Graphs: Introduction

Ali Shokoufandeh,

Department of Computer Science, Drexel University

Overview of this talk

Introduction:

Notations and Definitions

Graphs and Modeling

Algorithmic Graph Theory and Combinatorial Optimization

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- **Notations and Definitions**
- Graphs and Modeling
- Algorithmic Graph Theory and Combinatorial Optimization

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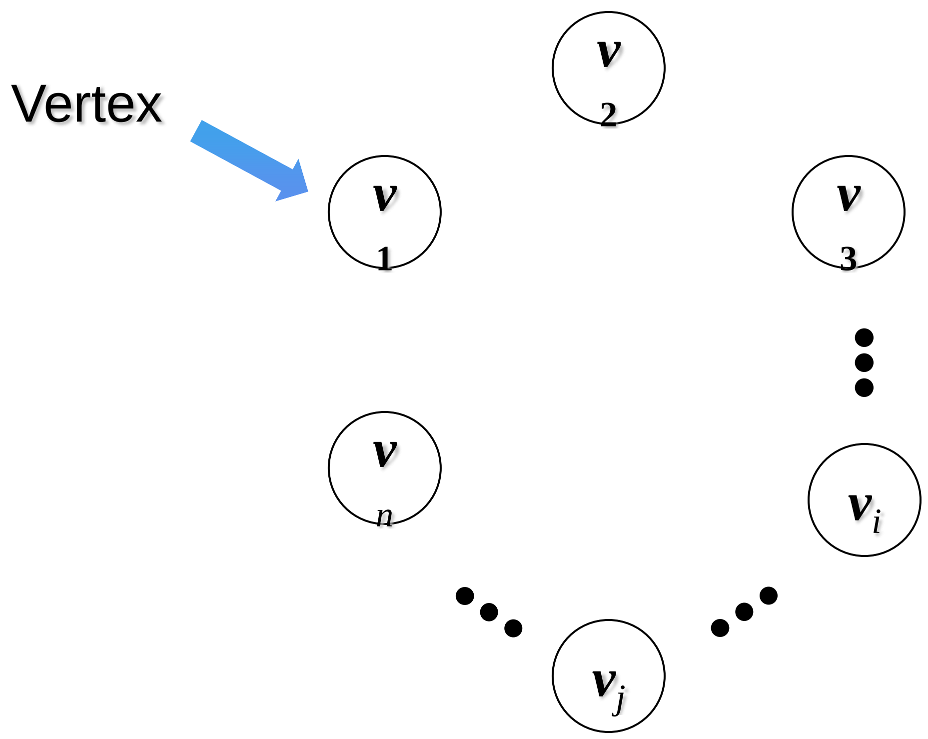
A graph G is a triple consisting of a ***vertex set*** $V(G)$, an ***edge set*** $E(G)$, and a ***relation (weight function)*** $W(G)$ associates with each edge. The two vertices of each edge will be called its ***endpoints*** (not necessarily distinct).



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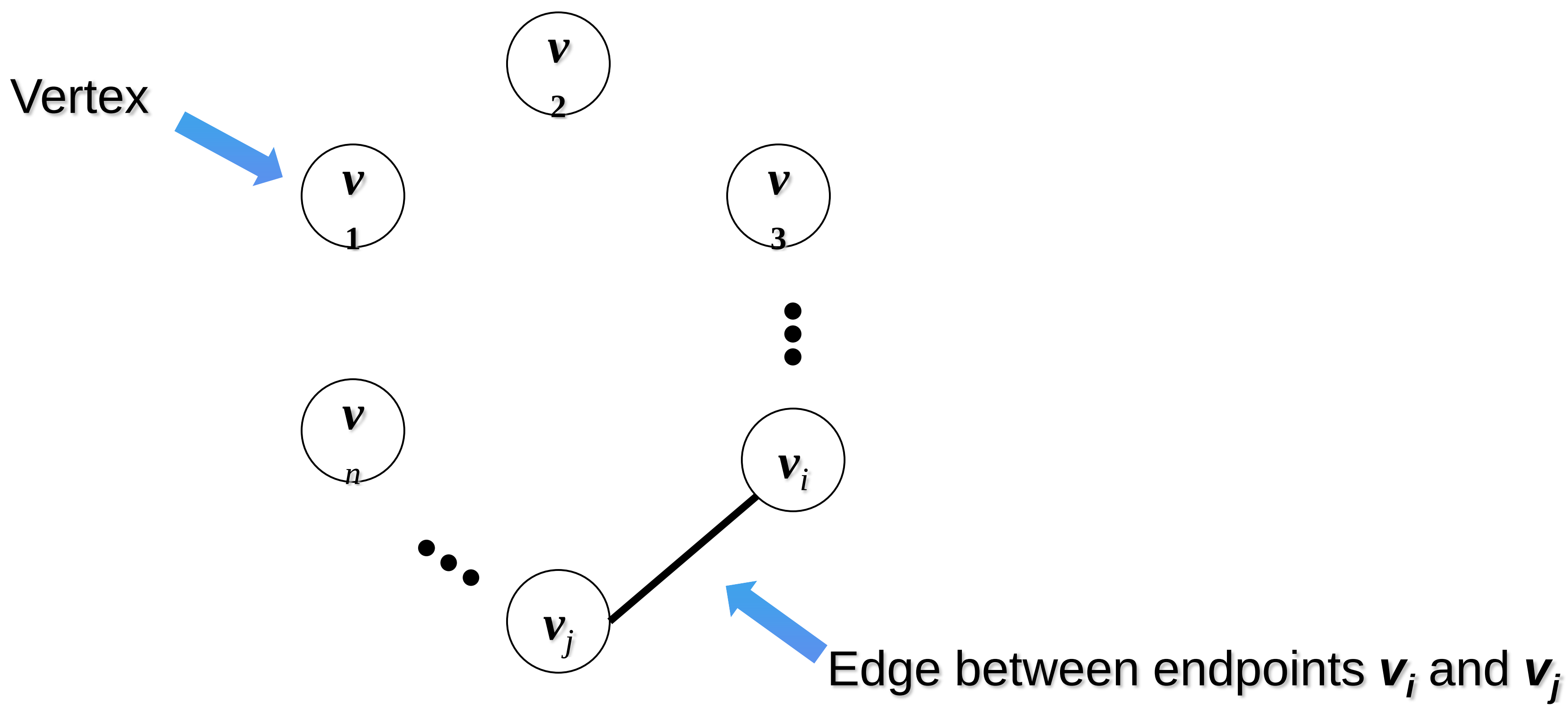

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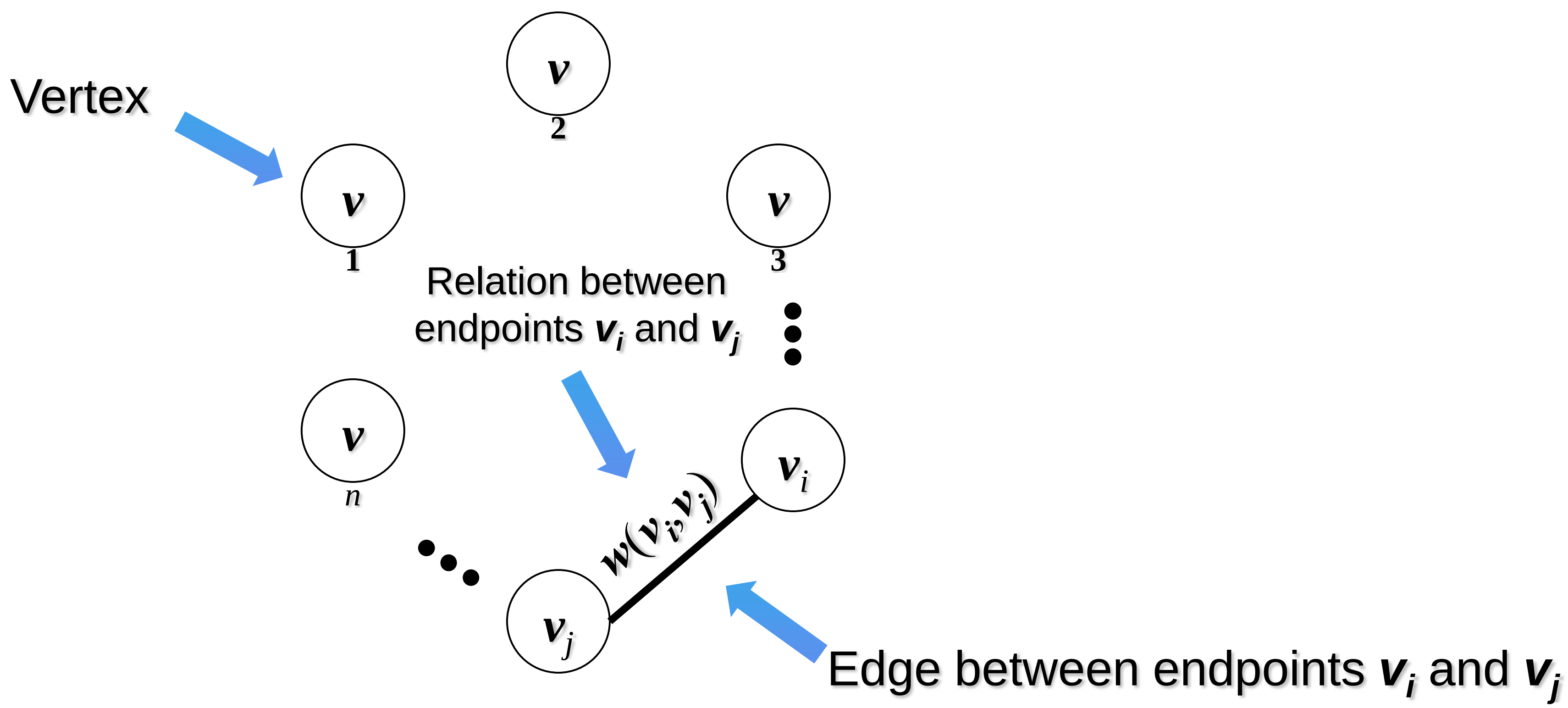

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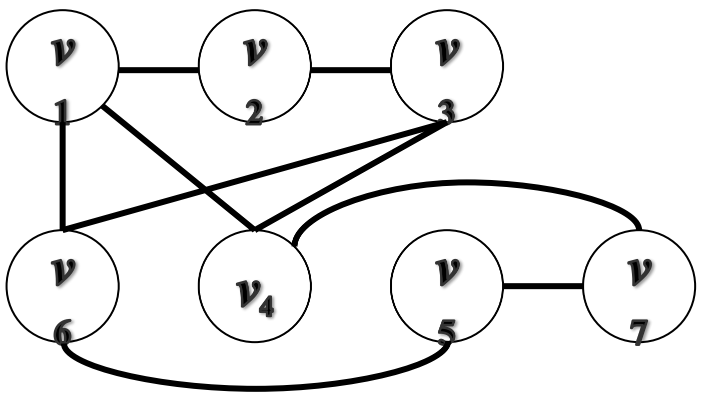

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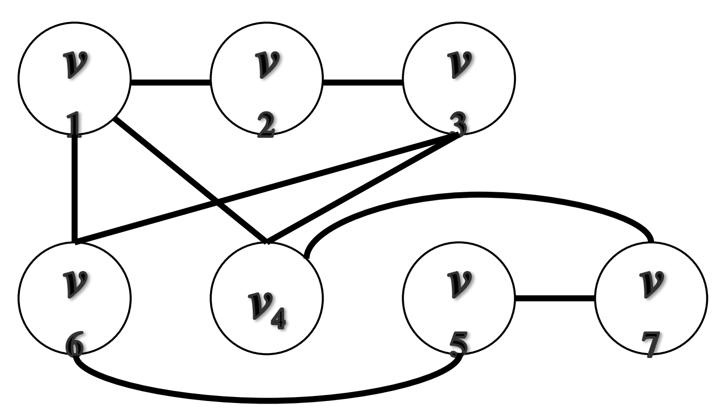

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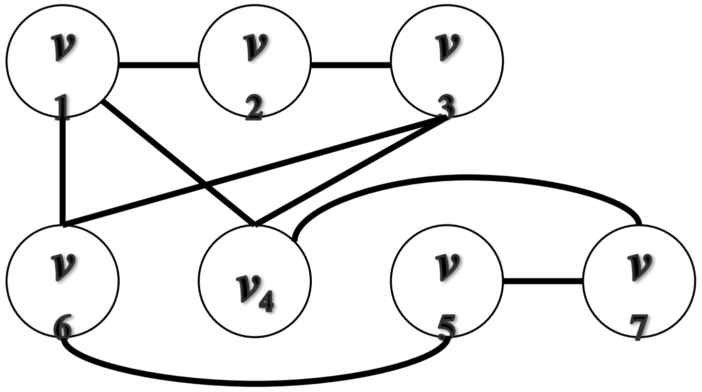

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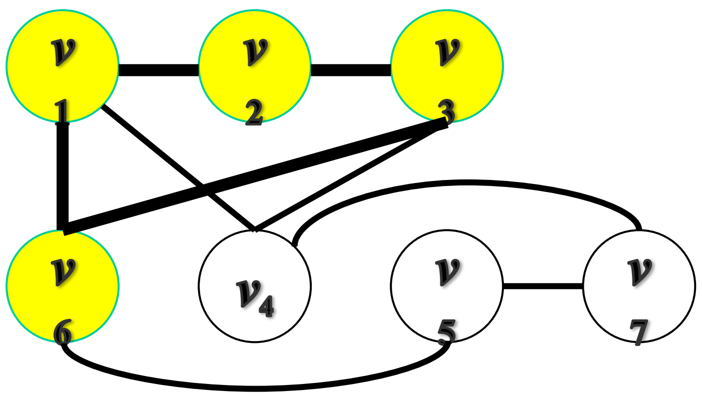

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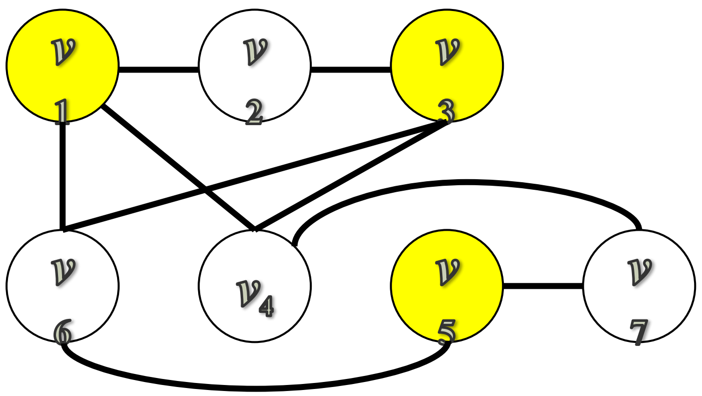
  - Example: an ***independent set*** in a graph is a set of vertices that are pairwise nonadjacent



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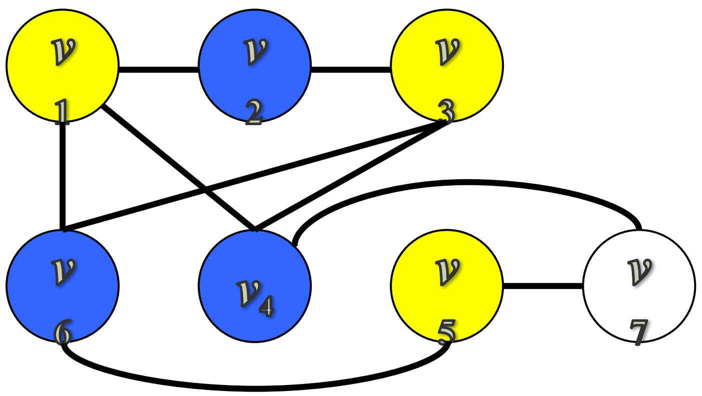

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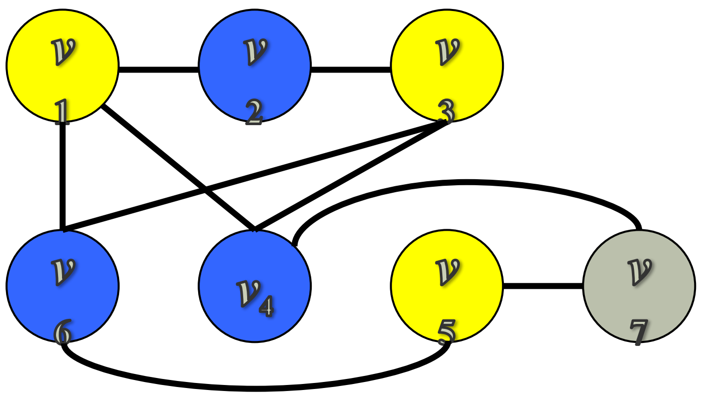
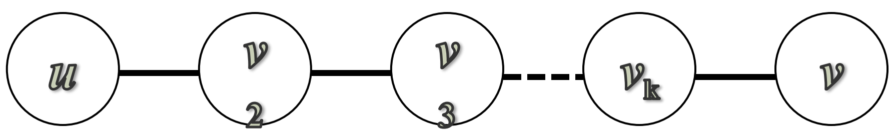
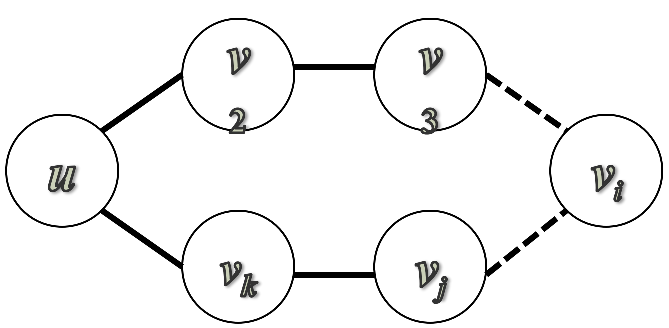


Notations:

- A ***<u,v>-path*** is a simple graph that begins at ***u*** and ends at ***v***, whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the ordering.



- A ***cycle*** is a simple path whose vertices can be cyclically ordered with overlapping endpoints (***u***=***v***).



Notations:

- Given a graph $\boldsymbol{G}=(\boldsymbol{V},\boldsymbol{E})$, where
  - $\boldsymbol{V}$ is its vertex set, $|\boldsymbol{V}|=\boldsymbol{n}$,
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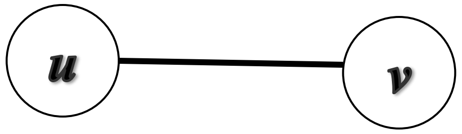
- In an undirected graph an edge $(\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{v},\boldsymbol{u})$.



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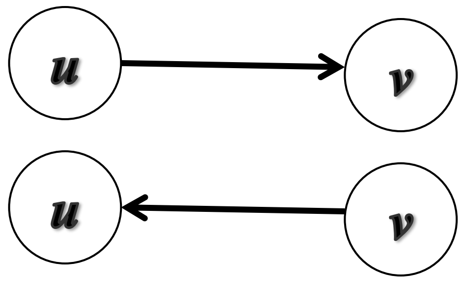

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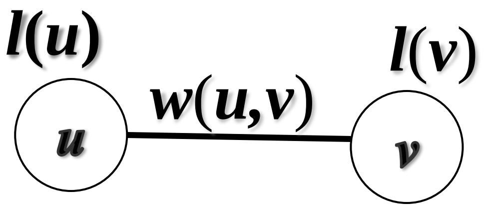

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- Running time of graph algorithms are usually expressed in terms of $\boldsymbol{n}$ or $\boldsymbol{m}$.

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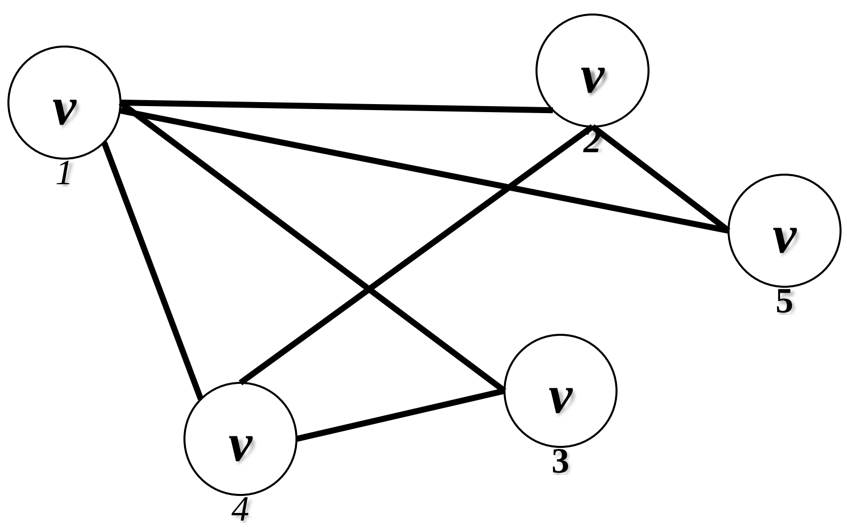

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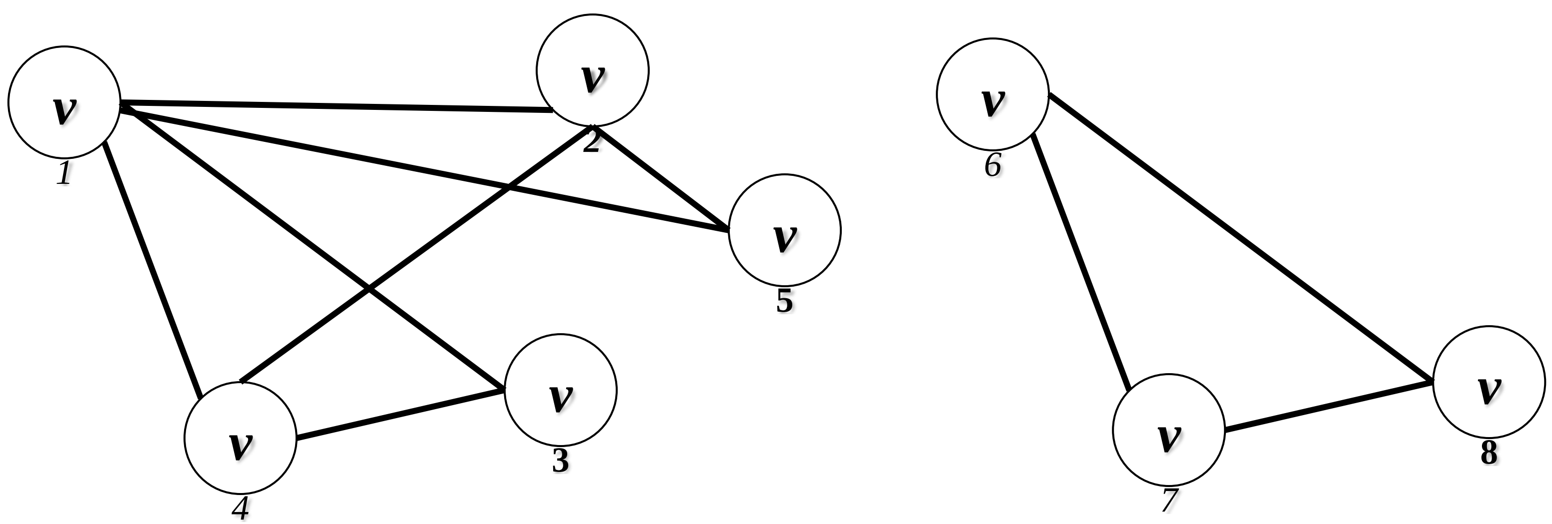

