R \frac{\Lambda(t)}{g(t)}\, dt} \,=\, \frac{m}{g(R)\, w(R)} \quad . \tag{6.19}
$$

□

A. Gray and M. Pinsky proved in [4] a nice result, which shows that Brownian diffusion is fast in negative curvature and slow in positive curvature, even on the level of scalar curvature.

In view of this reference to the work by Gray and Pinsky [4] we mention here the following consequence of the proof of Theorem 6.1, equations (6.18) and (6.11).

Theorem 6.3. *Let $\{N^n, P^m, C^m_{w,g,h}\}$ denote an isoperimetric comparison constellation which is $w-$balanced from below on $[\,0, R\,]$. Then for all $x \in D_R \subset P^m$, we have:*

$$
E(x) \,\geq\, E_W(s(r(x))) \,=\, \int_{s(r(x))}^{s(R)} q_W(t)\, dt \quad , \tag{6.20}
$$

where $E_W(s)$ is the mean exit time function for Brownian motion in the disc of radius R centered at the pole p_W in the model space $C^m_{w,g,h}$. The function $q_W(s)$ is the isoperimetric quotient function and $s(r)$ is the stretching function of the $W-$model comparison space.

7. Consequences

We first observe the following volume comparison in consequence of Theorem 6.1, inequality (6.1):

Corollary 7.1. *Let $\{N^n, P^m, C^m_{w,g,h}\}$ be a comparison constellation on the interval $[\,0, R\,]$, as in Theorem 6.1. Then*

$$\mathrm{Vol}(D_r) \,\leq\, \mathrm{Vol}(B^W_{s(r)}) \quad \textit{for every } r \in [\,0, R\,]\ . \tag{7.1}$$

Proof. Let $\mathcal{G}(r)$ denote the following function

$$\mathcal{G}(r) \,=\, \log\left(\frac{\mathrm{Vol}(D_r)}{\mathrm{Vol}(B^W_{s(r)})}\right) \quad . \tag{7.2}$$

Since $W(s)$ is a warping function for an $m-$dimensional model space we have $\mathcal{G}(0) \,=\, \lim_{r\to 0}\mathcal{G}(r) \,=\, 0$. Then, from isoperimetric inequality (6.1) and inequality (7.5) below, we have

$$\mathcal{G}'(r) \,\leq\, \frac{1}{g(r)}\left(\frac{\mathrm{Vol}(\partial D_r)}{\mathrm{Vol}(D_r)} - \frac{\mathrm{Vol}(\partial B^W_{s(r)})}{\mathrm{Vol}(B^W_{s(r)})}\right) \,\leq\, 0 \quad . \tag{7.3}$$

In consequence we therefore have $\mathcal{G}(r) \,\leq\, \mathcal{G}(0) \,=\, 0$, or equivalently:

$$\mathrm{Vol}(D_r) \,\leq\, \mathrm{Vol}(B^W_{s(r)}) \quad \text{for every } \ r \in [\,0, R\,] \quad . \tag{7.4}$$

□

Proposition 7.2. *Let $\{N^n, P^m, C^m_{w,g,h}\}$ be a comparison constellation on the interval $[\,0, R\,]$, as in Theorem 6.1. Then we have:*

$$g(r)\frac{\partial}{\partial r}\,\mathrm{Vol}(D_r) \,\leq\, \mathrm{Vol}(\partial D_r) \quad \textit{for all} \quad r \in [\,0, R\,]\ . \tag{7.5}$$

Proof. Let $\Psi(x) = \psi(r(x))$ denote the radial mean exit time function, transplanted into D_R.

