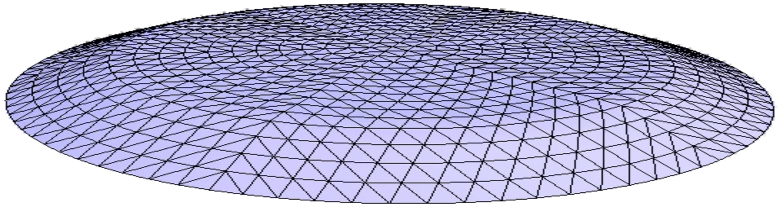
Laplace-Beltrami Eigenstuff
Part 2 - Computation



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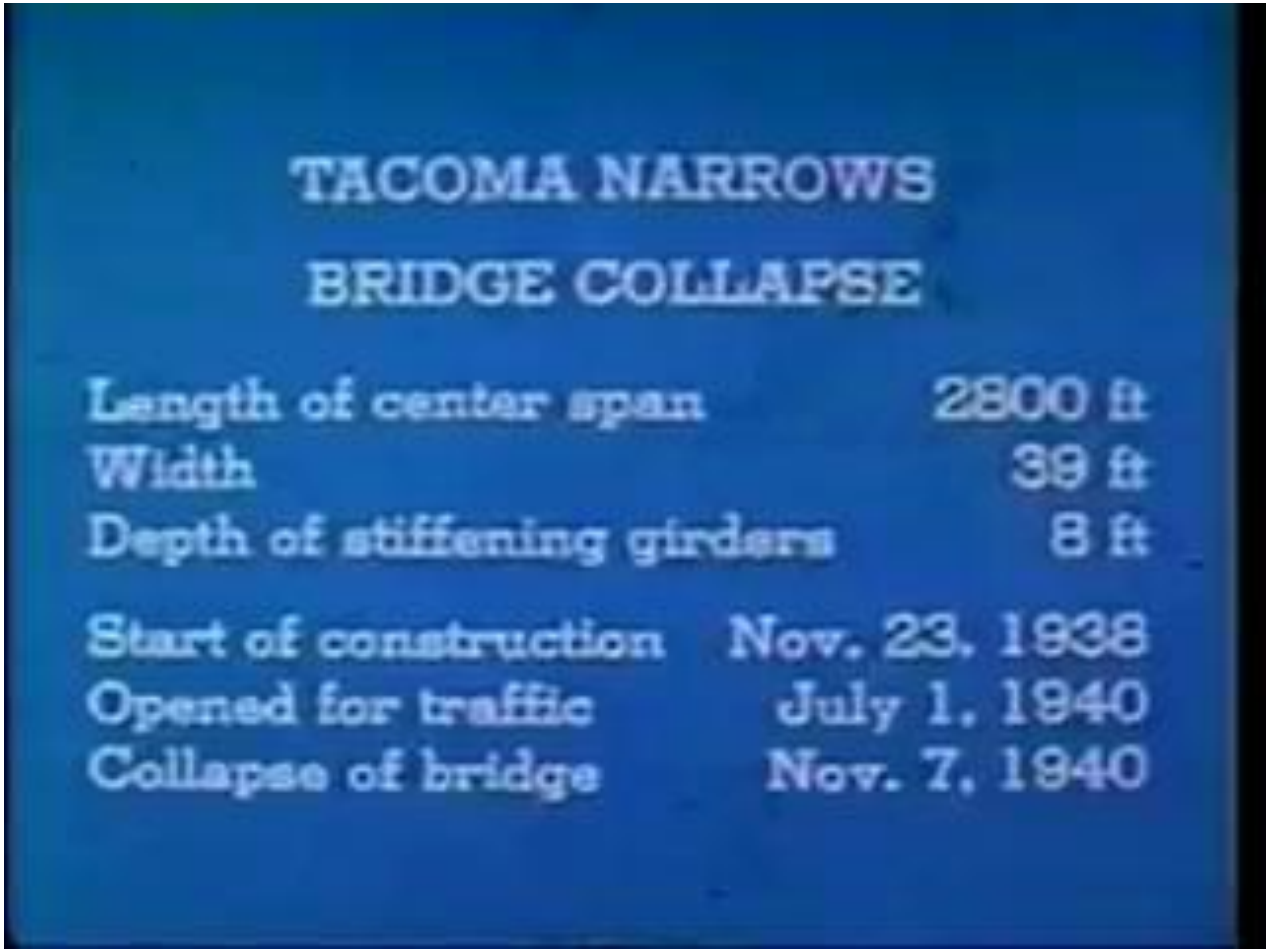
Know your Eigenvalues



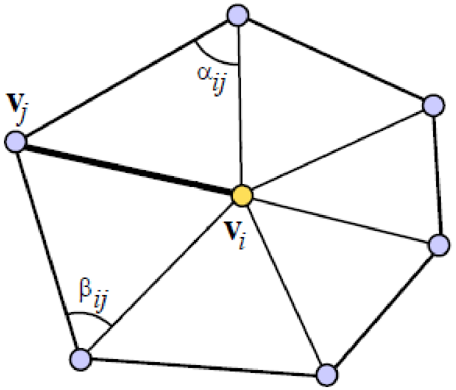
Discrete LBO (Graph)

Discrete Laplace-Beltrami operators:

$$\Delta f(p_i) := \frac{1}{d_i} \sum_{j \in N(i)} w_{ij} \left[f(p_i) - f(p_j) \right]$$

- $N(i)$ set of one ring neighbors of vertex i
- $f(p_i)$ (or simply f_i) value of the real function f at vertex p_i
- The d_i are the masses associated to vertex i
- and the w_{ij} are edge weights.