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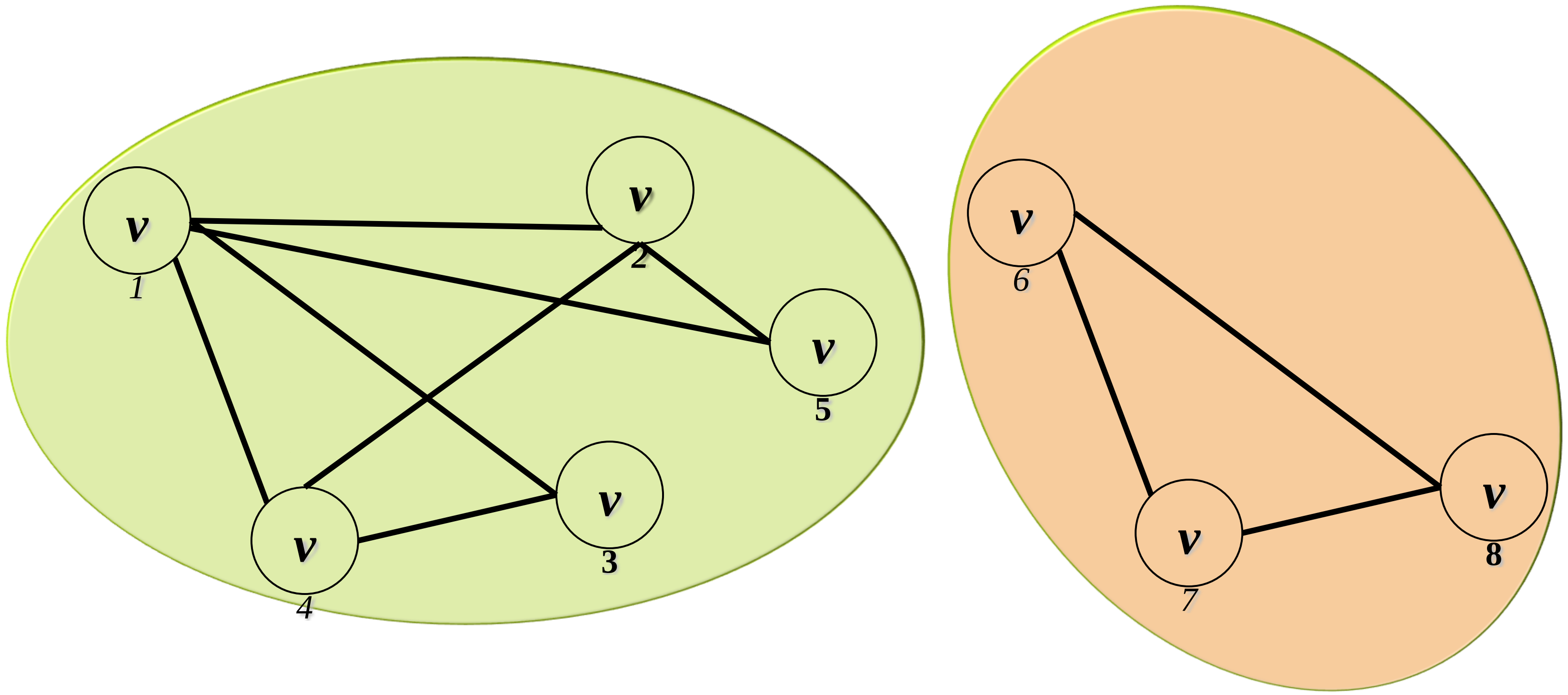

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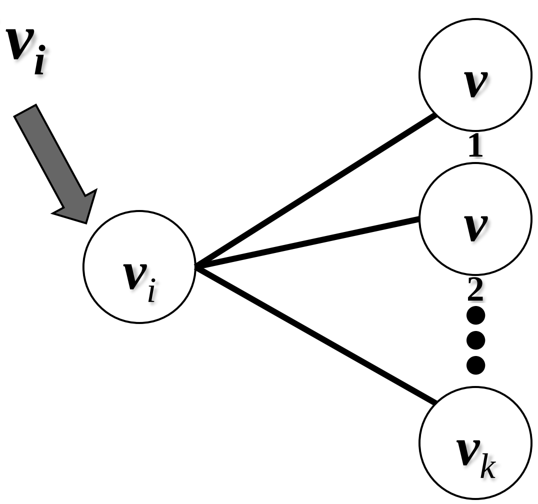
k is the degree of $\boldsymbol{v_i}$



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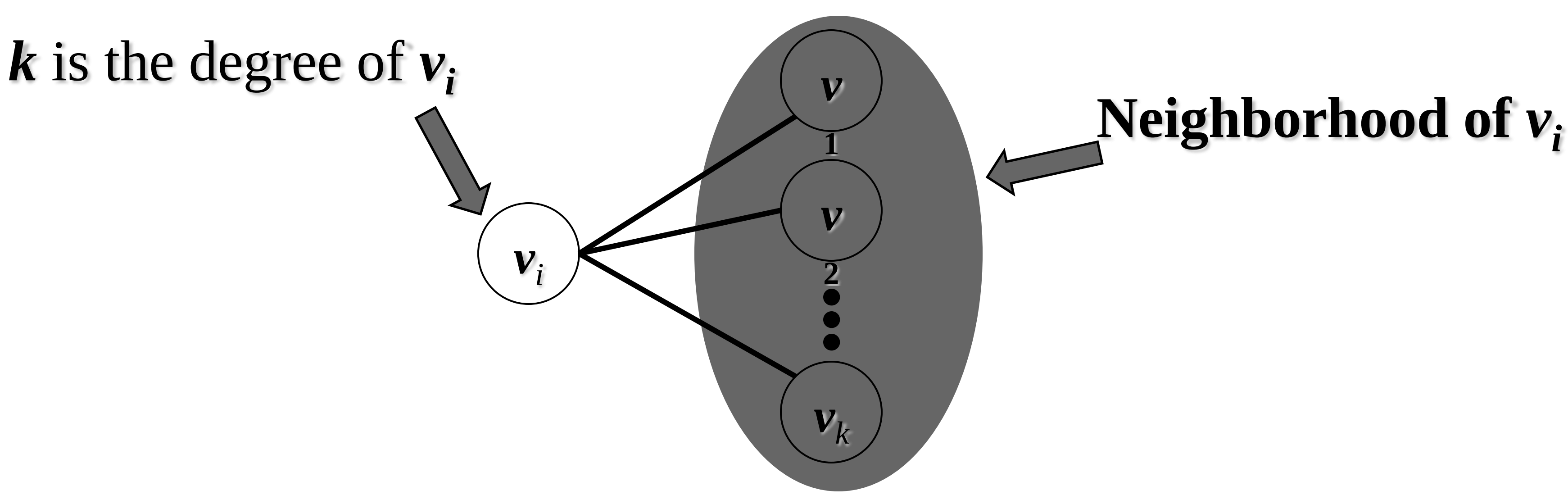
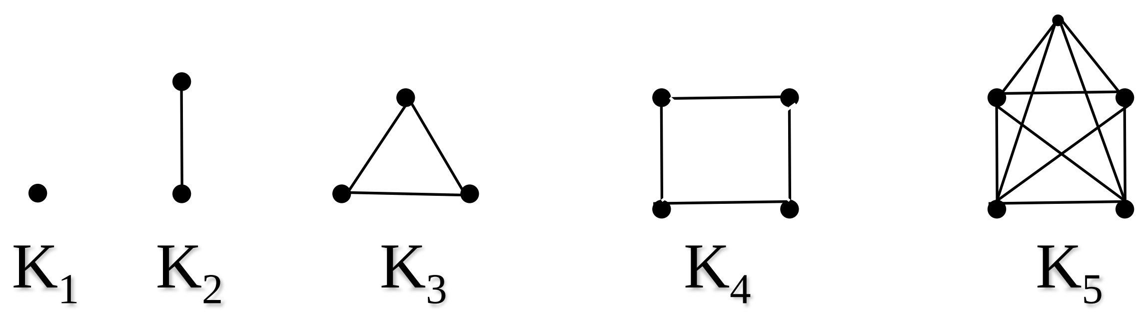
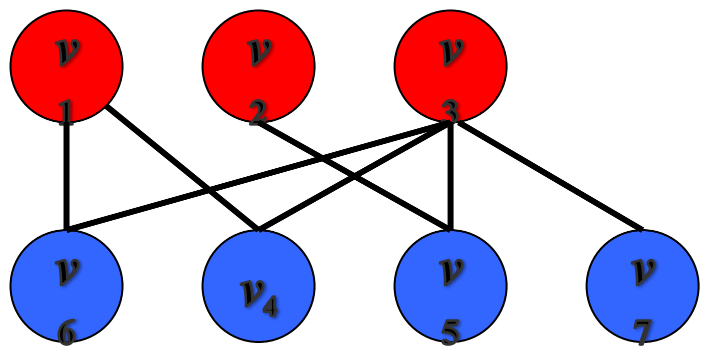


Important Graphs:

- A ***complete graph*** is a simple graph whose vertices are pairwise adjacent. The complete graph with ***n*** vertices is denoted $\mathbf{K}_n$.



- A graph ***G*** is ***bipartite*** if ***V***(***G***) is the union of two disjoint (possibly empty) independent sets, called partite sets of ***G***.



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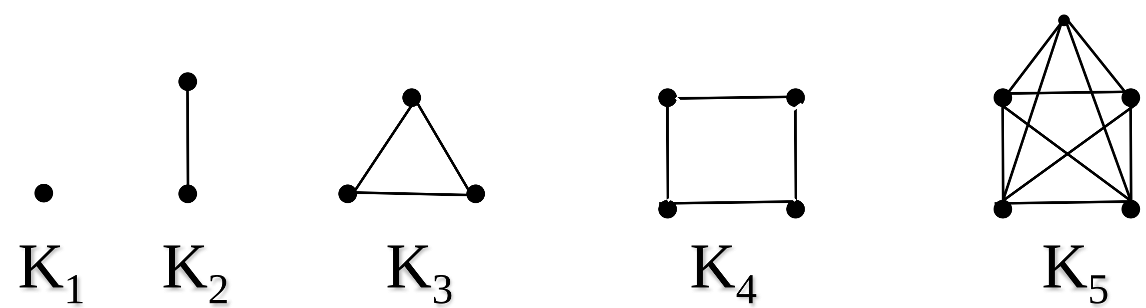

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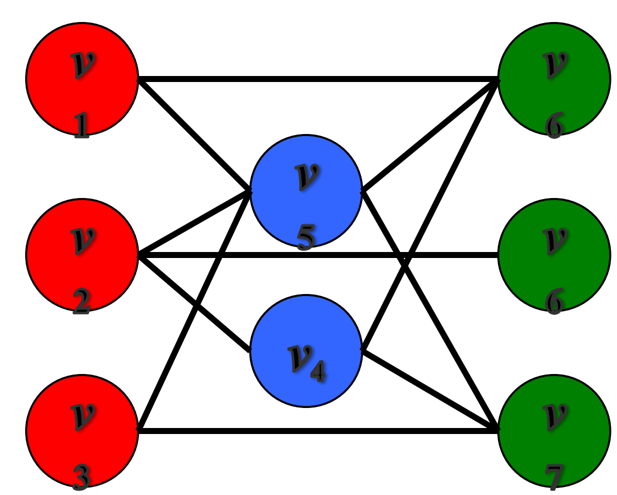

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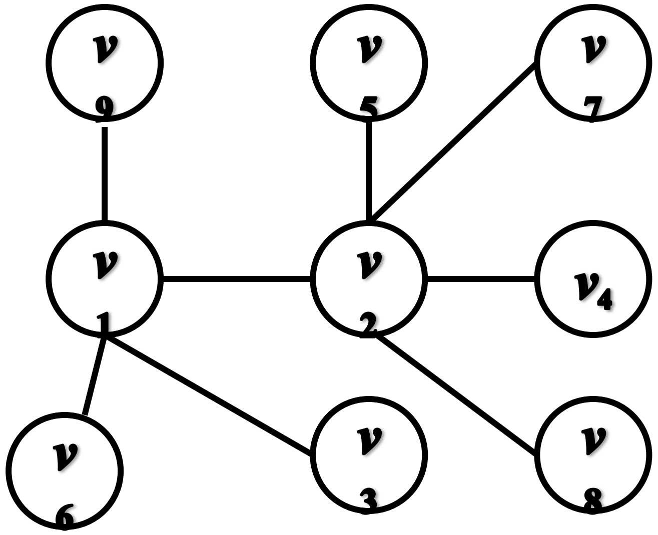
- A Tree is a connected acyclic graph.



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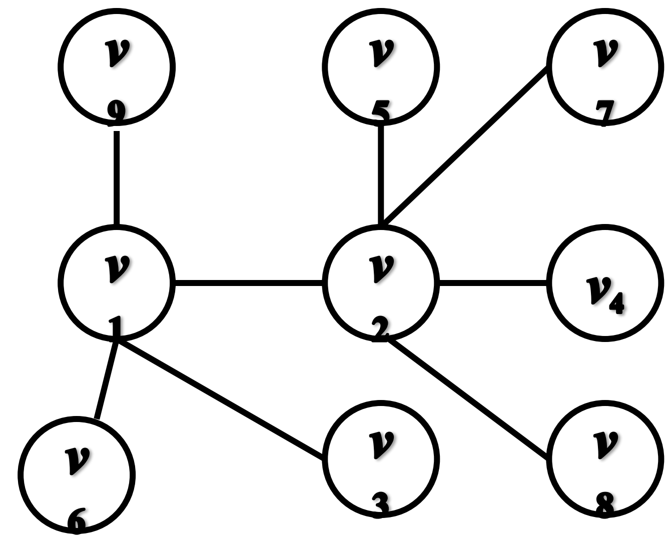
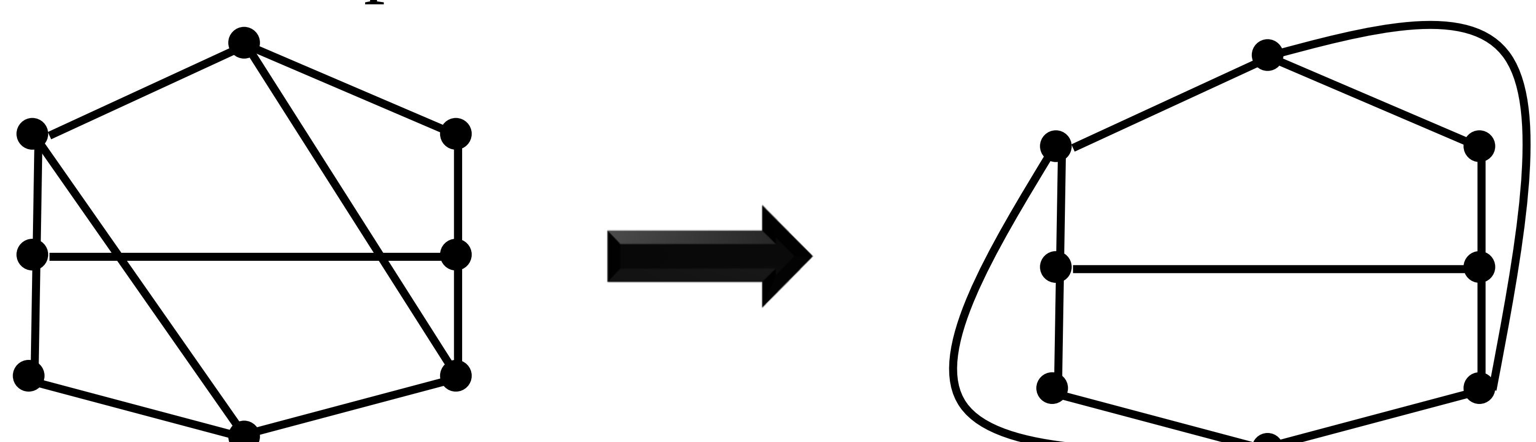


- A graph is ***planar*** if it can be drawn in the plane without crossings.
- A graph that is so drawn in the plane is also said to be embedded in the plane.

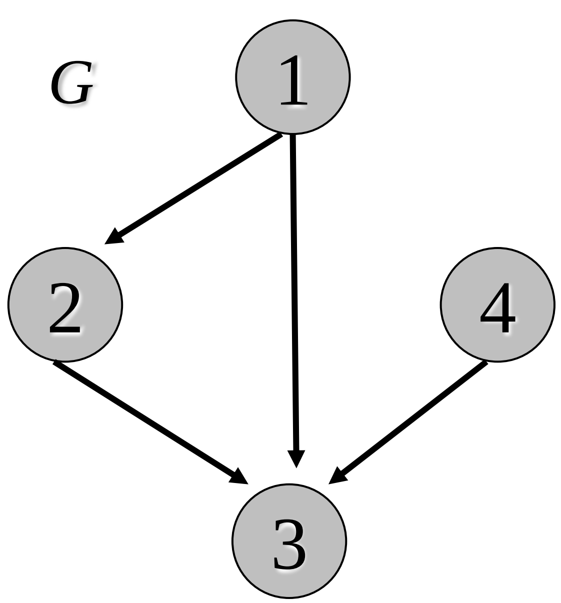


Graph Representation:

- The adjacency matrix of a graph G, denoted by A_G is an $n \times n$ defined as follows:

$$A_G[i,j] = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases}$$

- If G is directed then A_G is asymmetric.



$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Notes:

- Number of 1's in A_G is m if G is directed; if its undirected, then number of 1's is $2m$.
- Degree of a vertex is the sum of entries in corresponding row of A_G
- If G is undirected then sum of all degree is $2m$.
- In a directed graph sum of the out degrees is equal to m.

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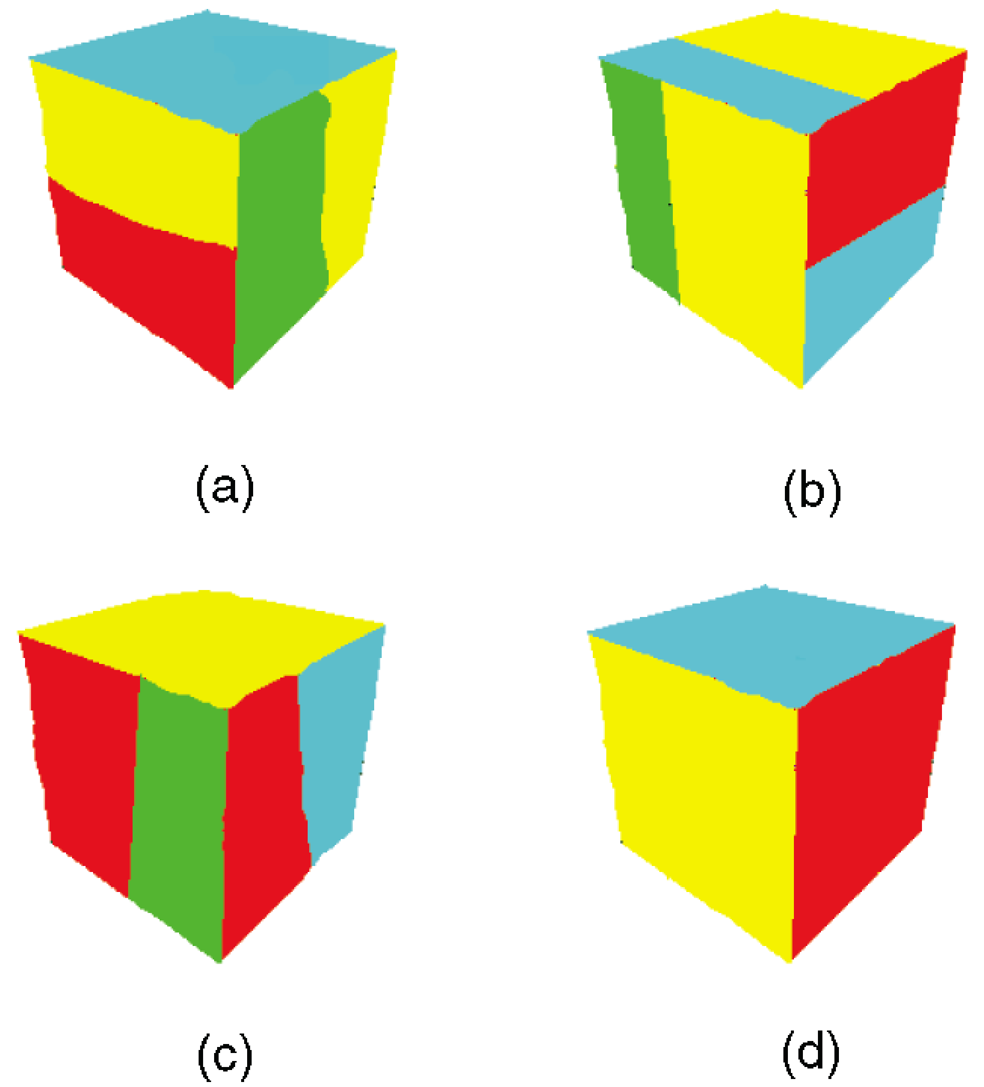