With the notation of [1] we then have

$$\begin{aligned} \Omega(t) &= \{x \in P \,|\, \psi(r(x)) > t\} = D_{\psi^{-1}(t)} \\ V(t) &= \mathrm{Vol}(D_{\psi^{-1}(t)}) \\ \text{and } \ \Sigma(t) &= \partial D_{\psi^{-1}(t)} \ \text{ for all } \ t \in]\,0, \psi(0)\,[\quad . \end{aligned}$$

The co-area formula states that

$$V'(t) = -\int_{\partial D_{\psi^{-1}(t)}} \|\nabla^P \Psi(x)\,\|^{-1} d\sigma_t \quad . \tag{7.6}$$

On the other hand, we know that on D_R we have for all r:

$$\|\nabla^P \Psi\| \, = \, -\Gamma(r)\, \|\nabla^P r\| \, \geq -g(r)\,\Gamma(r) \quad , \tag{7.7}$$

Therefore

$$\begin{aligned} V'(t) \, &\geq \int_{\partial D_{\psi^{-1}(t)}} \frac{1}{g(r)\,\Gamma(r)} d\sigma_t \\ &= \frac{1}{g(r)\,\Gamma(r)} \operatorname{Vol}(\partial D_{\psi^{-1}(t)}) \quad . \end{aligned} \tag{7.8}$$

We define $F(r) = \operatorname{Vol}(D_r)$ and have

$$V(t) = \operatorname{Vol}(D_{\psi^{-1}(t)}) = F \circ \psi^{-1}(t) \quad , \tag{7.9}$$

then

$$\begin{aligned} V'(t) \, &= \, F'(\psi^{-1}(t))\, \frac{d}{dt}\psi^{-1}(t) \\ &= \frac{\frac{d}{dr}\operatorname{Vol}(D_r)}{\frac{d}{dr}\psi(r)} \\ &= \frac{\frac{d}{dr}\operatorname{Vol}(D_r)}{\Gamma(r)} \quad . \end{aligned} \tag{7.10}$$

Since we also know from equation (7.8) that

$$V'(t) \, \geq \, \frac{\operatorname{Vol}(\partial D_r)}{g(r)\,\Gamma(r)} \quad , \tag{7.11}$$

and since $\Gamma(r) < 0$ on $]\,0, R[\,$, we finally get, as claimed:

$$g(r)\frac{d}{dr}\operatorname{Vol}(D_r) \, \leq \, \operatorname{Vol}(\partial D_r) \quad \text{for all } \; r \in [\,0, R\,] \quad . \tag{7.12}$$

□

Corollary 7.3. *Let $\{N^n, P^m, C^m_{w,g,h}\}$ be a comparison constellation on $[\,0, R\,]$, as in Theorem 6.1. Then*

$$\int_{\partial D_r} \|\nabla^P r\|\, d\nu \, \leq \, g(r)\, W^{m-1}(s(r)) \;\; \textit{for every } r \in [\,0, R\,] \quad . \tag{7.13}$$

Proof. Follows from equations (6.10), (6.17) in the proof of Theorem 6.1 and Corollary 7.1. □

Moreover, as direct applications of the volume comparison in Corollary 7.1, using the obvious fact that the geodesic balls in P, B^P_R are subsets of the extrinsic balls D_R of the same radius, we get from the general non-explosion condition and parabolicity condition in [7, Theorem 9.1]:

Proposition 7.4. *Let $\{N^n, P^m, C^m_{w,g,h}\}$ denote an isoperimetric comparison constellation which is $w-$balanced from below on $[\,0, \infty\,[$ 0. Suppose that*

$$\int^{\infty} \frac{r(s)\, g(r(s))}{\log(\mathrm{Vol}(B^W_s))}\, ds \;=\; \infty \quad . \tag{7.14}$$

Then P^m is stochastically complete, i.e. the Brownian motion in the submanifold is non-explosive.

From the same volume comparison result we get

Proposition 7.5. *Let $\{N^n, P^m, C^m_{w,g,h}\}$ denote an isoperimetric comparison constellation which is $w-$balanced from below on $[\,0, \infty\,[$. Suppose that*

$$\int^{\infty} \frac{r(s)\, g(r(s))}{\mathrm{Vol}(B^W_s)}\, ds \;=\; \infty \quad . \tag{7.15}$$

Then P^m is parabolic.