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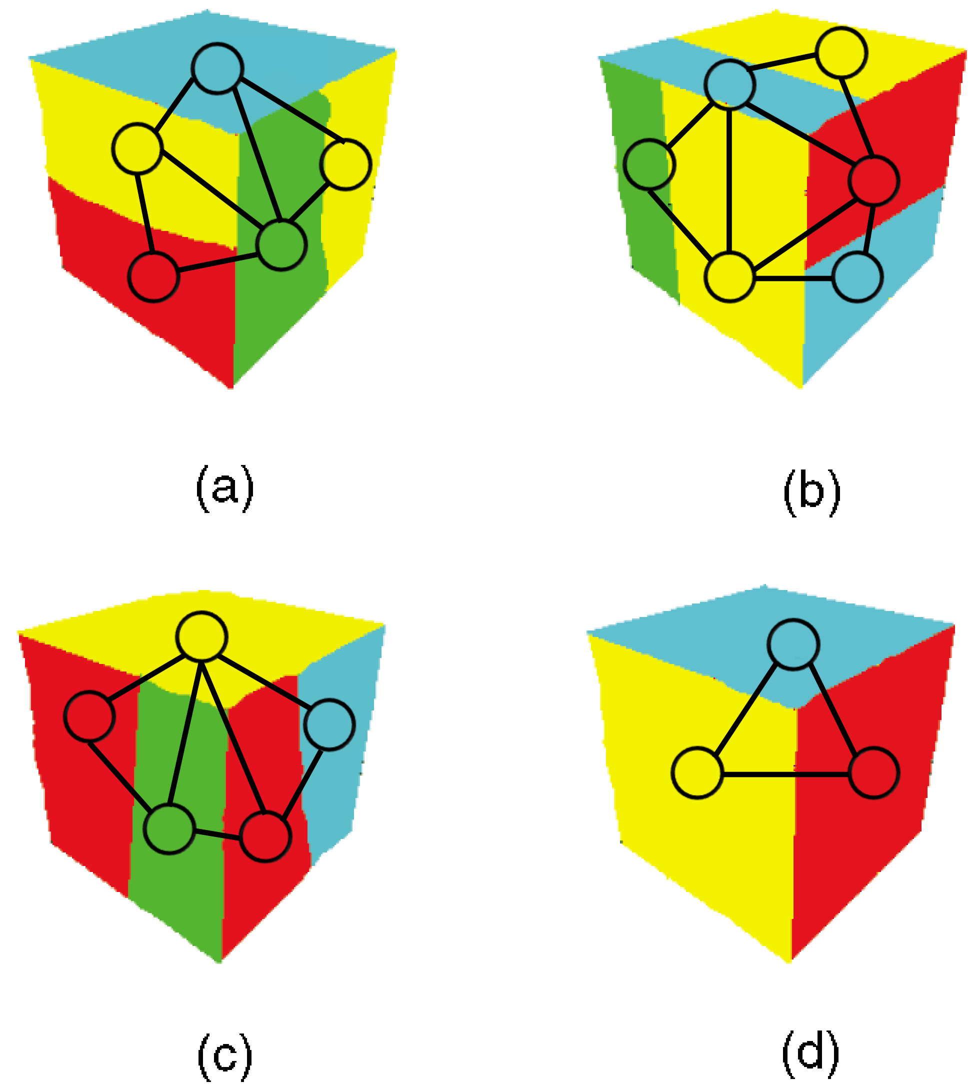
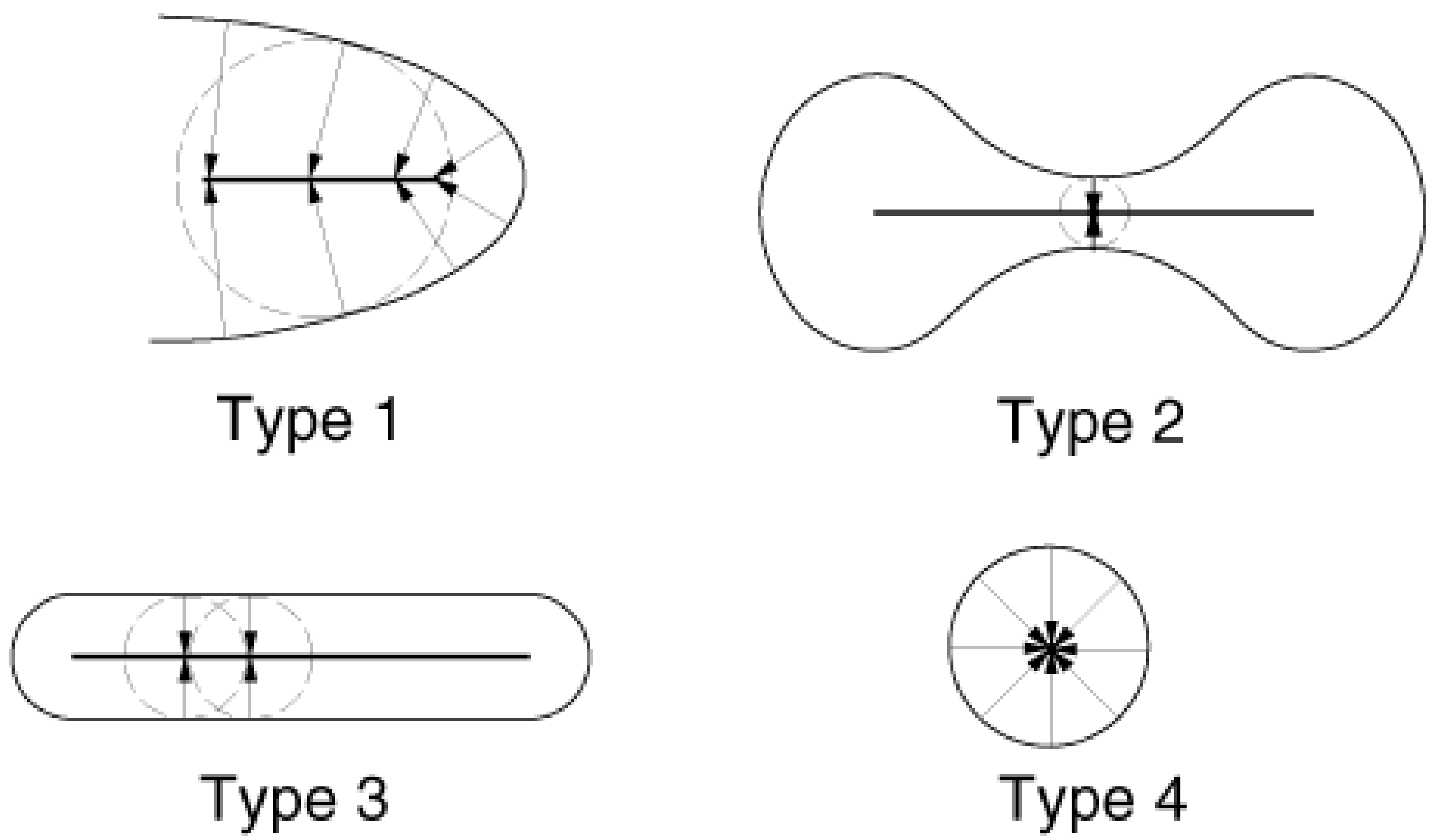
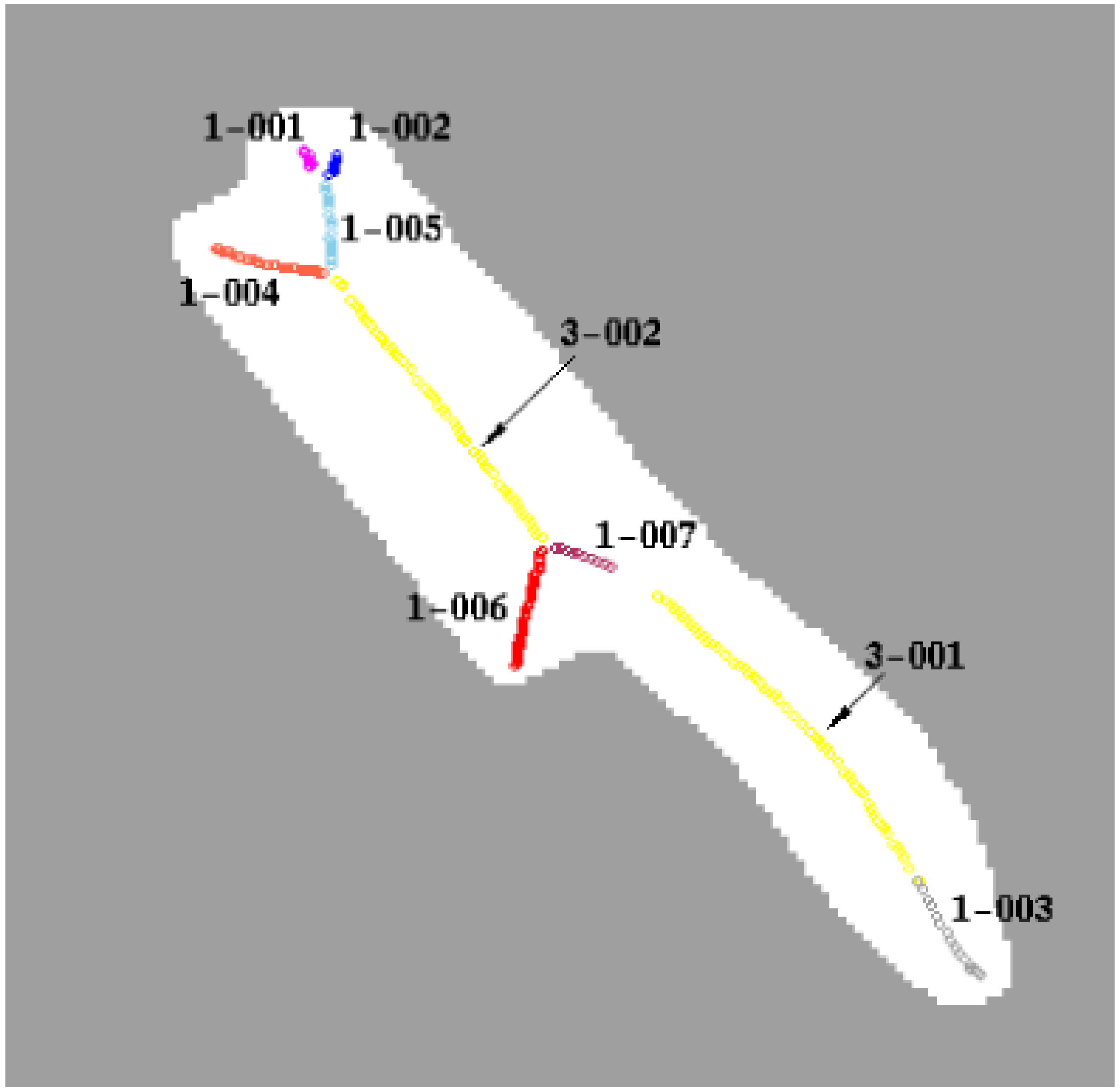


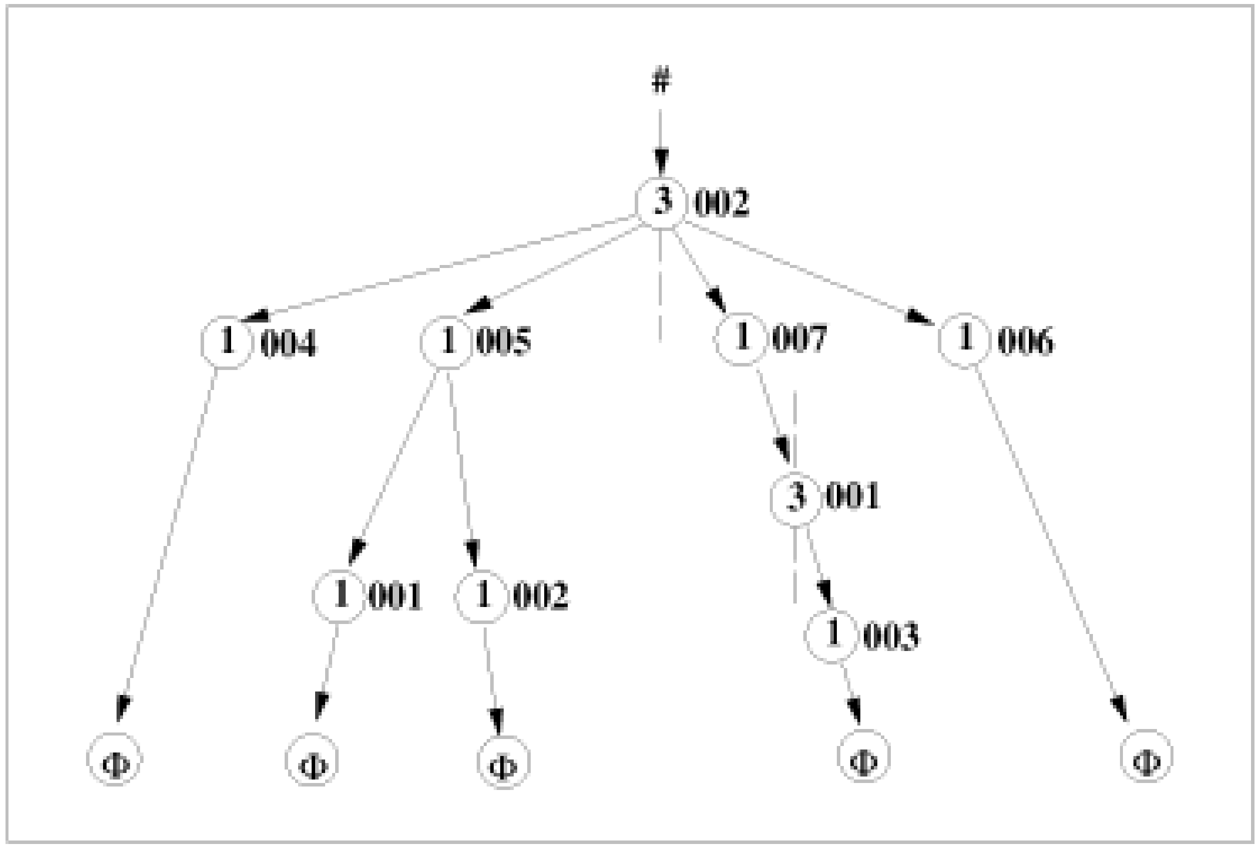
Shock:



Shock Graph (Siddiqi et. al. 1999)

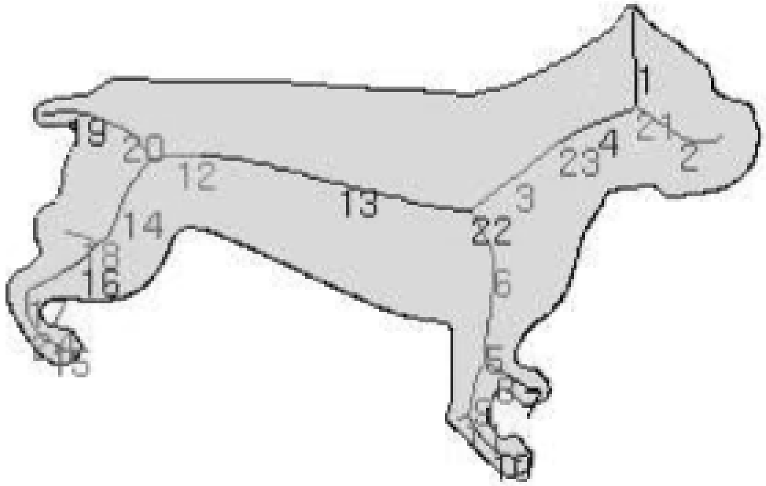
Shock Graphs

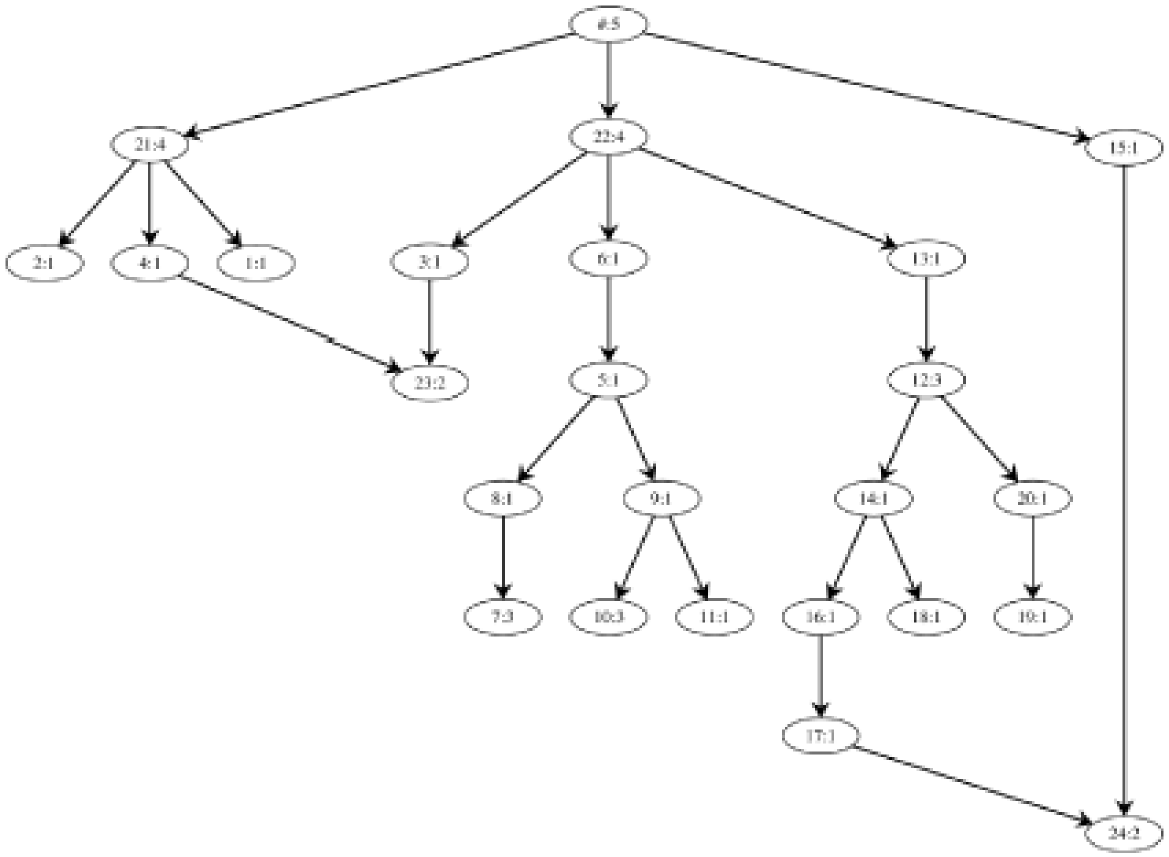




Shock Graphs

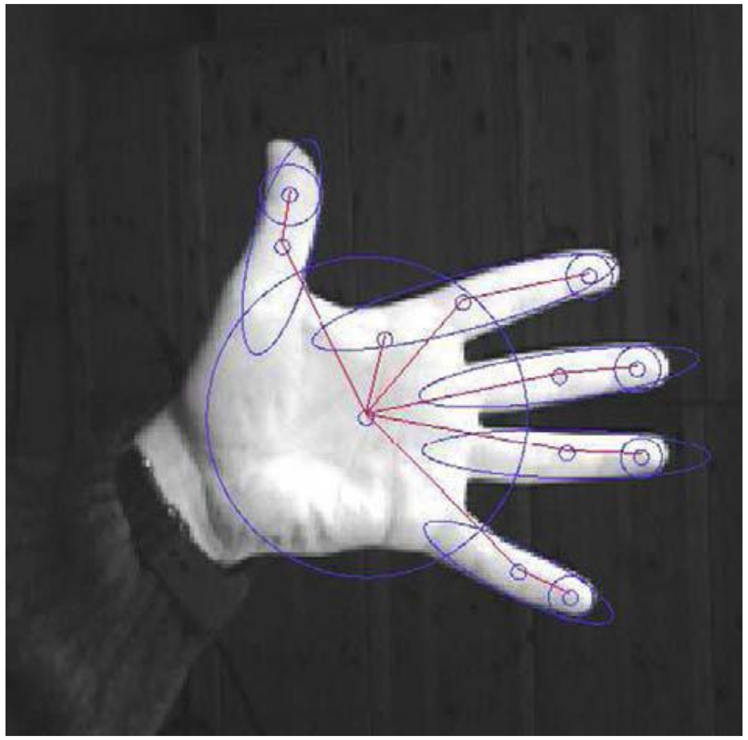
- Representing shape properties.

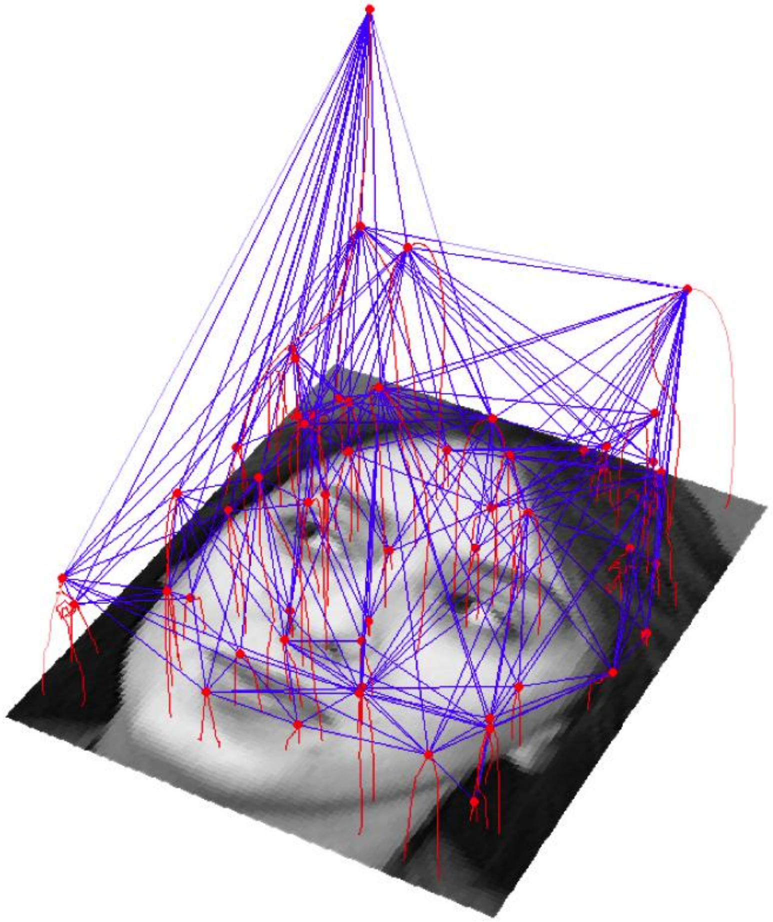




Variations in Reresentation:

- Representing shape properties.
- Representing appearance features.
- ...