When the submanifold P^m is minimal and the ambient space N^n is a Cartan-Hadamard manifold (with sectional curvatures bounded from above by 0), we get the following two-sided isoperimetric inequality:

Corollary 7.6. *Let P^m be a minimal submanifold in N^n. Let $p \in P^m$ be a point which is a pole of N. Suppose that the p-radial sectional curvatures of N are bounded from above and from below as follows:*

$$-\frac{w_2''(r(x))}{w_2(r(x))} \;\leq\; \mathcal{K}(\sigma_x) \;\leq\; -\frac{w_1''(r(x))}{w_1(r(x))} \;\leq\; 0 \quad . \tag{7.16}$$

Then

$$\frac{\mathrm{Vol}(\partial B^{w_1}_R)}{\mathrm{Vol}(B^{w_1}_R)} \;\leq\; \frac{\mathrm{Vol}(\partial D_R)}{\mathrm{Vol}(D_R)} \;\leq\; \frac{\mathrm{Vol}(\partial B^{W_2}_{s(R)})}{\mathrm{Vol}(B^{W_2}_{s(R)})} \;\leq\; \frac{m}{g(R)}\eta_{w_2}(R) \;,$$

where $W_2(s)$ is the warping function of the comparison model space $C^m_{w_2,g,0}$, see Theorem 6.1.

Proof. Since P is minimal it is radially mean $0-$convex, so $h = 0$. The upper isoperimetric bound follows then directly from Theorem 6.1. Indeed, $\{N^n, P^m, C^m_{w_2,g,0}\}$ is a comparison constellation with the model comparison space $C^m_{w_2,g,0}$ which is w_2-balanced from below. This latter claim follows because $w_2(r)$ satisfies the strong balance condition (4.28) in view of the assumptions $-w_2''(r(x))/w_2(r(x)) \leq 0$ and $h = 0$. The lower isoperimetric bound follows directly from Theorem B in [21]. Also for this to hold we need the corresponding balance condition from below, which is again satisfied because of the curvature assumption: $-w_1''(r(x))/w_1(r(x)) \leq 0$, see [21, Observation 5.12]. □

8. The intrinsic Viewpoint

In this section we consider the intrinsic versions of the isoperimetric and volume comparison inequalities (6.1) and (7.1) assuming that $P^m = N^n$. In this case, the extrinsic distance to the pole p becomes the intrinsic distance in the ambient manifold, so, for all $r > 0$ the extrinsic domains D_r become the geodesic balls B^N_r of N^n. We have for all $x \in P$:

$$\begin{aligned} \nabla^P r(x) &= \nabla^N r(x) \\ H_P(x) &= 0 \quad . \end{aligned} \tag{8.1}$$

Thus $\|\nabla^P r\| = 1$, so $g(r(x)) = 1$ and $\mathcal{C}(x) = h(r(x)) = 0$, the *stretching* function becomes the identity $s(r) = r$, $W(s(r)) = w(r)$, and the isoperimetric comparison space $C^m_{w,g,h}$ is reduced to the auxiliary model space M^m_w. Following the lines of the proof of Theorem 6.1 in this setting and taking into account that Lemma 6.2 is now obsolete for the statements of (6.16) to hold true, we have the following isoperimetric inequality for geodesic balls:

Corollary 8.1. *Let N^n denote a complete riemannian manifold with a pole p. Suppose that the p-radial sectional curvatures of N^n are bounded from below by the p_w-radial sectional curvatures of a w-model space M^n_w for all $r > 0$. Then*

$$\frac{\operatorname{Vol}(\partial B^N_R)}{\operatorname{Vol}(B^N_R)} \leq \frac{\operatorname{Vol}(\partial B^w_R)}{\operatorname{Vol}(B^w_R)} \leq n\eta_w(R) \quad . \tag{8.2}$$

As a consequence of Corollary 7.3 and Corollary 8.1, we obtain the following well-known comparisons for the volume of geodesic balls and spheres, (see [27]).

Corollary 8.2. *Let N^n denote a complete Riemannian manifold with a pole p. Suppose that the p-radial sectional curvatures of N^n are bounded from below by the p_w-radial sectional curvatures of a w-model space M^n_w for all $r > 0$. Then*

$$\operatorname{Vol}(B^N_r) \leq \operatorname{Vol}(B^w_r) \quad \textit{for every } r \in [0, R]\ . \tag{8.3}$$

Corollary 8.3. *Let N^n denote a complete Riemannian manifold with a pole p. Suppose that the p-radial sectional curvatures of N^n are bounded from below by the p_w-radial sectional curvatures of a w-model space M^n_w for all $r > 0$. Then*

$$\operatorname{Vol}(\partial D_r) \leq w^{n-1}(r) = \operatorname{Vol}(\partial B^w_r) \ \textit{for every } r \in [0, R] \quad . \tag{8.4}$$

9. Capacity Analysis

Given the extrinsic balls with radii $\rho < R$, D_ρ and D_R, the annulus $A_{\rho,R}$ is defined as

$$A_{\rho,R} = D_R - D_\rho \quad .$$

The unit normal vector field on the boundary of this annulus $\partial A_{\rho,R} = \partial D_\rho \cup \partial D_R$ is denoted by ν and defined by the following normalized P-gradient of the distance function restricted to ∂D_ρ and ∂D_R, respectively:

$$\nu = \nabla^P r(x)/\|\nabla^P r(x)\| \quad , \quad x \in \partial A_{\rho,R} \quad .$$

We apply the previously mentioned result by Greene and Wu, [6], with a lower radial curvature bound again, but now we consider radial functions with $f'(r) \geq 0$ which will change the inequality in the Laplace comparison, equation (4.11).

Theorem 9.1. *Let* $\{N^n, P^m, C^m_{w,g,h}\}$ *denote an isoperimetric comparison constellation which is* $w-$*balanced from below on* $[\,0, R\,[$. *Then*

$$\mathrm{Cap}(A_{\rho,R}) \leq \left(\int_{s(\rho)}^{s(R)} \frac{1}{W^{m-1}(t)}\, dt \right)^{-1} \quad . \tag{9.1}$$

Proof. We consider again the second order differential operator L but now we look for the smooth solution $\xi(r)$ to the following Dirichlet–Laplace problem on the interval $[\,\rho, R\,]$, $\rho > 0$:

$$\begin{aligned} \mathrm{L}\,\xi(r) = 0 \ \text{ on } \ [\,\rho, R\,] \quad , \\ \xi(\rho) \,=\, 0 \ , \ \xi(R) \,=\, 1 \quad . \end{aligned} \tag{9.2}$$

The solution is again constructed via the function $\Lambda(r)$ defined in equation (4.12):

$$\xi'(r) \,=\, \Xi(r) \,=\, \frac{1}{g(r)\Lambda(r)} \left(\int_\rho^R \frac{1}{g(t)\Lambda(t)}\, dt \right)^{-1} \quad , \tag{9.3}$$

Then we have

$$\xi(r) \,=\, \left(\int_\rho^r \frac{1}{g(t)\Lambda(t)}\, dt \right) \left(\int_\rho^R \frac{1}{g(t)\Lambda(t)}\, dt \right)^{-1} \quad . \tag{9.4}$$

Applying the Laplace inequality (4.11) on the radial functions $\xi(r)$ - now with a non-negative derivative - transplanted into P^m in N^n now gives the following comparison inequality, using the assumptions stated in the theorem:

$$\begin{aligned} \Delta^P \xi(r(x)) \,&\leq\, (\xi''(r(x)) - \xi'(r(x))\eta_w(r(x)))\ g^2(r(x)) \\ &\quad + m\xi'(r(x))\,(\eta_w(r(x)) - h(r(x))) \\ &=\, \mathrm{L}\,\xi(r(x)) \\ &=\, 0 \\ &=\, \Delta^P v(x) \quad , \end{aligned} \tag{9.5}$$

where $v(x)$ is the Laplace potential function for the extrinsic annulus $A_{\rho,R} \,=\, D_R - D_\rho$, setting $v_{\partial D_\rho} \,=\, 0$ and $v_{\partial D_R} \,=\, 1$.