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- Many reasons:
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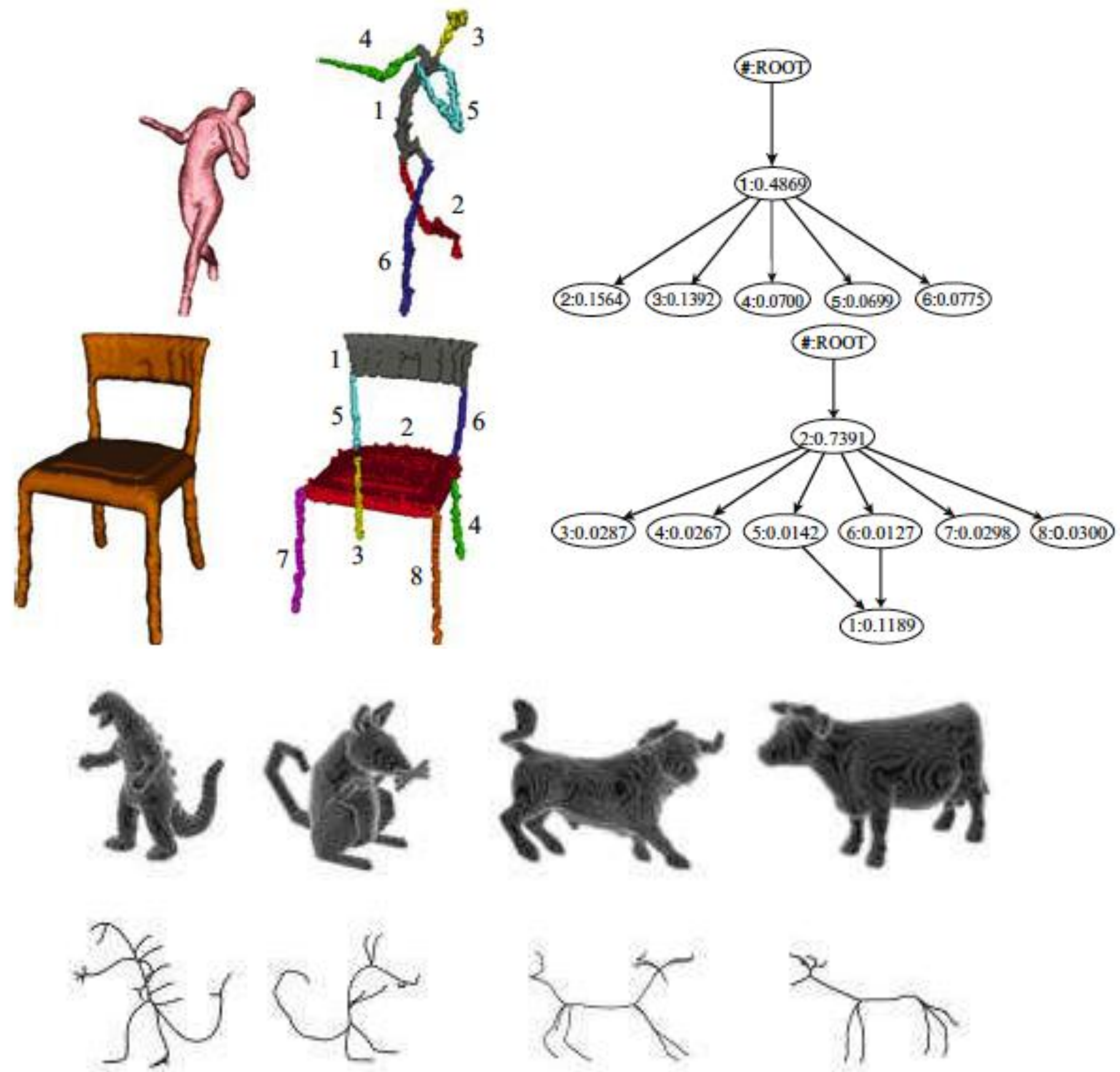

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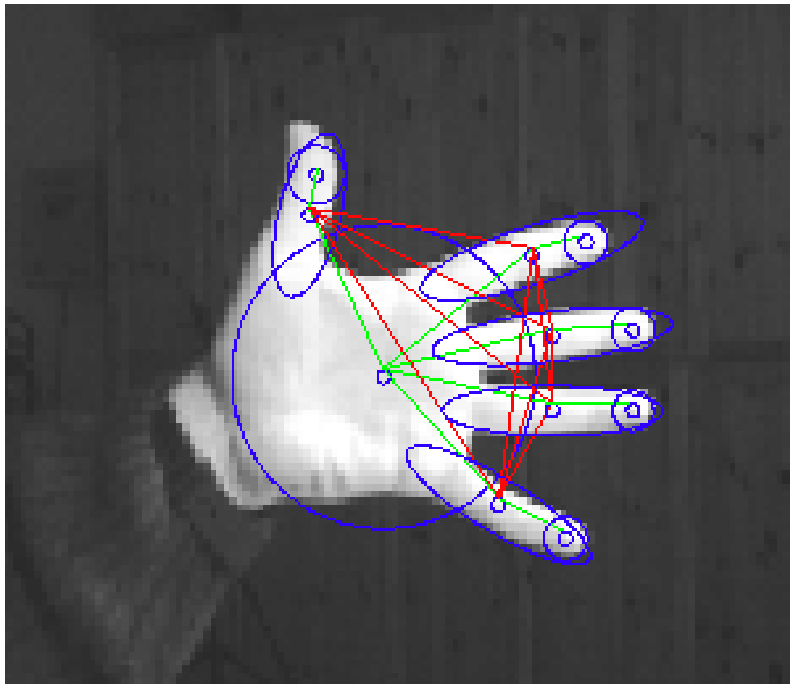

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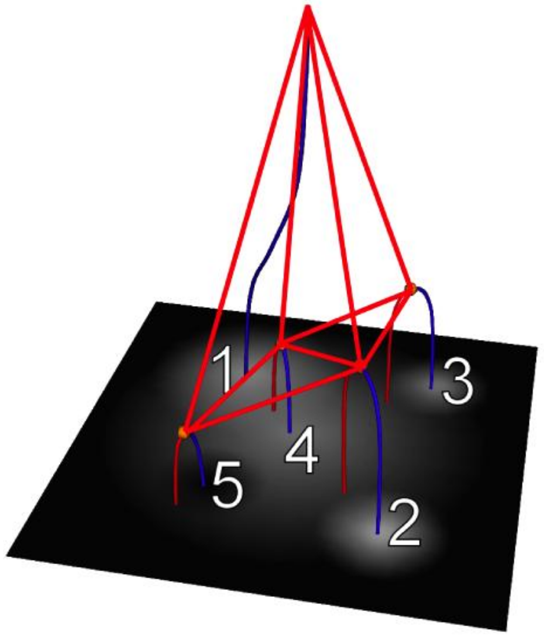

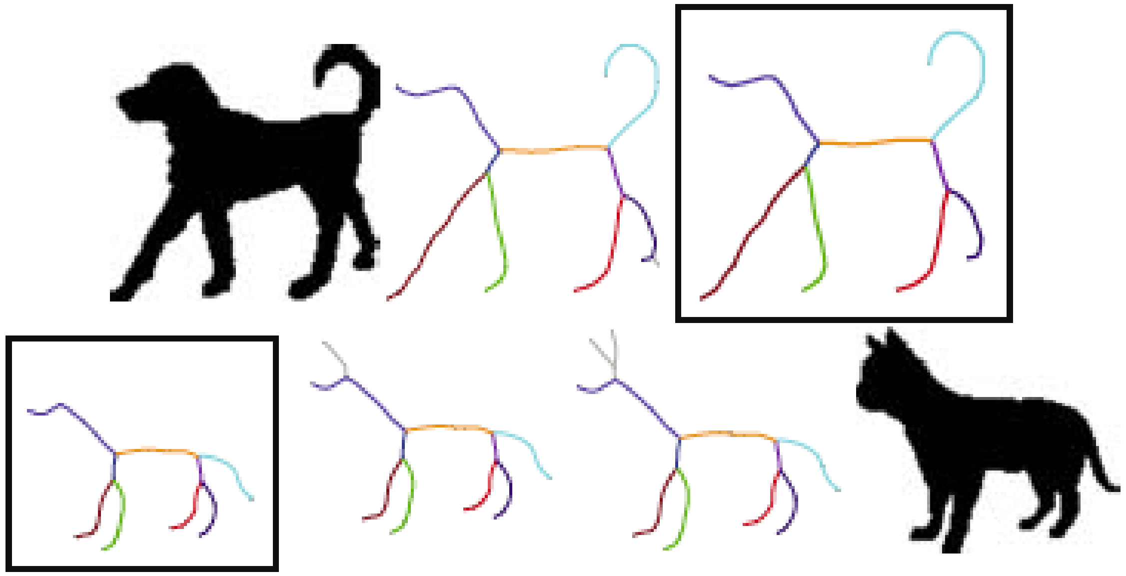

Sebastian et al., 2004

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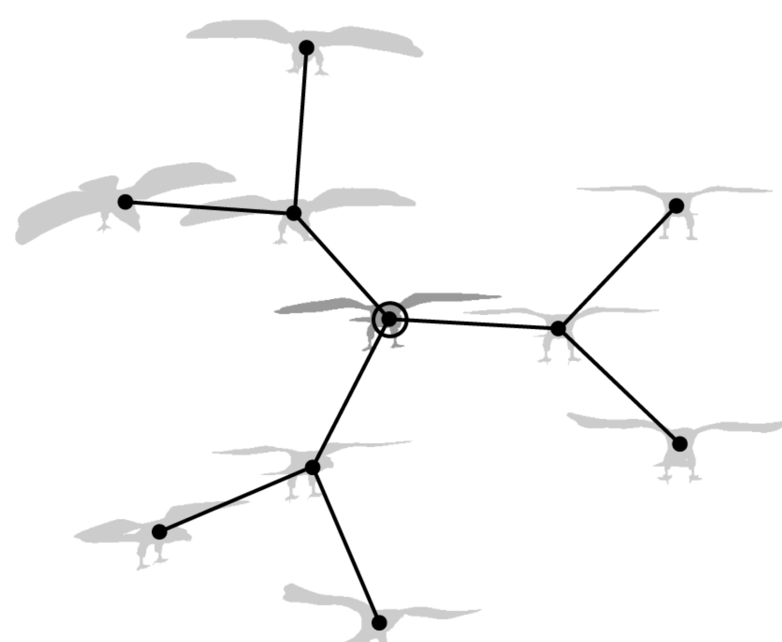
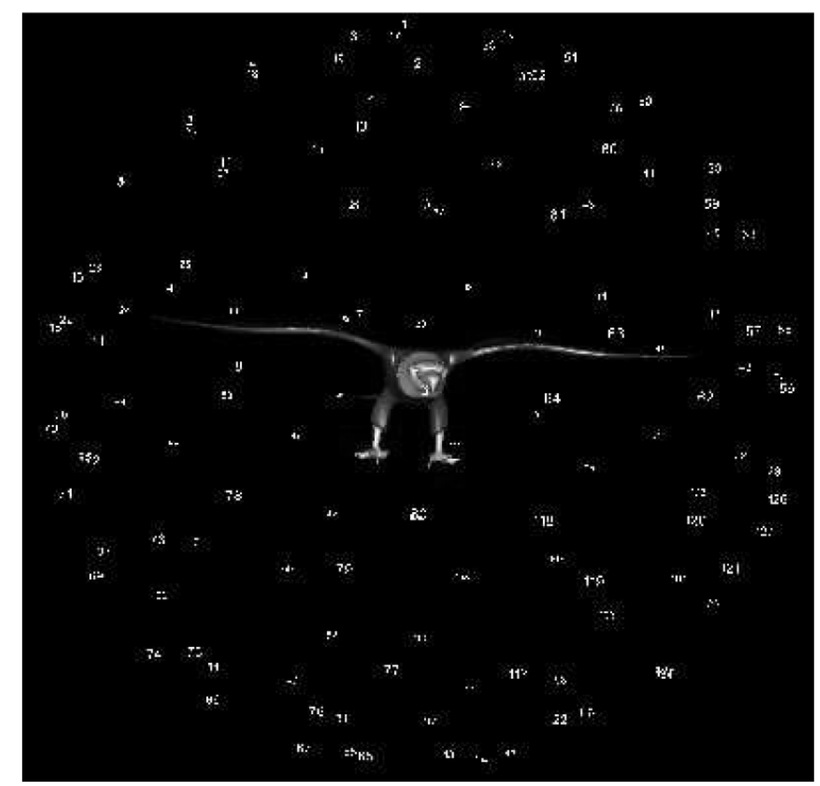
  - Capturing distributions.





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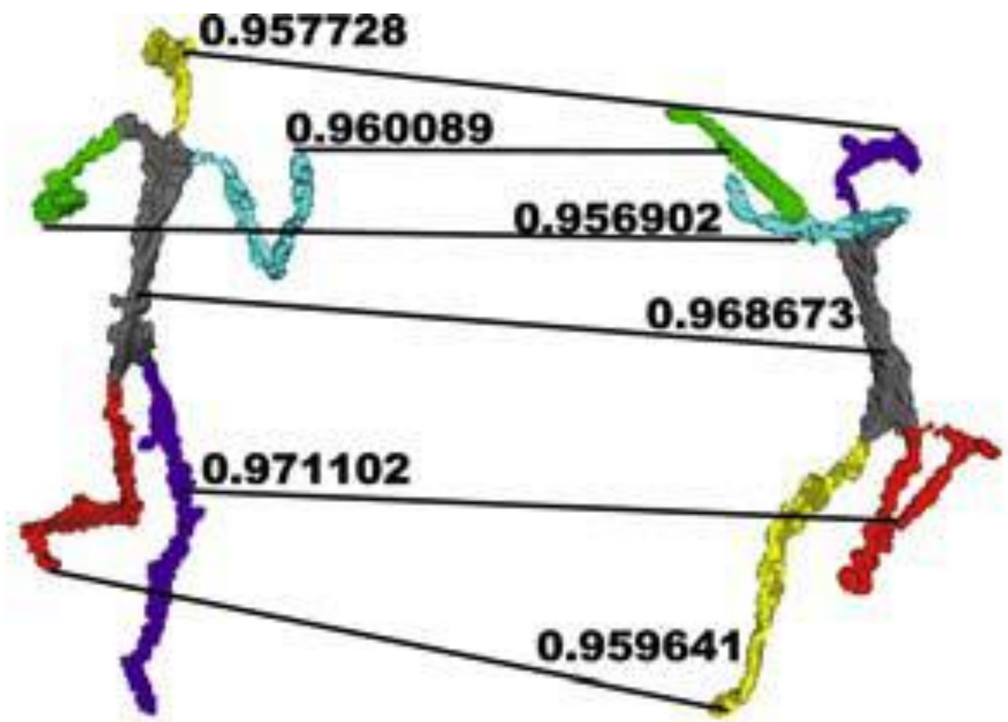
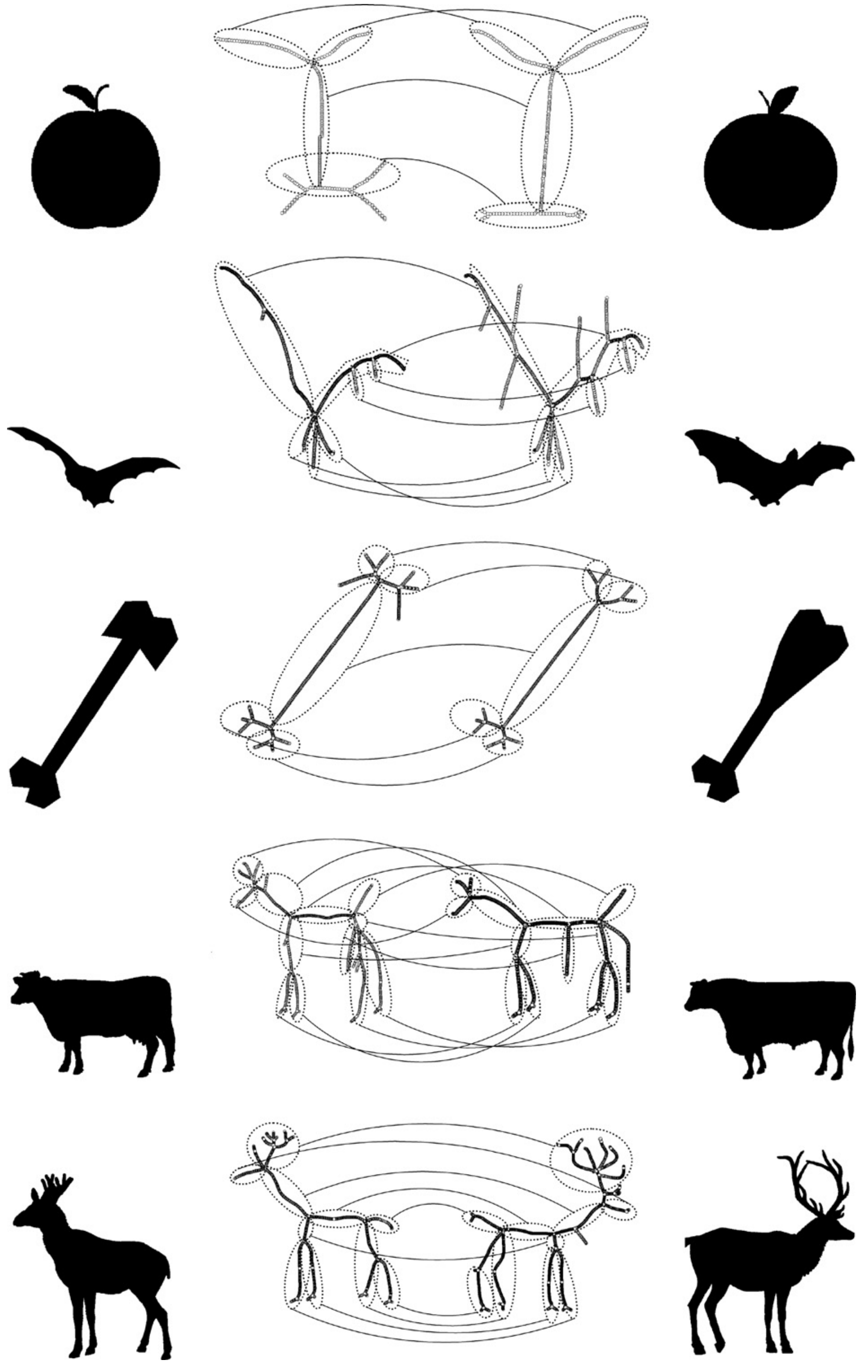
  - **Makes computational tasks easier!**

Sample Computational Tasks:

- Shape matching reduced to graph matching





Sample Computational Tasks:

- Shape matching reduced to graph matching.
- Object recognition (affinity matrix).

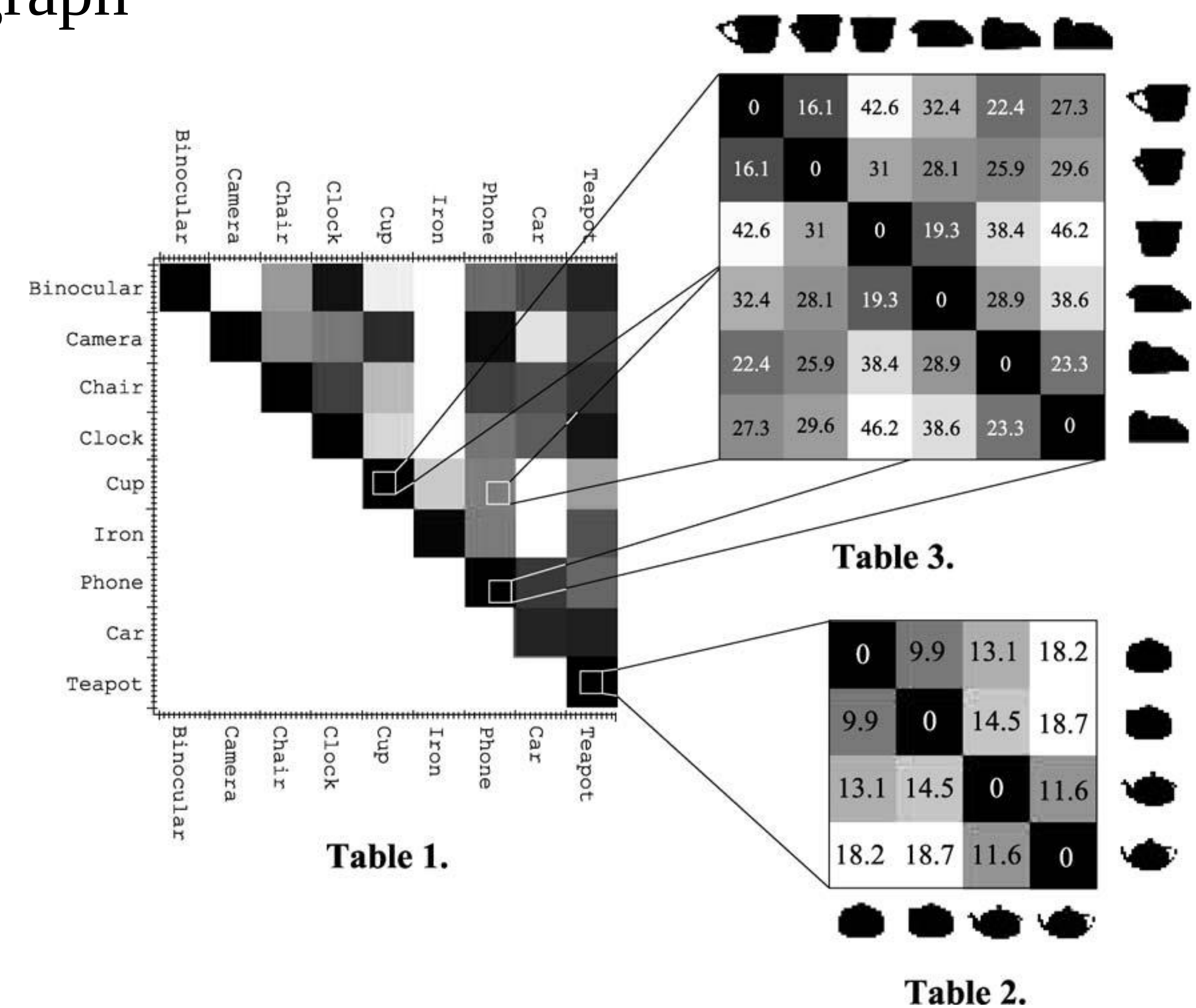


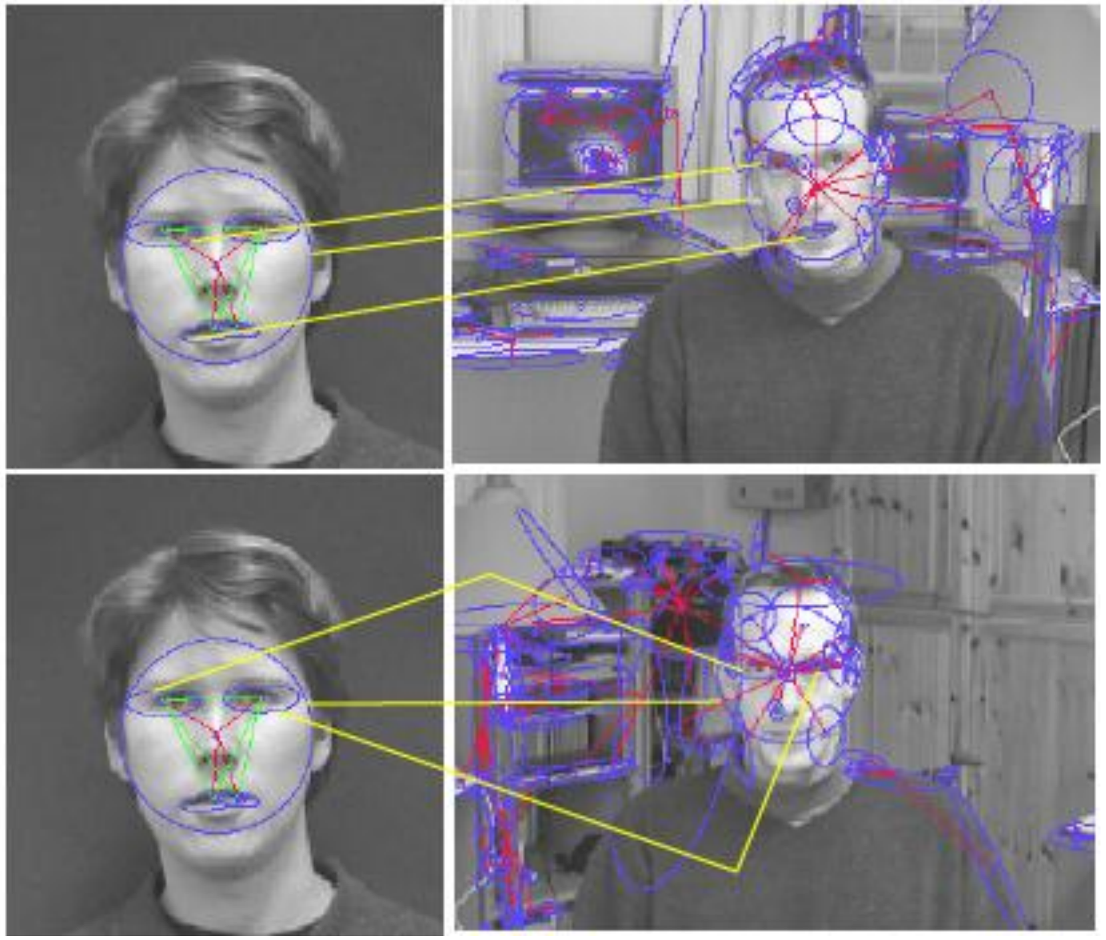
Table 1.

Table 3.

Table 2.

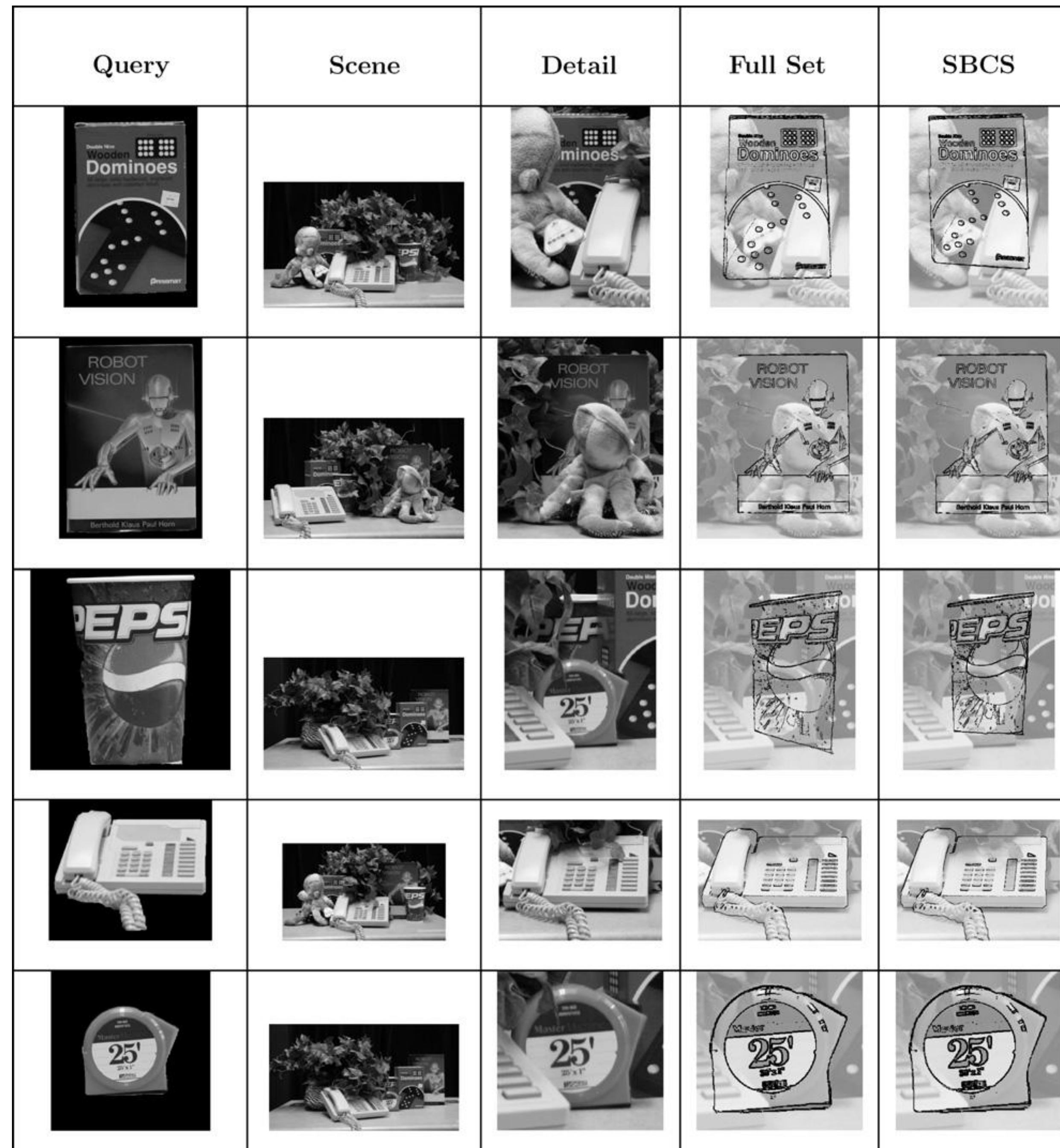
Sample Computational Tasks:

- Shape matching reduced to graph matching.
- Object recognition (cluttered scene).



Sample Computational Tasks:

- Shape matching reduced to graph matching.
- Object recognition
- Localization.



Sample Computational Tasks:

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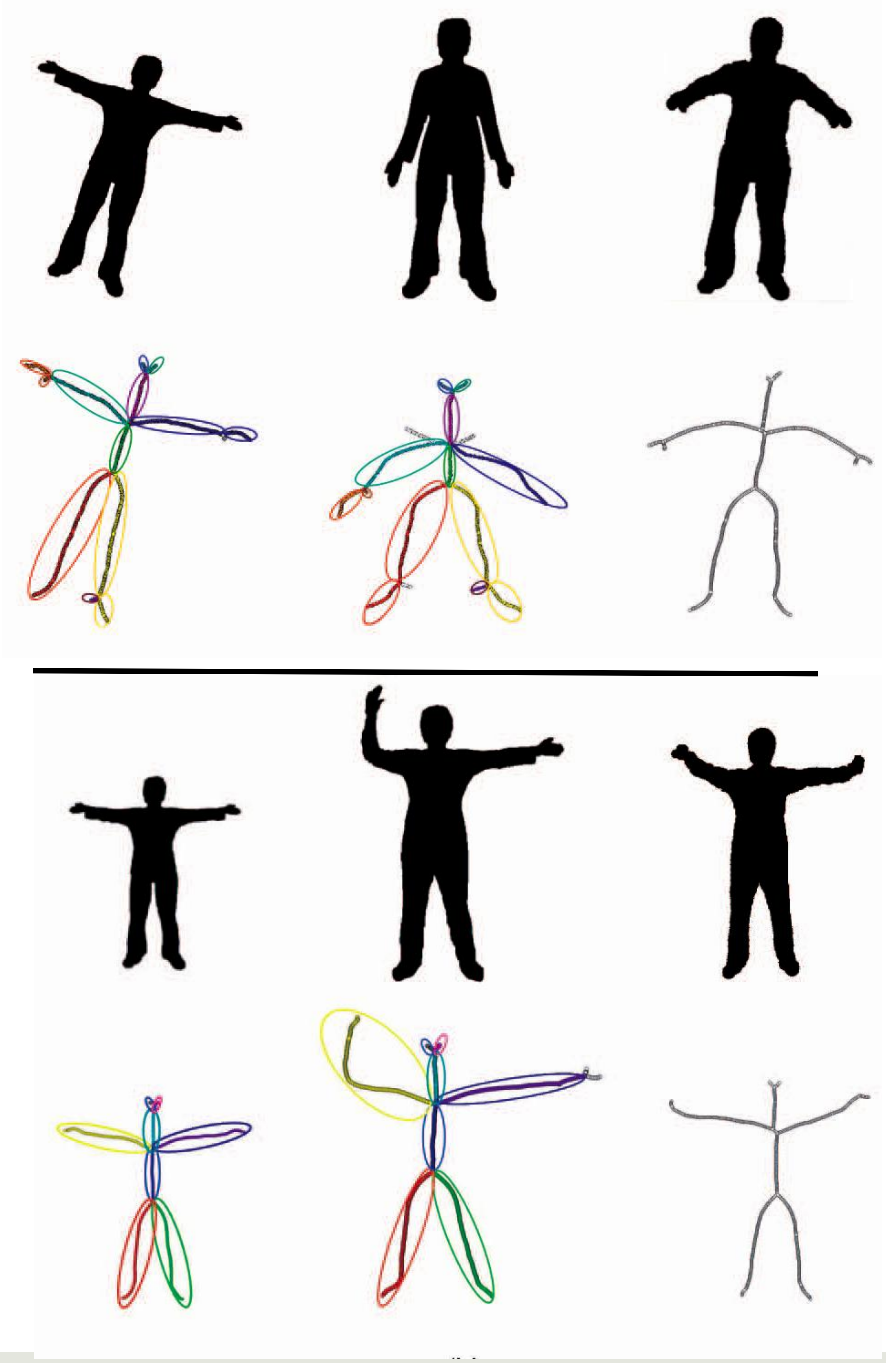

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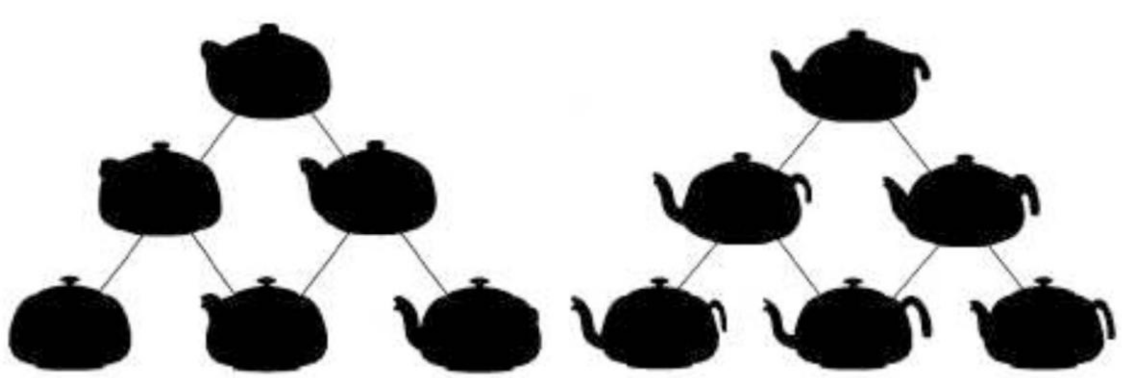
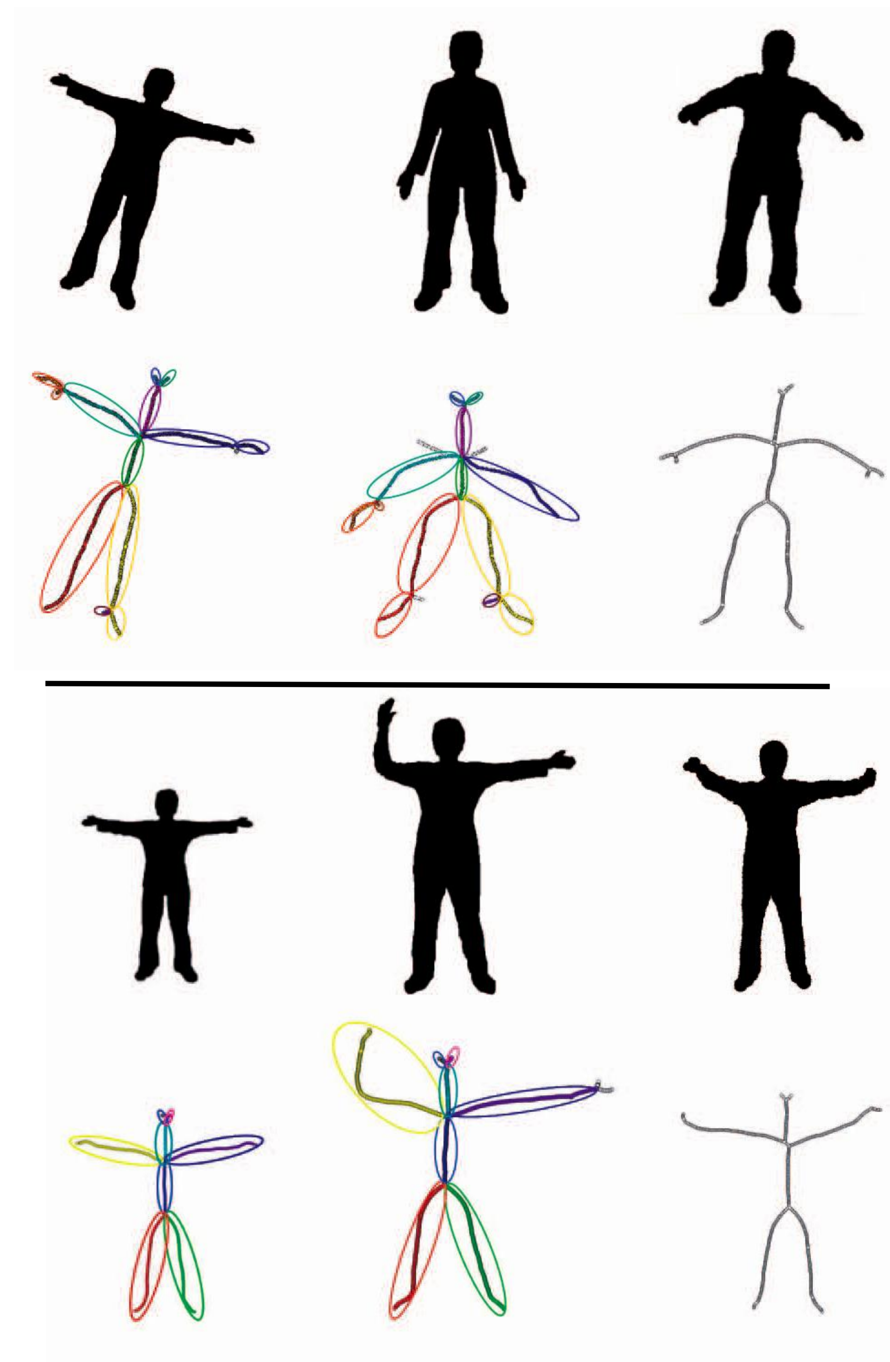




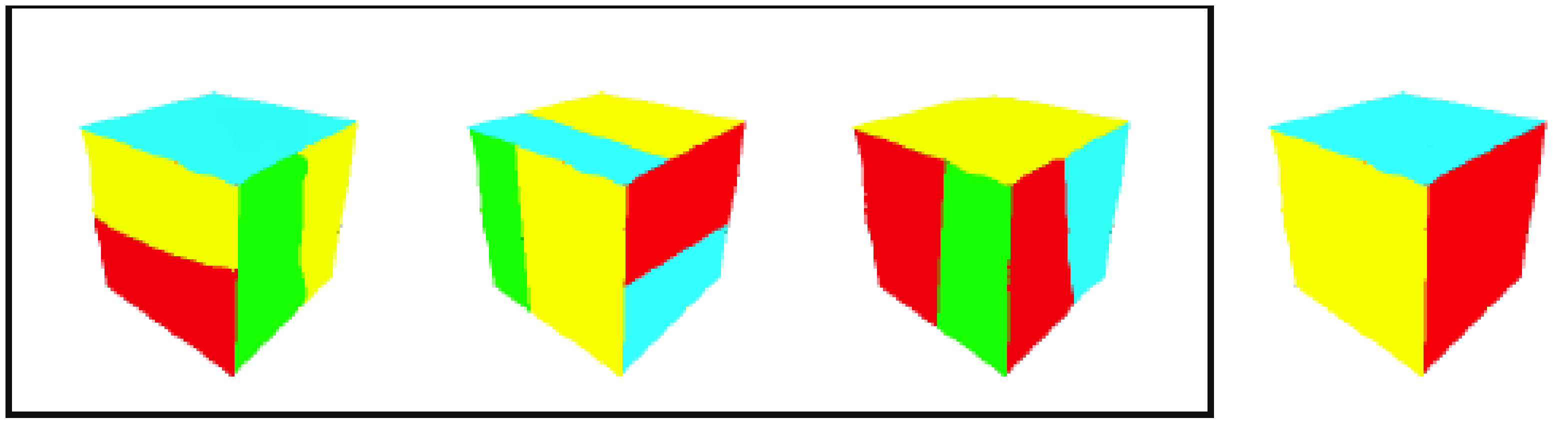
Sample Computational Tasks:

- Shape matching reduced to graph matching.
- Object recognition
- Localization.
- Shape abstraction
- Segmentation

Felzenszwalb and Huttenlocher 2004

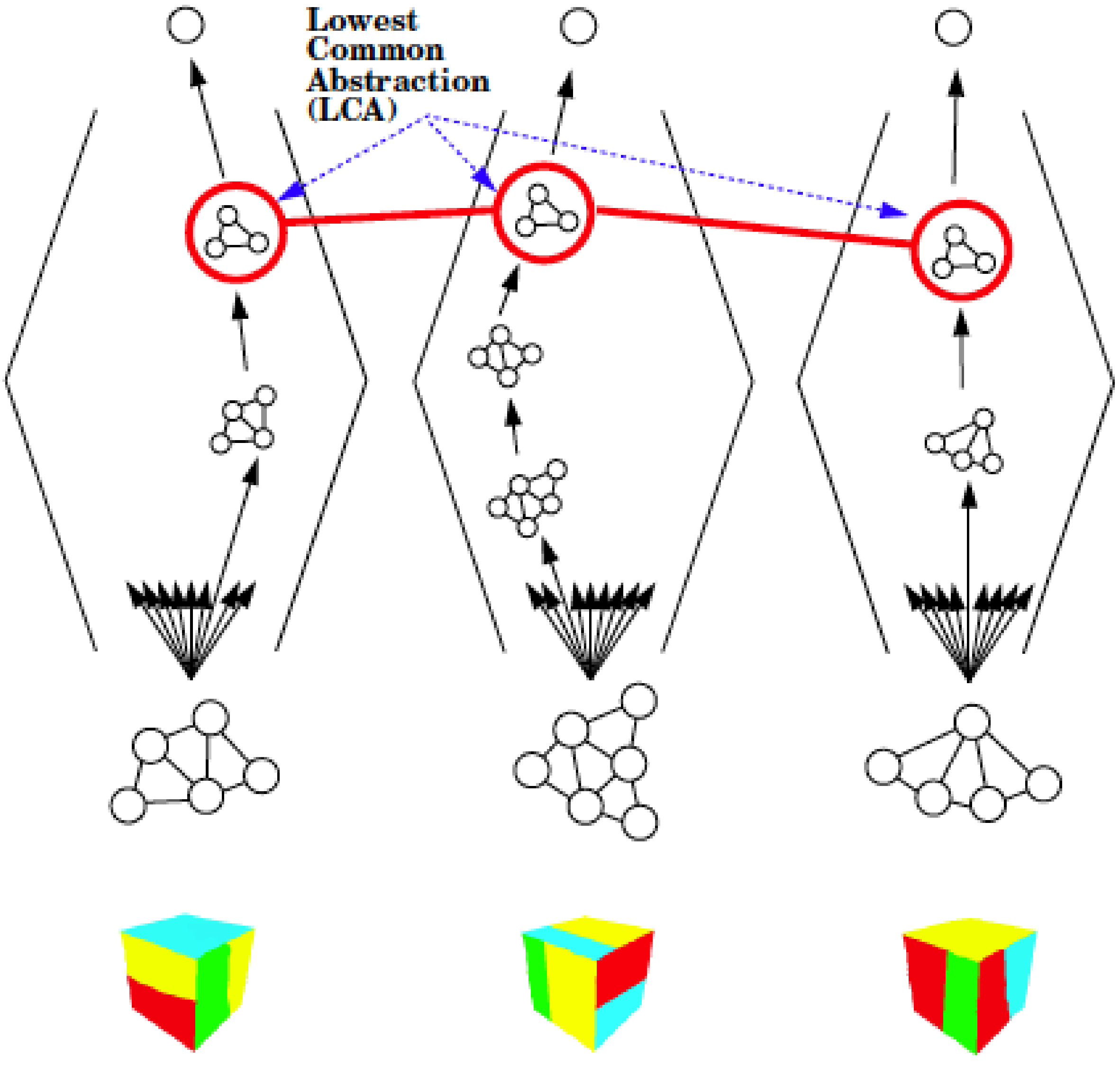
Why is representation hard?

- What is the right level of abstraction?



Hardness of Representation

- What is the right level of abstraction?
- Generic model construction requires complex grouping algorithms.