For this inequality to hold we need that

$$\xi''(r) - \xi'(r)\eta_w(r) \leq 0 \quad . \tag{9.6}$$

This follows from the Laplace equation itself together with the consequence (4.29) of the balance condition that $h(r) \leq \eta_w(r)$:

$$(\xi''(r) - \xi'(r)\eta_w(r))\, g^2(r) = -m\xi'(r)\left(\eta_w(r) - h(r)\right) \leq 0 \quad . \tag{9.7}$$

The maximum principle then applies again and gives:

$$v(x) \leq \xi(r(x)) \text{ , for all } x \in A_{\rho,R} \quad . \tag{9.8}$$

Moreover, the function v must be nonnegative in the annular domain $A_{\rho,R}$. Otherwise v would have an intrinsic (negative) minimum in $A_{\rho,R}$, and since v is harmonic this is ruled out by the minimum principle.

From the inequalities $0 \leq v(x) \leq \xi(r(x))$ for $x \in A_{\rho,R}$ and the identities $v = \xi = 0$ on ∂D_ρ we get $\|\nabla^P v(x)\| \leq \|\nabla^P \xi(r(x))\|$ for all $x \in \partial D_\rho$. But since $\|\nabla^P \xi(r(x))\| = |\xi'(\rho)| \|\nabla^P r(x)\|$ we then have at ∂D_ρ:

$$\Xi(\rho) = |\xi'(\rho)| \geq \|\nabla^P v(x)_{|\partial D_\rho}\| \cdot \|\nabla^P r(x)_{|\partial D_\rho}\|^{-1} \quad , \tag{9.9}$$

or equivalently:

$$\Xi(\rho)\|\nabla^P r(x)_{|\partial D_\rho}\| \geq \|\nabla^P v(x)_{|\partial D_\rho}\| \quad . \tag{9.10}$$

For the capacity we get via an application of the divergence theorem, see e.g. [7, p. 152]:

$$\mathrm{Cap}(A_{\rho,R}) = \int_{\partial D_\rho} \langle \nabla^P v, n_{\partial D_\rho} \rangle_{\partial D_\rho} \, d\mu \quad , \tag{9.11}$$

where $n_{\partial D_\rho}$ denotes the unit normal vector field along ∂D_ρ pointing *into* the domain $A_{\rho,R}$.

Now, since v is nonnegative in the annular domain and $v = 0$ at the inner boundary, then the inwards directed gradient $\langle \nabla^P v, n_{\partial D_\rho} \rangle_{\partial D_\rho}$ is also nonnegative. Since ∂D_ρ is a level hypersurface (of value $v = 0$) for v in P, we have that $n_{\partial D_\rho}$ is proportional to $\nabla^P v$. It therefore follows that

$$\langle \nabla^P v, n_{\partial D_\rho} \rangle_{\partial D_\rho} = \|\nabla^P v(x)\| \quad . \tag{9.12}$$

Using (4.23) and Corollary 7.3, we then have

$$
\begin{aligned}
\operatorname{Cap}(A_{\rho,R}) &= \int_{\partial D_\rho} \|\nabla^P v(x)\|\, d\nu \\
&\leq \Xi(\rho) \int_{\partial D_\rho} \|\nabla^P r\|\, d\nu \\
&\leq \Xi(\rho)\, g(\rho)\, \Lambda(\rho) \\
&= \left(\int_\rho^R \frac{1}{g(t)\Lambda(t)}\, dt \right)^{-1} \\
&= \left(\int_{s(\rho)}^{s(R)} \frac{1}{W^{m-1}(t)}\, dt \right)^{-1} \quad ,
\end{aligned} \tag{9.13}
$$

which shows the general capacity bound (9.1).

□

As a consequence we have the following extrinsic version of Ichihara's [9, Theorem 2.1].

Theorem 9.2. *Let $\{N^n, P^m, C^m_{w,g,h}\}$ denote an isoperimetric comparison constellation which is $w-$balanced from below on $[\,0,\infty\,[$. Suppose that*

$$
\int^\infty \frac{1}{\operatorname{Vol}(\partial B^W_s)}\, ds = \infty \quad . \tag{9.14}
$$

Then P^m is parabolic.