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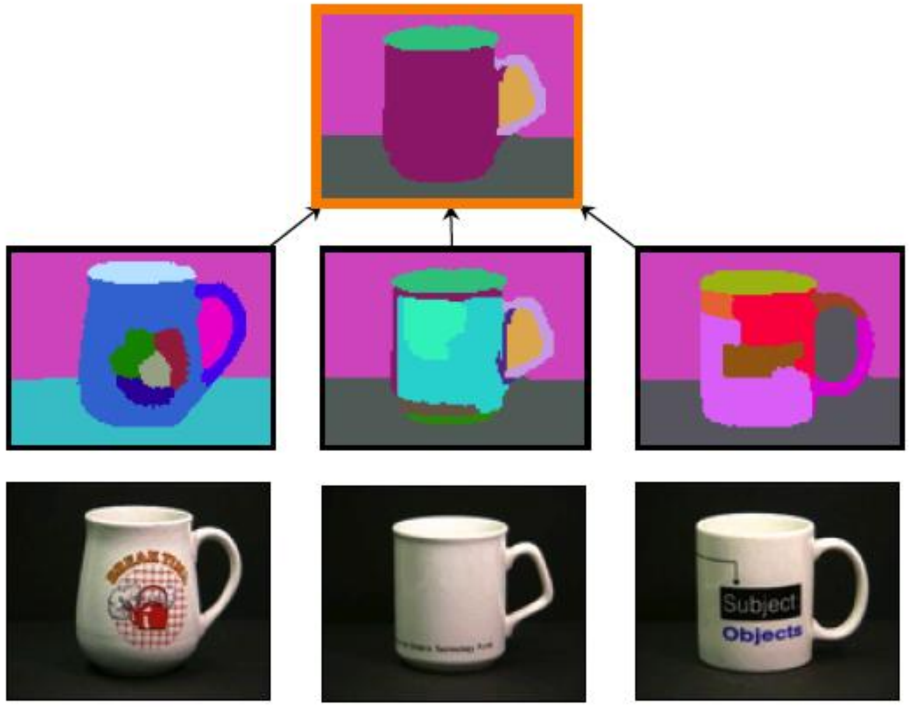

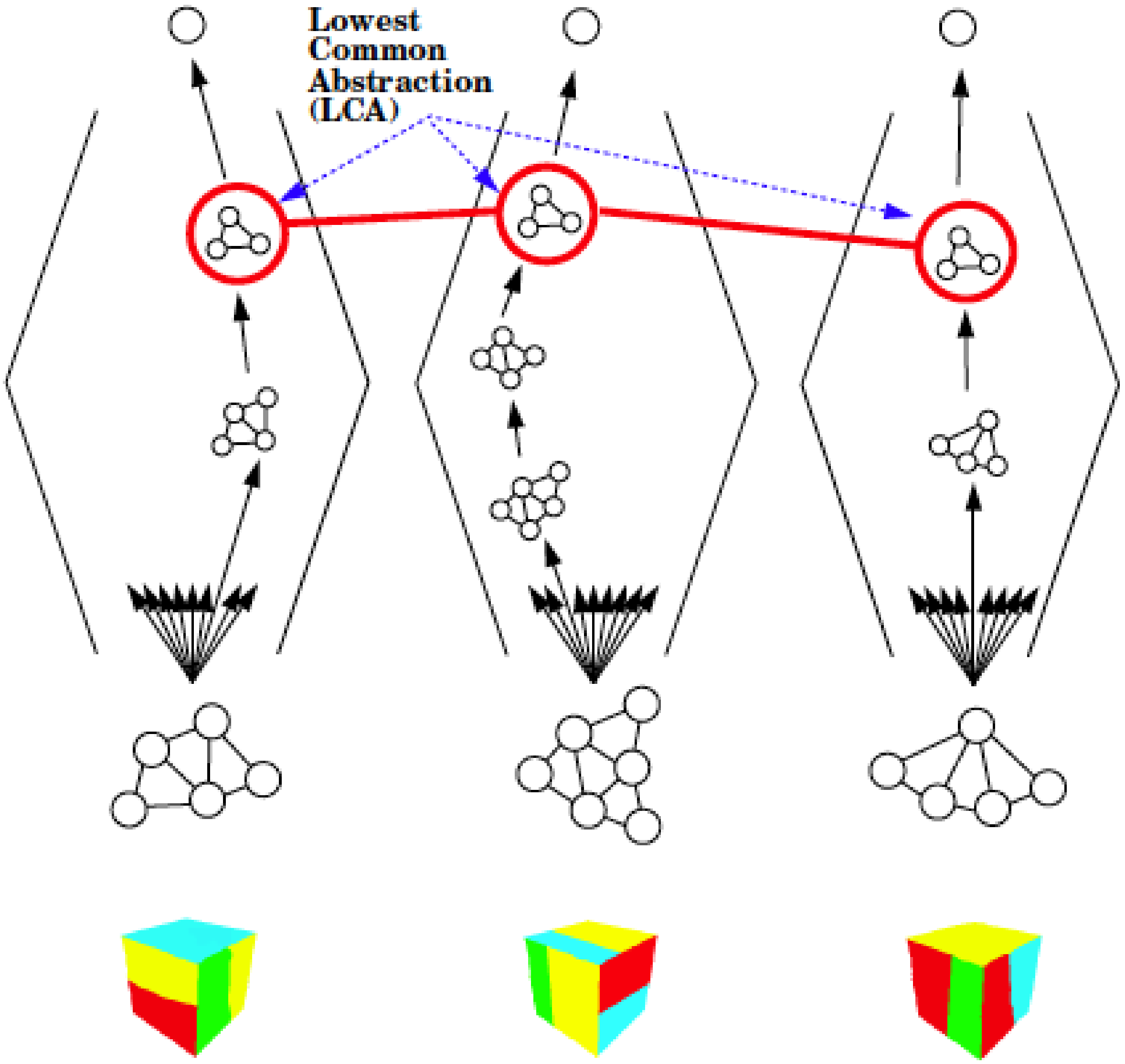

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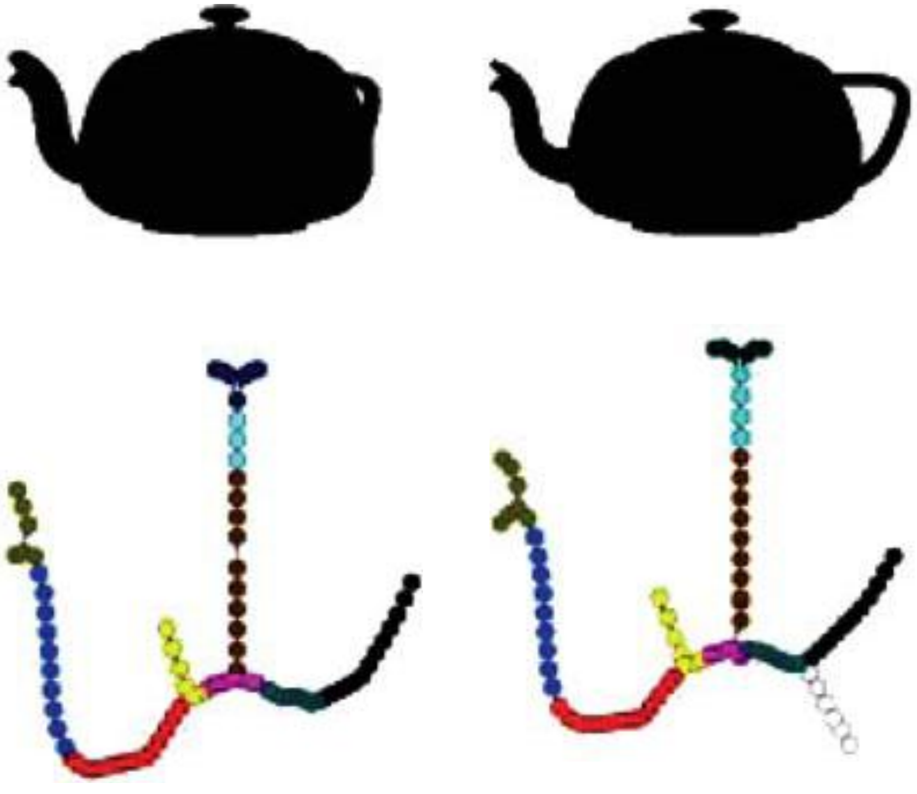

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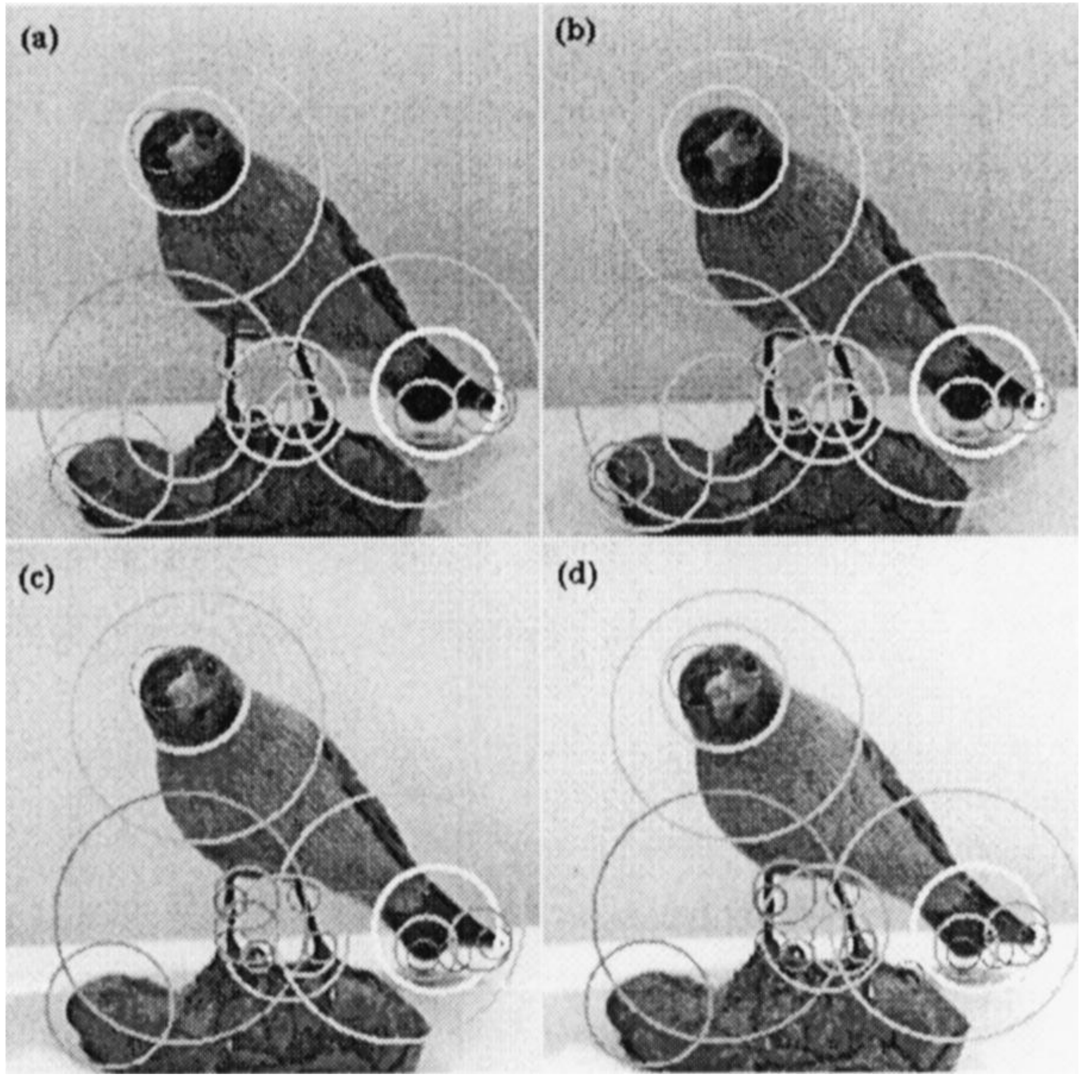

Overview of this talk

Introduction:

Notations and Definitions

Graphs and Modeling

Algorithmic Graph Theory and Combinatorial Optimization

Problems:

Decision (yes or no) **problems**:

- Does graph ***G*** contain an induced copy of graph ***H***?

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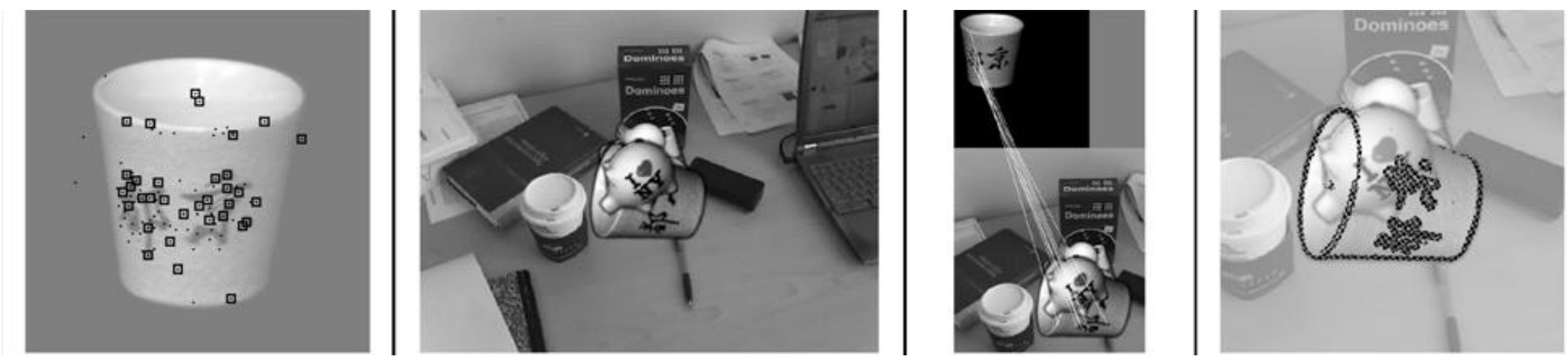
Decision (yes or no) **problems**:

- Does graph G contain an induced copy of graph H?
- **Localization problem**



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Optimization problems:

- Find the optimal induced substructure in a graph:
  - Maximum cardinality minimum weight matching, minimum spanning tree, maximum clique, maximum hitting set, etc.

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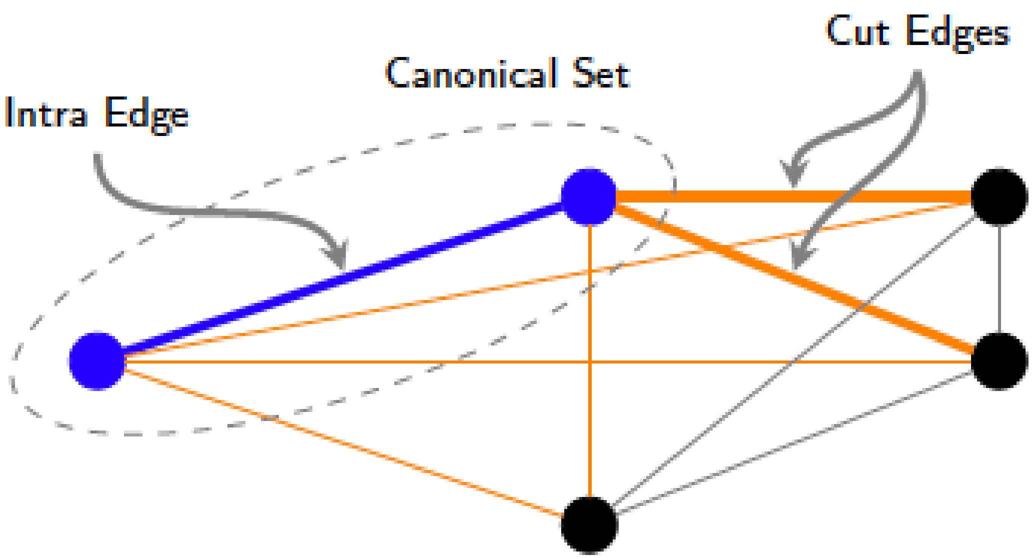

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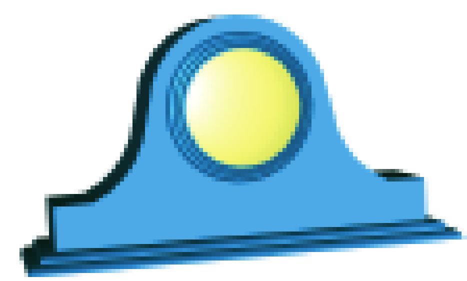
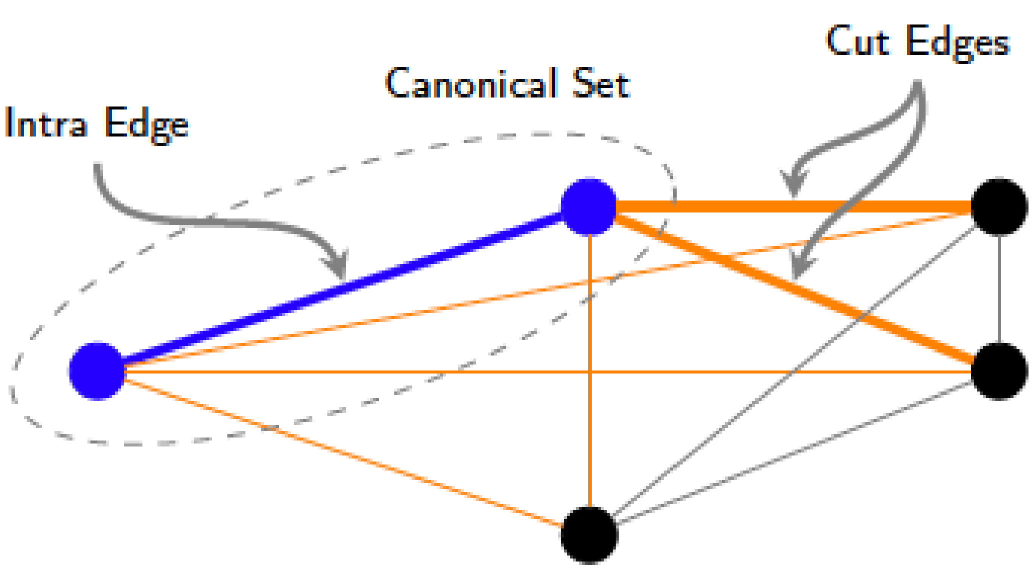
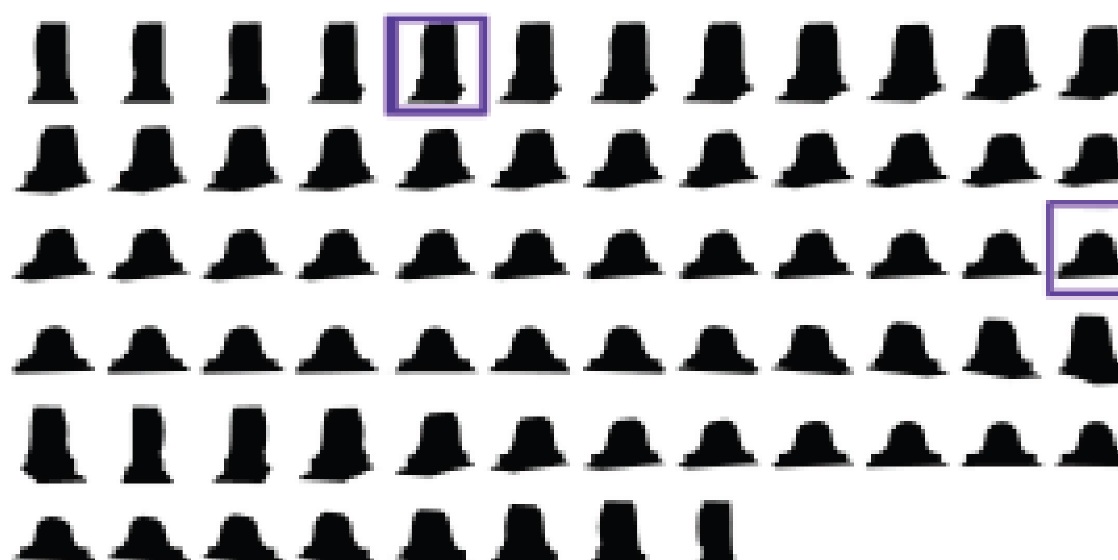
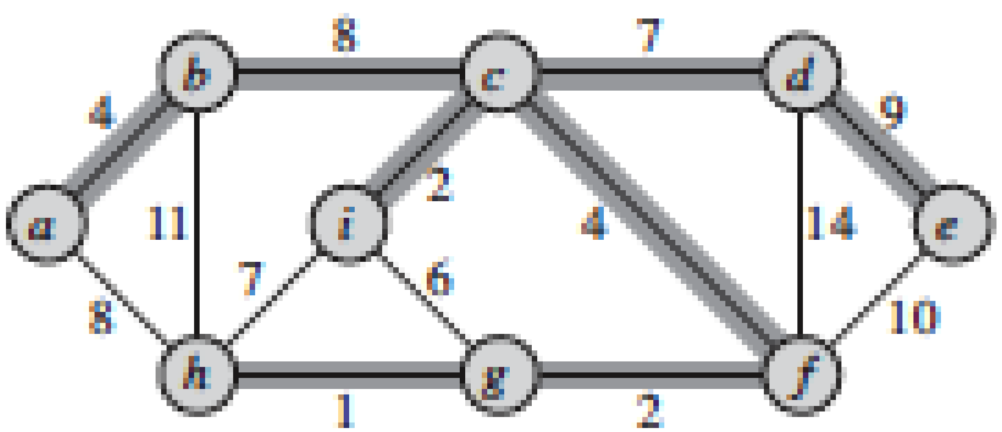






Algorithmic Graph Theory:

- **Objective:** Designing efficient combinatorial methods for solving decision or optimization problems.
  - Runs in polynomial number of steps in terms of size of the graph; $\boldsymbol{n}=|\boldsymbol{V}(\mathbf{G})|$ and $\boldsymbol{m}=|\boldsymbol{E}(\boldsymbol{G})|$.
  - **Example**: Minimum Spanning Tree (MST): $\boldsymbol{T}(\boldsymbol{m},\boldsymbol{n})=\boldsymbol{O}(\boldsymbol{m}+\boldsymbol{n}\ \mathbf{log}\ \boldsymbol{n})$.



Intro. to Algorithms. Corman et al.

```
GENERIC-MST(G, w)
A = ∅
while A does not form a spanning tree
    find an edge (u, v) that is safe for A
    A = A ∪ {(u, v)}
return A
```

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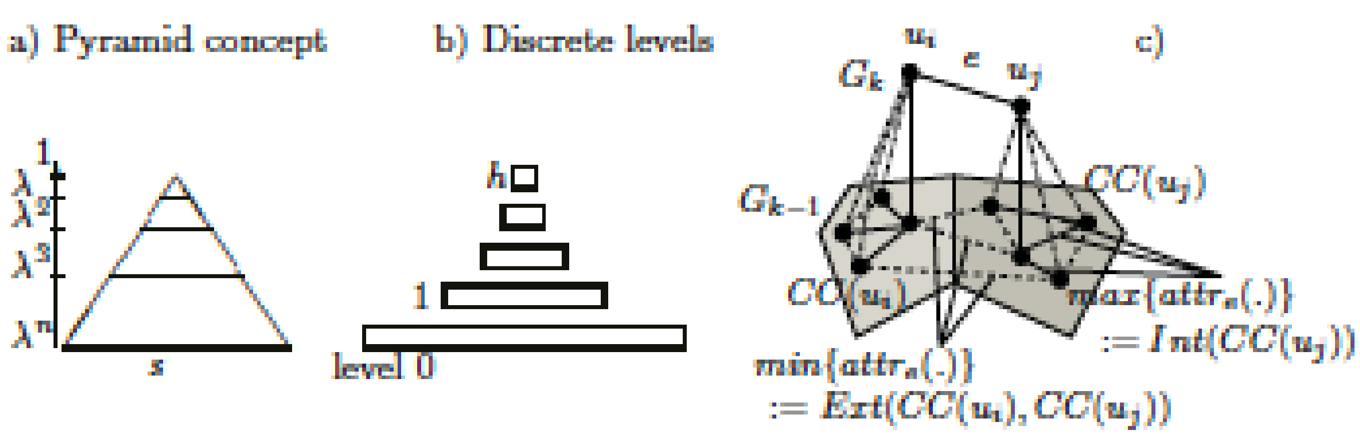

Haxhimusa and Kropatsch, 2004

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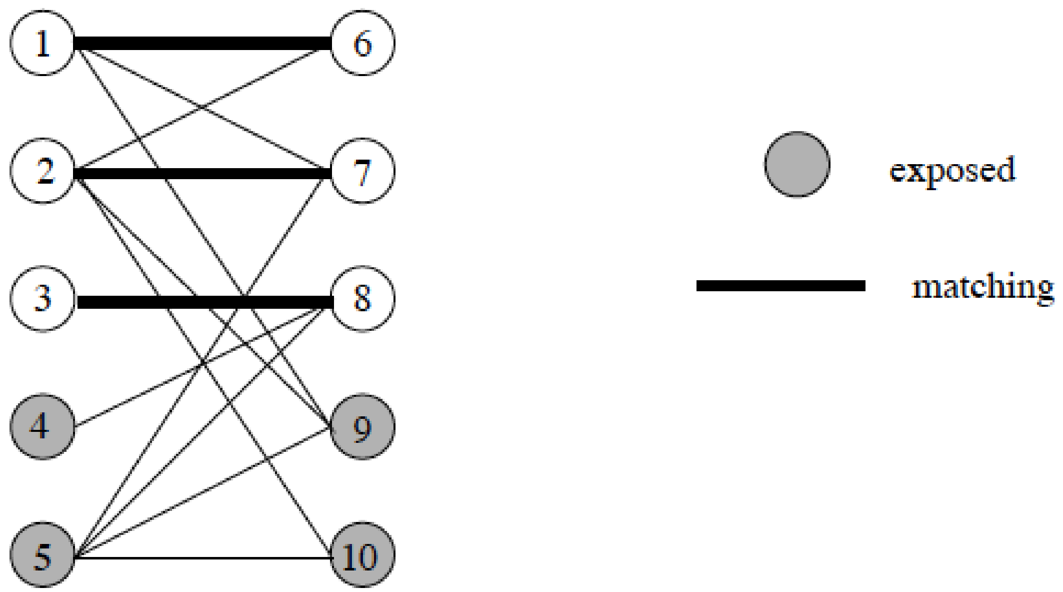

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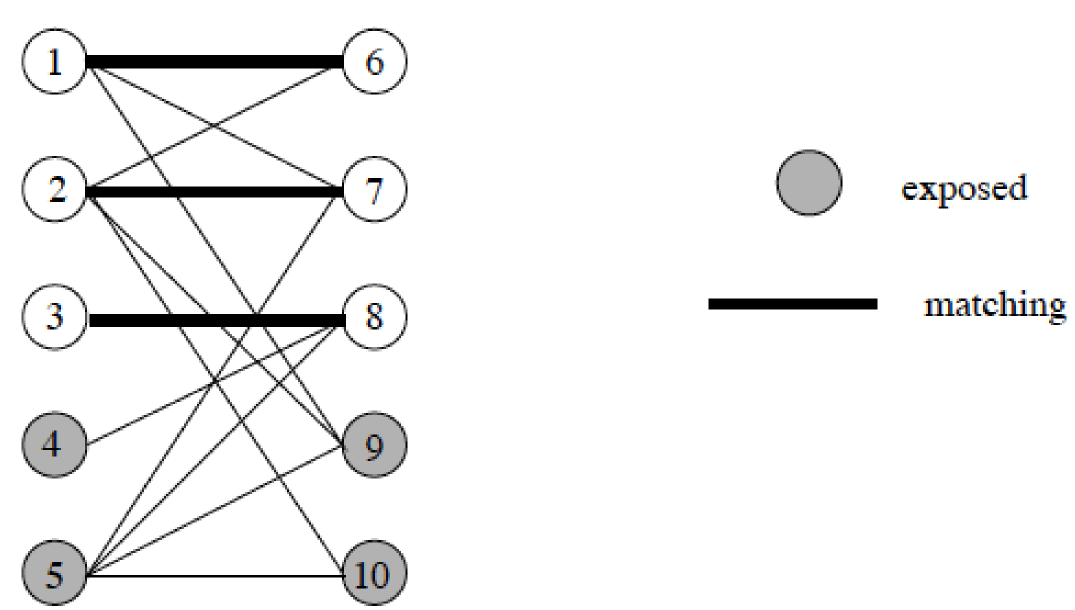

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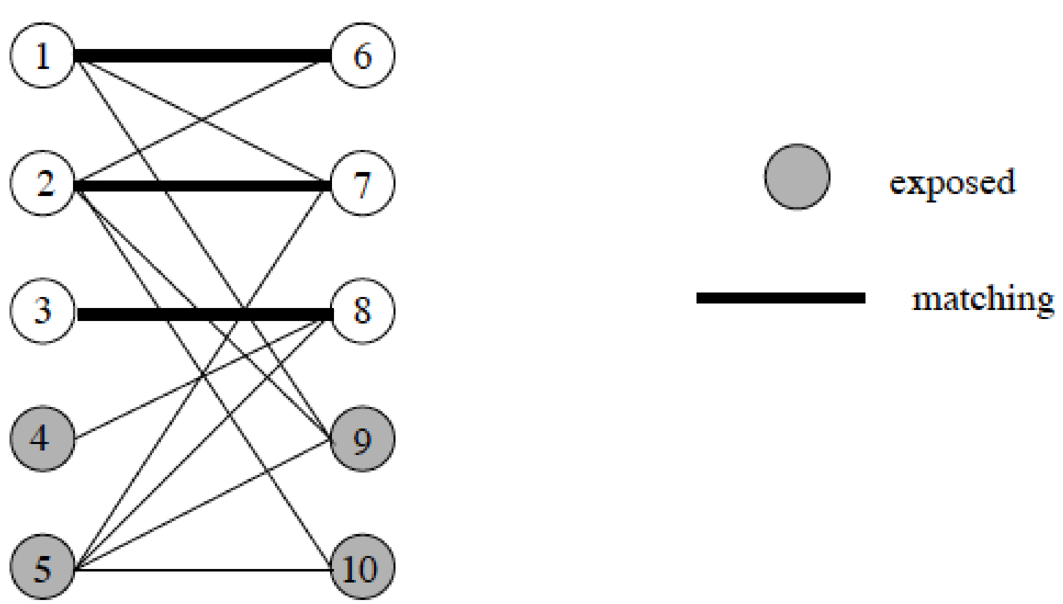

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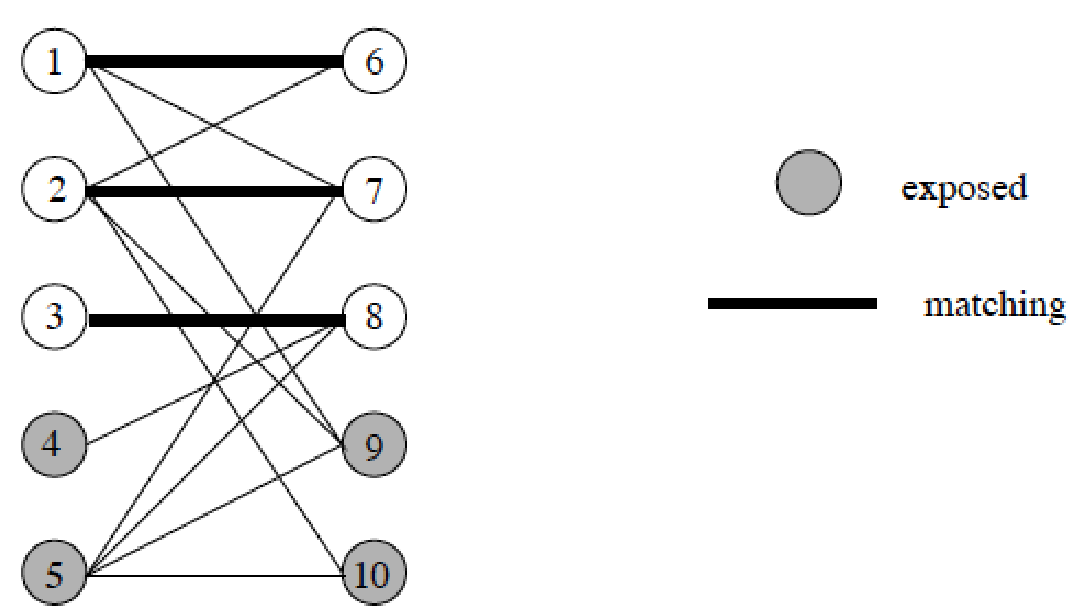

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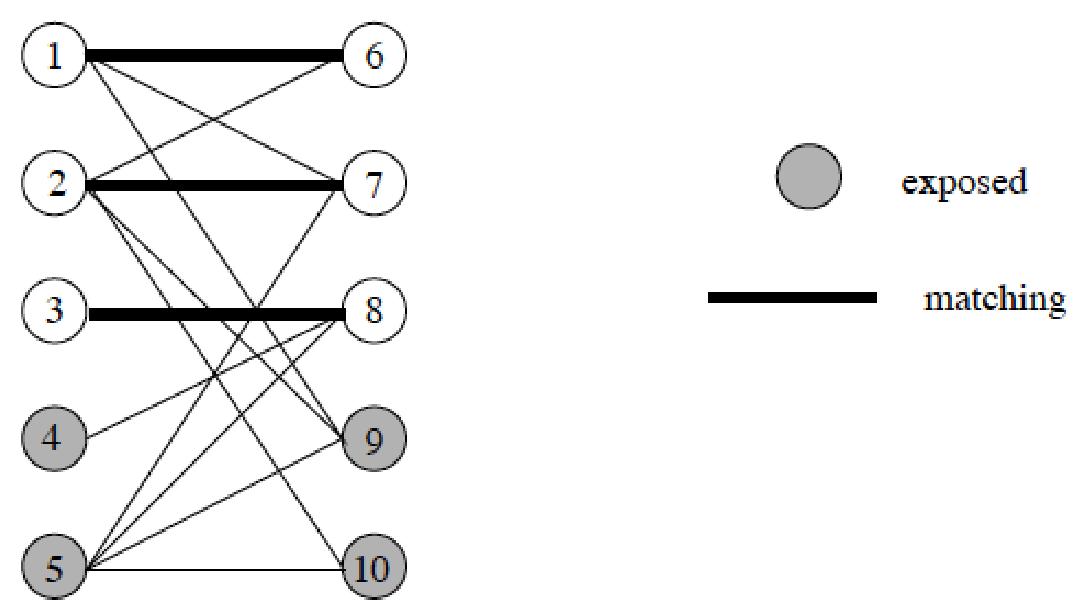

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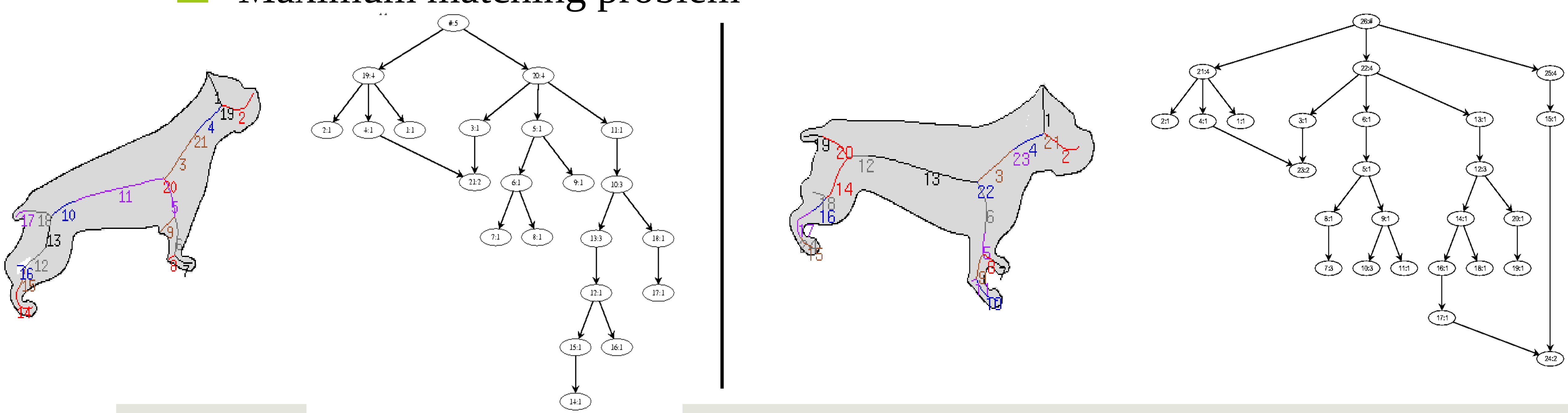

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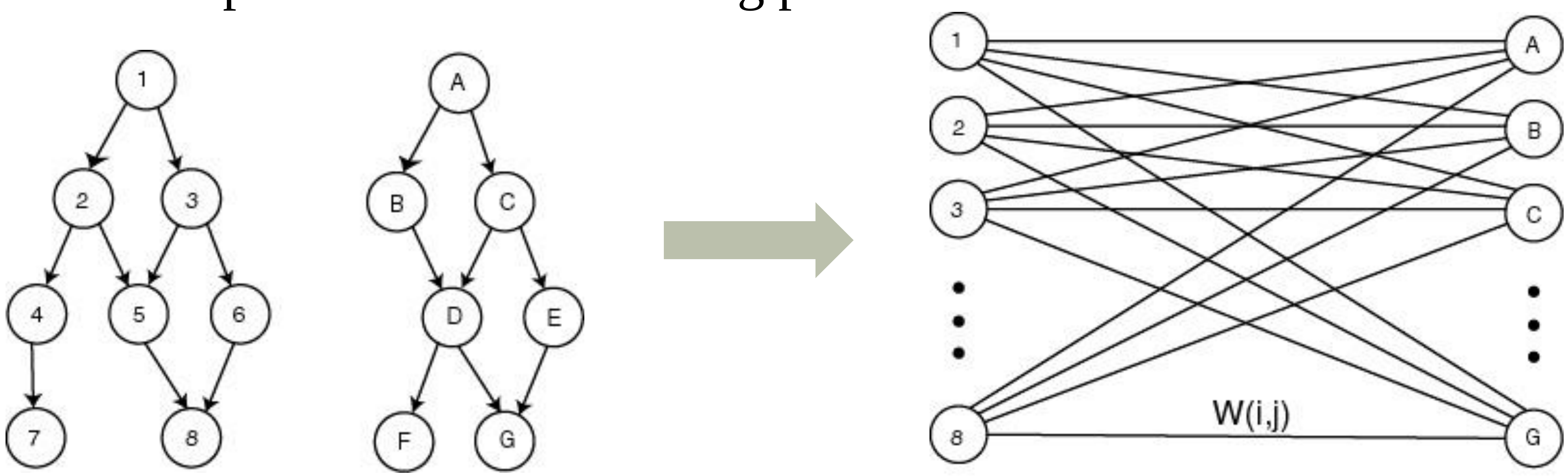

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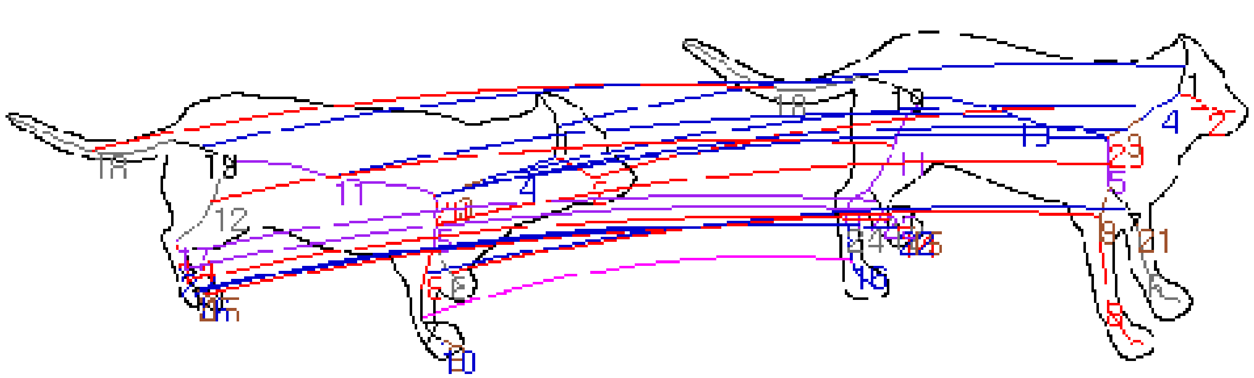

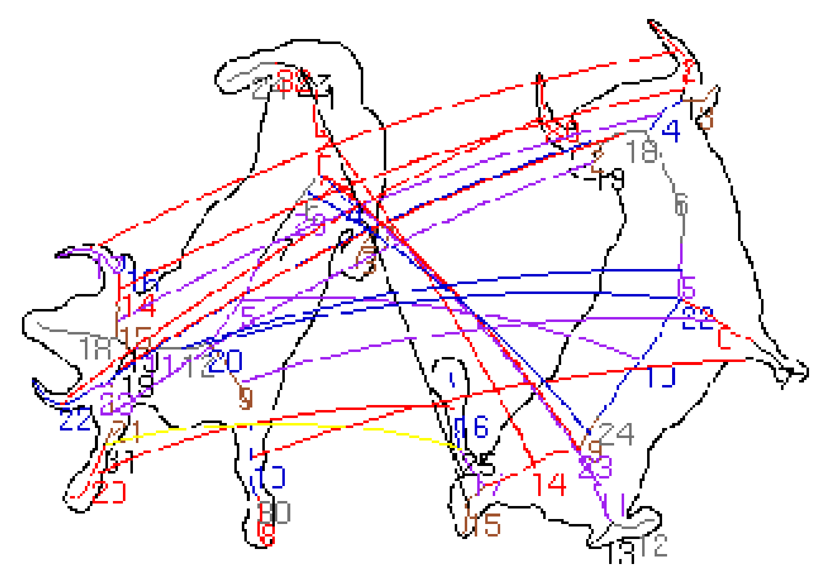

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  - Optimality of solution.

- **Bad news:** most of the combinatorial optimization problems involving graphs are computationally intractable:
  - traveling salesman problem, maximum cut problem, independent set problem, maximum clique problem, minimum vertex cover problem, maximum independent set problem, multidimensional matching problem,…

Algorithmic Graph Theory:

- **Dealing with the intractability:**
  - Bounded approximation algorithms
  - Suboptimal heuristics.

Algorithmic Graph Theory:

Bounded approximation algorithms

- Example: Vertex cover problem:
  - **A *vertex cover* of an undirected graph $G=(V,E)$ is a subset V' of V such that if (u,v) is an edge in E, then u or v (or both) belong to V'.**

Algorithmic Graph Theory:

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  - **The *vertex cover problem* is to find a vertex cover of minimum size in a given undirected graph.**

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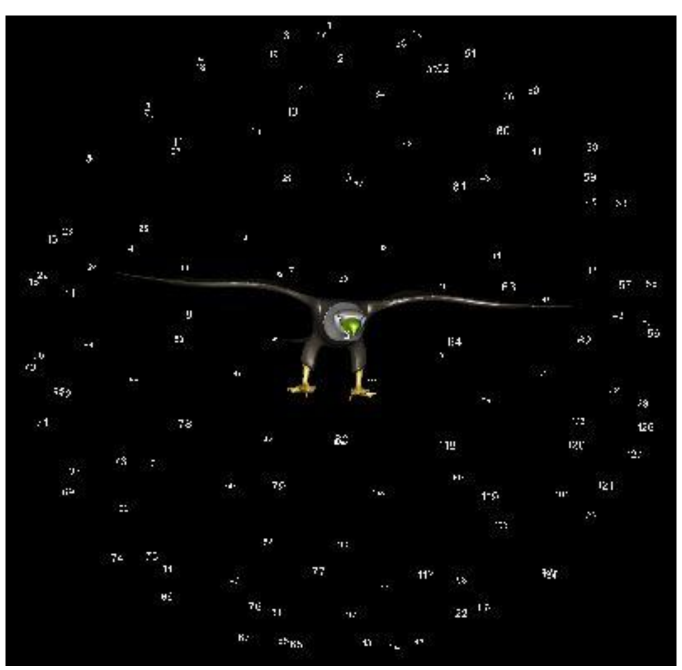

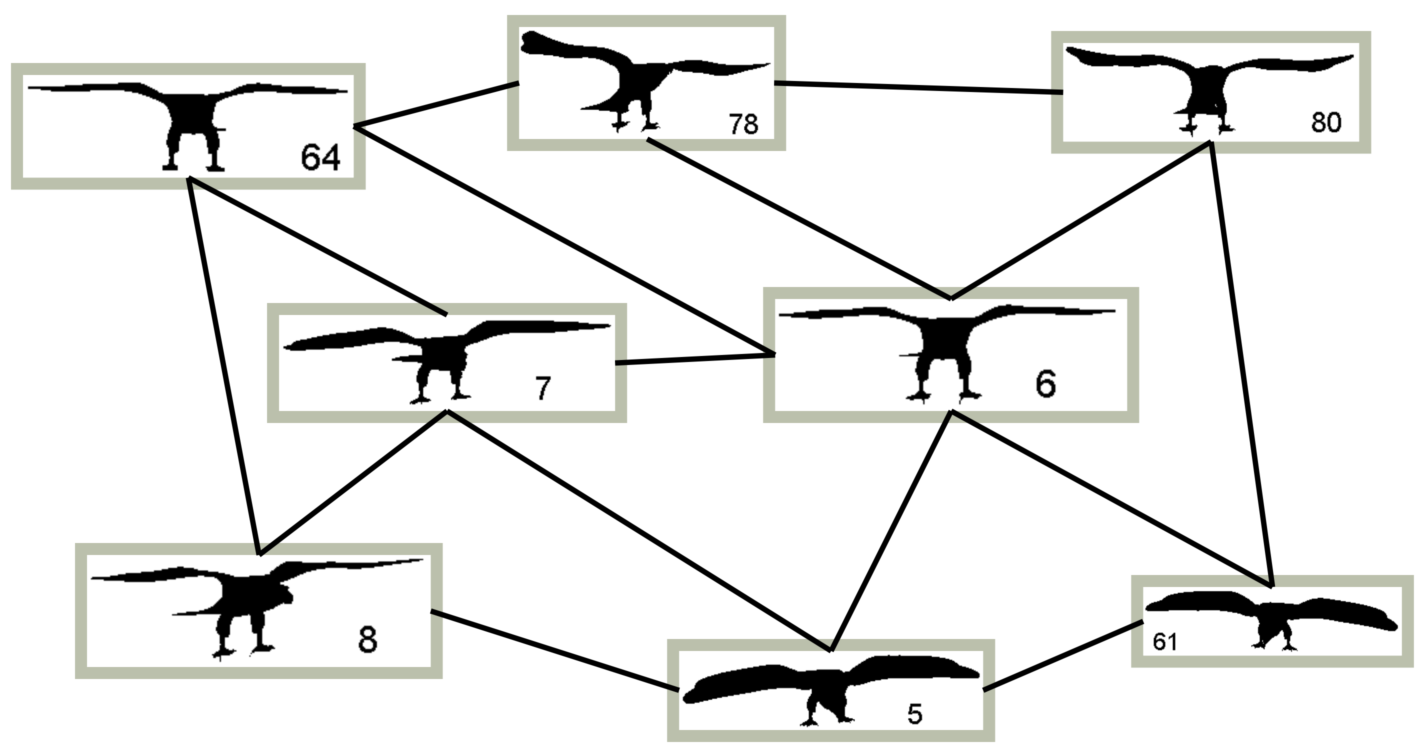

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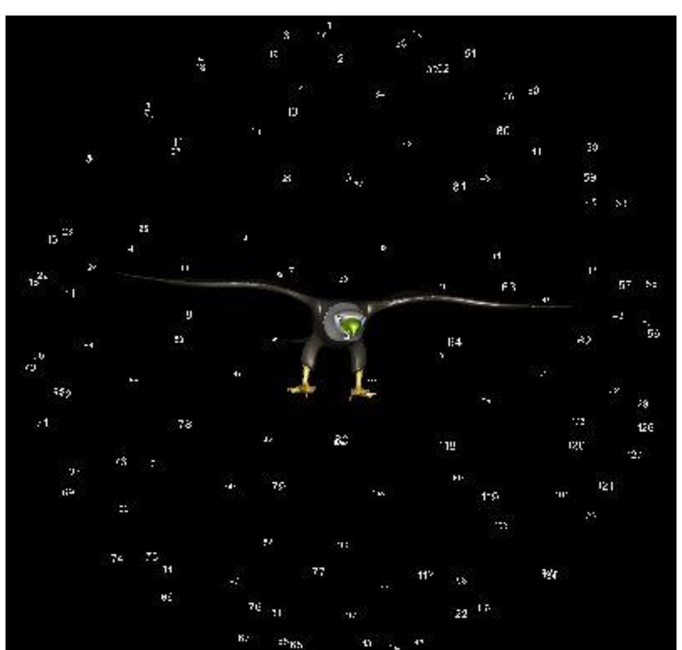