Proof. First of all, we know that $\operatorname{Vol}(\partial B^W_s) = W^{m-1}(s)$ for all s. On the other hand, and referring to the above capacity inequality (9.1), we see that the capacity is forced to 0 in the limit $s \to \infty$ because of the condition (9.14). According to the Kelvin–Nevanlinna–Royden criterion (b) of the introduction, see subsection 1.1, this corresponds to parabolicity of the submanifold. □

Remark 9.3. One first pertinent remark is that Theorem 9.2 is *not* a consequence of the previous Proposition 7.5 as is also pointed out by A. Grigor′yan in [7, p. 180–181]. Indeed, he shows by a nice example, that (7.15) is not equivalent to parabolicity of the model space, so, although (9.14) does imply parabolicity of the model space, we cannot conclude parabolicity of P^m via Proposition 7.5.

As a corollary to Theorem 9.2 we get Theorems A and B as follows:

Proof of Theorems A and B. The conditions $h(r) = C < \sqrt{-b}$ for $r \in [\,\rho = \rho_1, \infty[$ and $w(r) = Q_b(r)$, $b < 0$, imply via Lemma 4.17 the existence of a comparison constellation, $C^m_{w,g,\hat{h}}$ which is $w-$balanced from below on the interval $[\,0,\infty\,[$. We recall from that Lemma that $\hat{h}(r) = C$ for $r \geq \rho_2 \geq \rho_1 = \rho$.

For $b = 0$ the corresponding needed balance from below for Theorem B follows directly from equation (4.28). Furthermore, since we have in all cases that $1 \geq g(r) > 0$, the 'stretching' satisfies $s(r) \to \infty$ when $r \to \infty$. We have then according to Proposition 4.11, (4.12):

$$\begin{aligned}
&\int_{\rho_2}^{\infty} \frac{1}{\operatorname{Vol}(\partial B^W_s)}\, ds \\
&= \int_{\rho_2}^{\infty} \frac{1}{W^{m-1}(s(r))} \left(\frac{ds}{dr}\right) dr \\
&= \int_{\rho_2}^{\infty} \frac{1}{\Lambda(r)\, g(r)}\, dr \qquad (9.15)\\
&= \int_{\rho_2}^{\infty} \left(\frac{T}{Q_b(r)} \exp\left(-\int_r^1 \frac{m}{g^2(t)} \left(\eta_{Q_b}(t) - \hat{h}(t)\right) dt\right)\right)^{-1} dr \\
&= \int_{\rho_2}^{\infty} \left(\frac{T\,\lambda}{Q_b(r)} \exp\left(-\int_r^{\rho_2} \frac{m}{g^2(t)} \left(\eta_{Q_b}(t) - C\right) dt\right)\right)^{-1} dr\,,
\end{aligned}$$

where T is the fixed constant defined and found in Proposition 4.11, and where

$$\lambda = \exp\left(-\int_{\rho_2}^1 \frac{m}{g^2(t)} \left(\eta_{Q_b}(t) - \hat{h}(t)\right) dt\right) \quad , \tag{9.16}$$

and therefore the value of each of the integrals in 9.15 is ∞ if and only if

$$\int_{\rho}^{\infty} \left(\frac{1}{Q_b(r)} \exp\left(-\int_r^{\rho} \frac{m}{g^2(t)} \left(\eta_{Q_b}(t) - C\right) dt\right)\right)^{-1} dr = \infty \quad .$$

In this particular case the condition (9.14) therefore corresponds to (2.4) in Theorem A, and to (2.6) in Theorem B respectively. This proves the theorems. □

The intrinsic versions of these results (implying formally that $h(r) = 0$ and $g(r) = 1$ for all r) are well known and established by Ahlfors, Nevanlinna, Karp, Varopoulos, Lyons and Sullivan, and Grigor'yan, see [7, Theorem 7.3 and Theorem 7.5]. For warped product model space manifolds, the reciprocal boundary-volume integral condition is also necessary for parabolicity, but *in general* none of the conditions are necessary conditions for parabolicity.

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