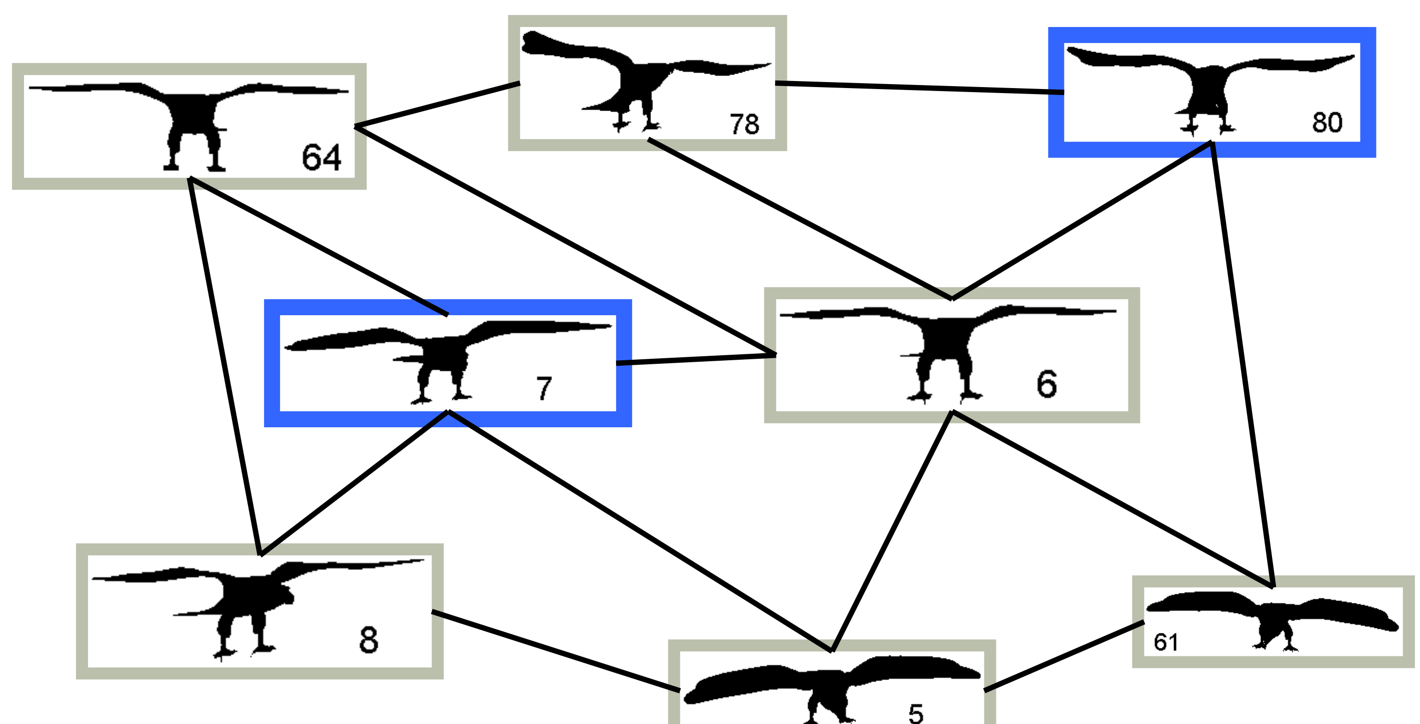

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  - The size of a vertex cover is the number of vertices in it.
  - The ***vertex cover problem*** is to find a vertex cover of **minimum** size in a given undirected graph.
  - **We call such a vertex cover an *optimal vertex cover*.**
  - **The vertex cover problem was shown to be NP-complete.**

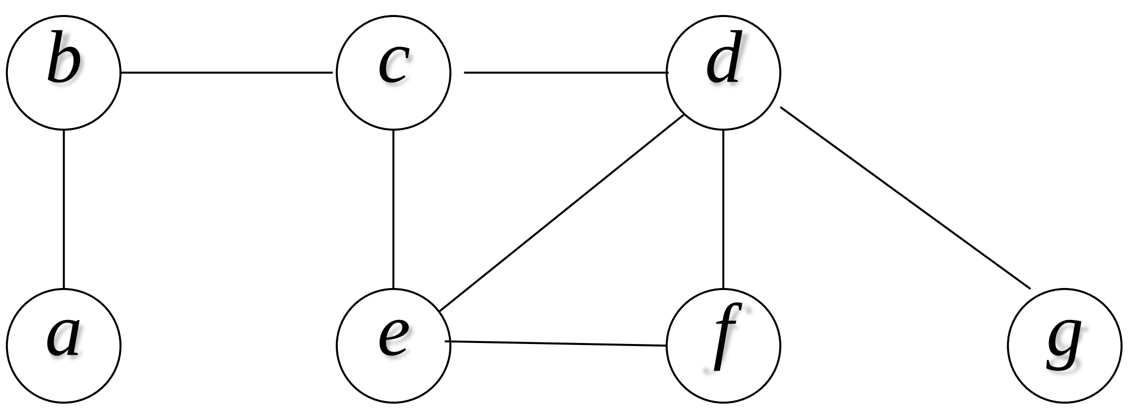
Algorithmic Graph Theory:

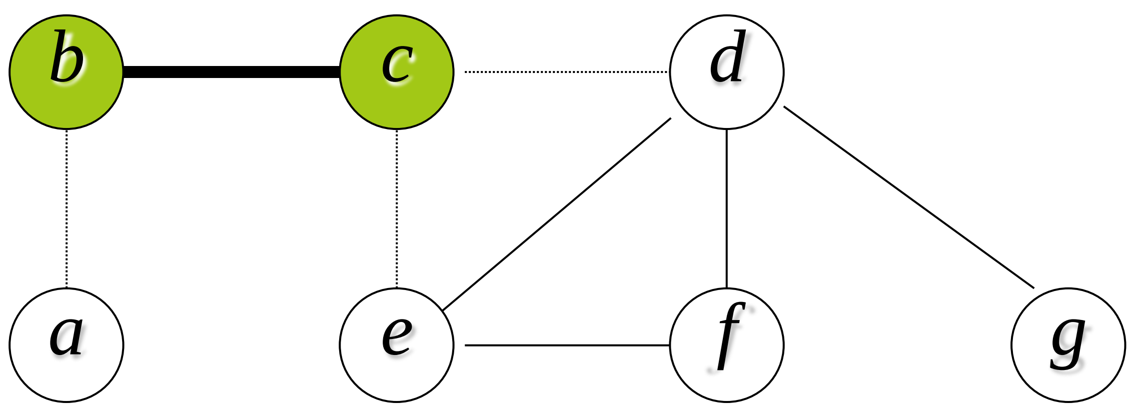
Vertex cover problem:

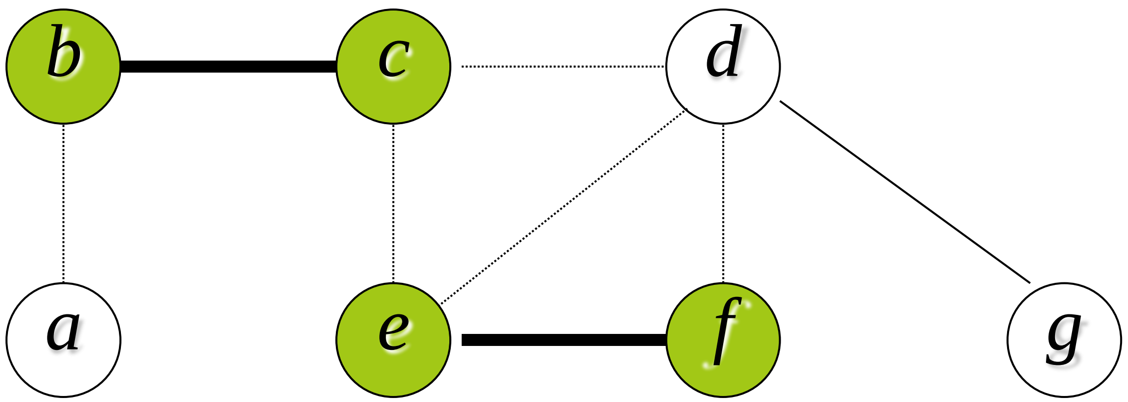
- The following approximation algorithm takes as input an undirected graph G and returns a vertex cover whose size is guaranteed no more than twice the size of optimal vertex cover:

1. $C \leftarrow \emptyset$
2. $E' \leftarrow E[G]$
3. While $E' \neq \emptyset$ do
4. Let (u, v) be an arbitrary edge in E'
5. $C \leftarrow C \cup \{u, v\}$
6. Remove from E' every edge incident on either u or v
7. Return C

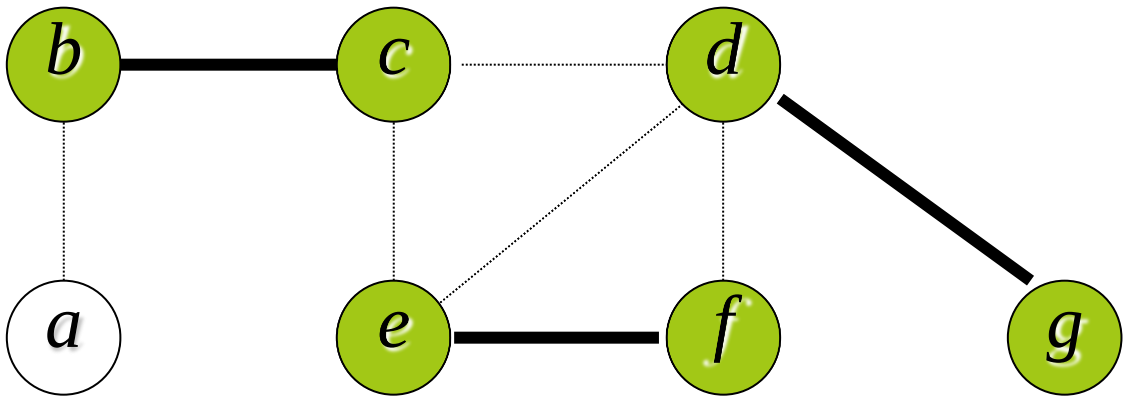
Algorithmic Graph Theory:

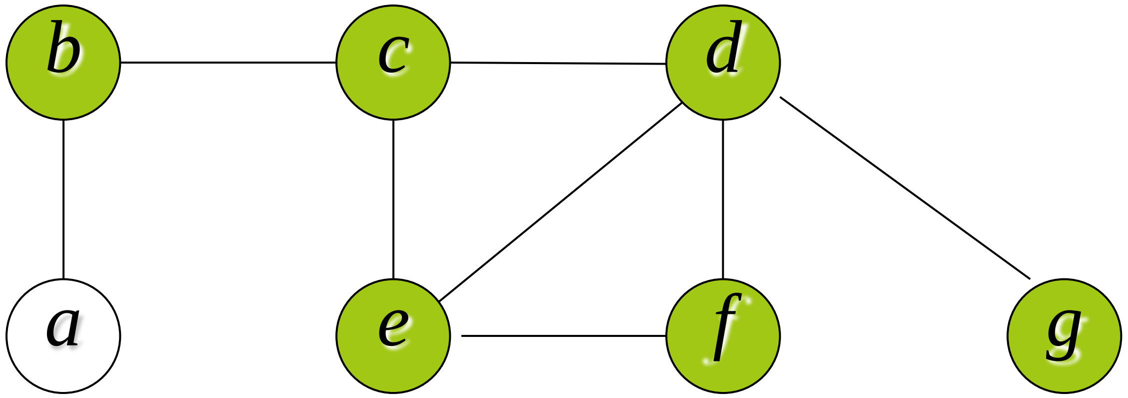






The Vertex Cover Problem





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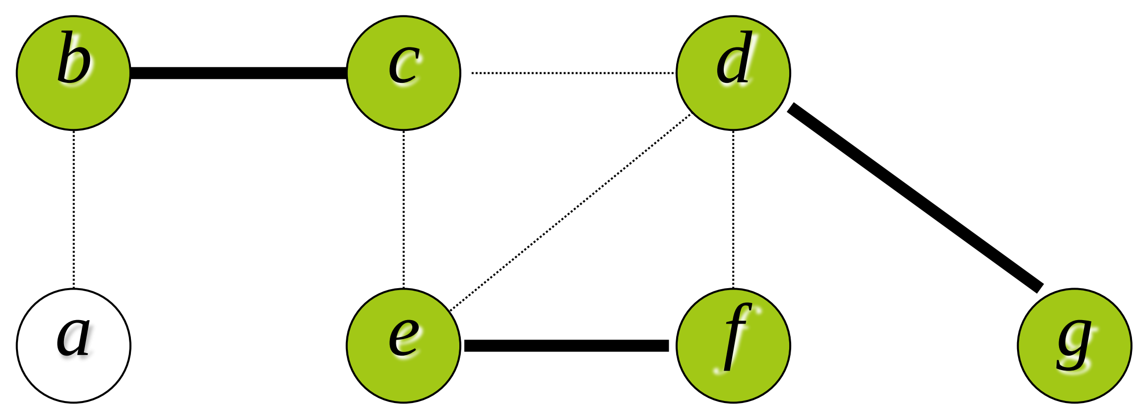

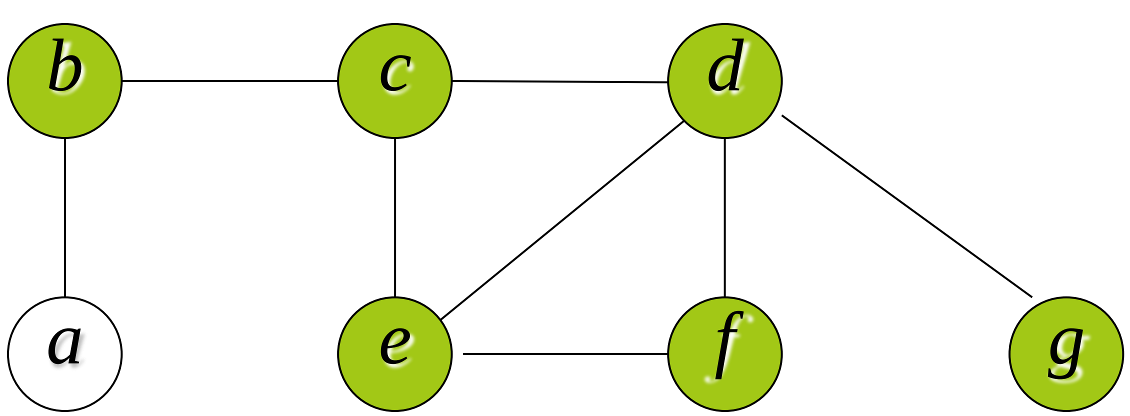

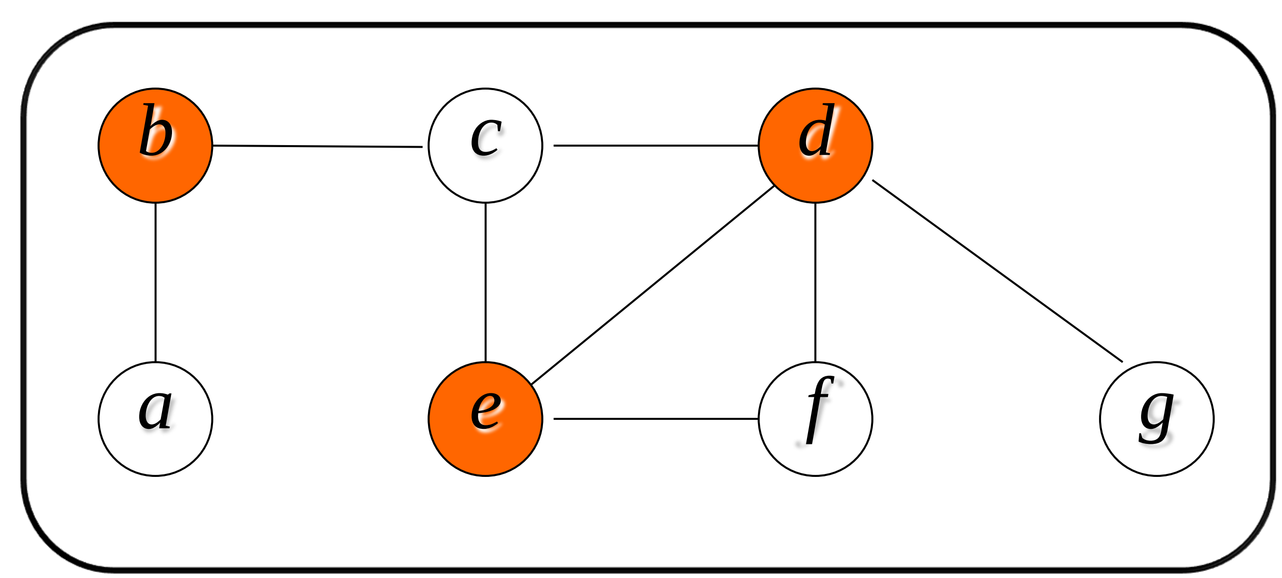

Algorithmic Graph Theory:

Theorem: Approximate vertex cover has a ratio bound of 2.

- ***Proof***:
  - It is easy to see that ***C*** is a vertex cover.
  - To show that the size of ***C*** is twice the size of optimal vertex cover.
  - Let ***A*** be the set of edges picked in line 4 of algorithm.
  - No two edges in ***A*** share an endpoint, therefore each new edge adds two new vertices to ***C***, so |***C***|=***2***|***A***|.
  - Any vertex cover should cover the edges in ***A***, which means at least one of the end points of each edge in *A* belongs to ***C*.***
  - So, |***A***|<=|***C****|, which will imply the desired bound.

Algorithmic Graph Theory:

Bounded approximation algorithms

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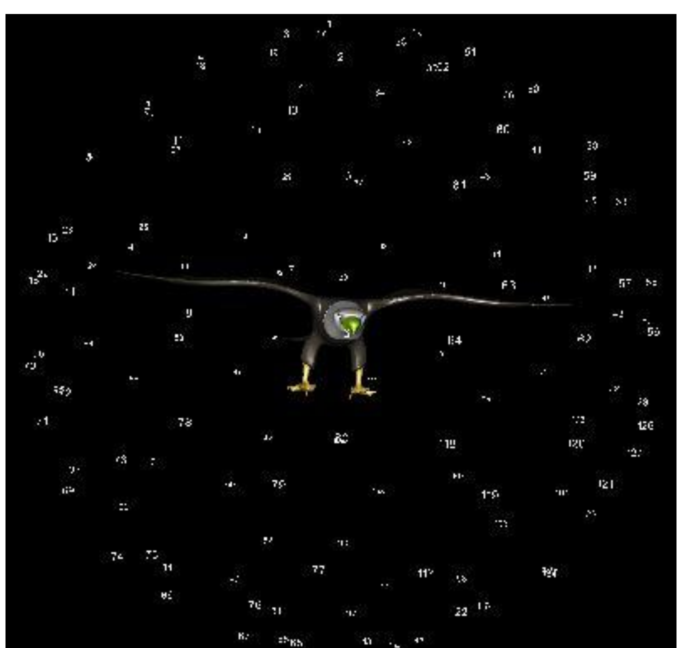

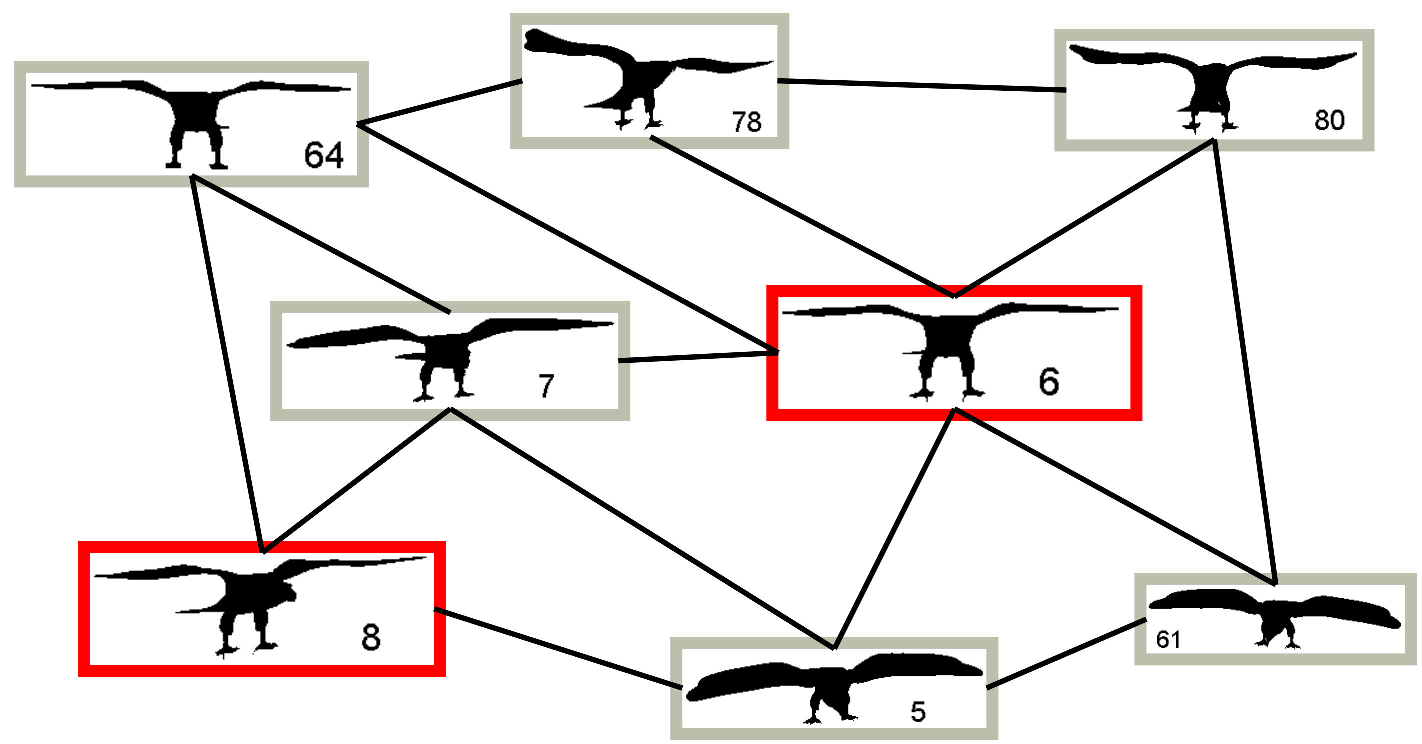

What Next?

Geometry of Graphs and Graphs Encoding the Geometry

Spectral Graph Theory