Discrete LBO



Different authors use different weights.

Eg. Desbrun (99):

$$w_{ij} = \frac{cot\alpha_{ij}\ cot\beta_{ij}}{2} \quad \text{and} \quad d_i = \frac{Area_i}{3}$$

$Area_i$ is the area of the 1 star around vertex i
α_{ij} and β_{ij} are the angles opposed to edge e_{ij}

It will turn out that this is a simplified version of linear FEM!

Discrete LBO (Matrix Form)

Equation in matrix form:

$$
\begin{aligned}
f &:= (f_1, f_2, \ldots, f_n)^T := (f(p_1), f(p_2), \ldots, f(p_n))^T \\
W &:= (w_{ij}) \quad \textit{(weighted adjacency matrix)} \\
V &:= diag(v_1, \ldots, v_n) \text{ with } v_i = \sum_j w_{ij} \\
A &:= V - W \\
D &:= diag(d_i) \quad \textit{(lumped mass matrix)} \\
L &:= D^{-1} A \quad \leftarrow \text{ not symmetric}
\end{aligned}
$$

then $\Delta f(p_i)$ is the i-th component of the vector $L\, f$:

$$(\Delta f_1, \ldots, \Delta f_n)^T = L\, f.$$

Note about Symmetry

- In case of node weights (also called masses) L cannot be represented as a symmetric matrix.
  - Slower matrix vector multiplication
  - Large NxN matrix difficult to handle / store
  - Eigenvalues can be imaginary!
- Instead keep Eigenvalue system symmetric and sparse (generalized EVP):

$$Lx = D^{-1}(W - V)x = -\lambda x \;\Leftrightarrow\; (W - V)x = -\lambda D x$$

- Or solve equivalently standard problem:

$$D^{-\frac{1}{2}}(W - V)D^{-\frac{1}{2}}y = -\lambda y \quad \text{with} \quad y := D^{\frac{1}{2}}x$$

Continuous Case

Definition

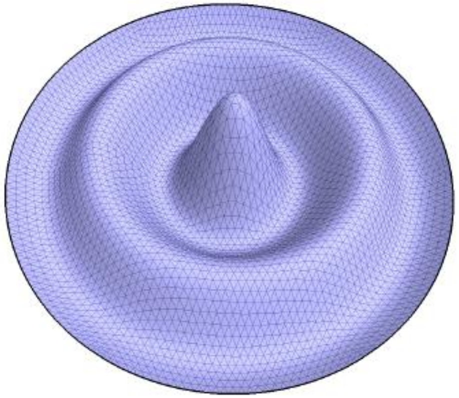
Helmholtz Equation (**Laplacian Eigenvalue Problem**):

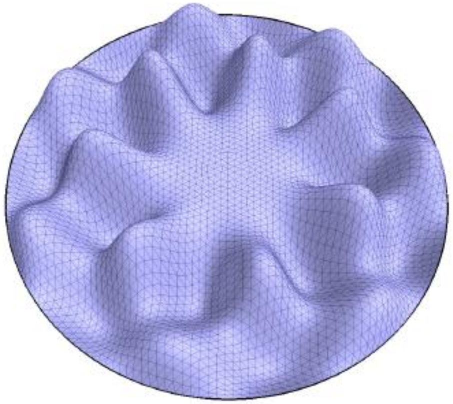
$$\Delta f = -\lambda f, \qquad f : M \to \mathbb{R}$$

Solution: Eigenfunctions f_i with corresponding family of eigenvalues (**Spectrum**):

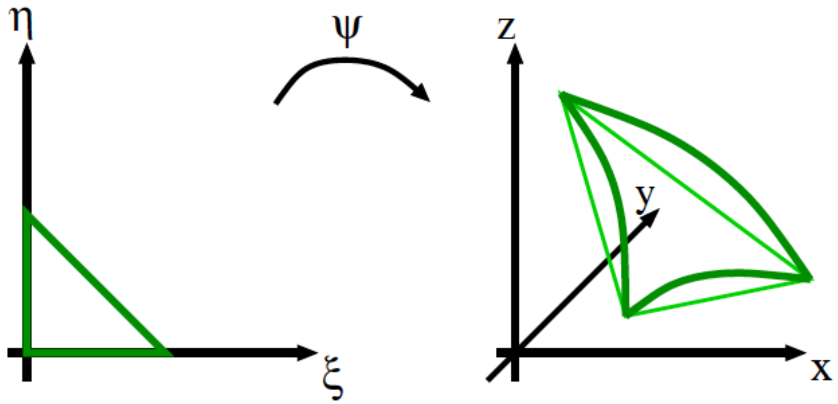
$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \uparrow +\infty$$

Here Laplace-Beltrami Operator: $\Delta f := div(grad\ f)$





LBO in local coordinates



Definition (1. fundamental matrix)

$\psi : \mathbb{R}^n \to \mathbb{R}^{n+k}$ be a (local) parametrization of a manifold M, then (with $i, j = 1, \ldots, n$ and det the determinant):

$$g_{ij} := \langle \partial_i \psi, \partial_j \psi \rangle, \quad G := (g_{ij}),$$
$$W := \sqrt{\det G}, \quad (g^{ij}) := G^{-1}.$$

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LBO in local coordinates

Definition (Laplace-Beltrami Operator)

The **Laplace-Beltrami Operator** in local coordinates:

$$\Delta f \quad = \frac{1}{W}\sum_{i,j}\partial_i(g^{ij}W\partial_j f)$$

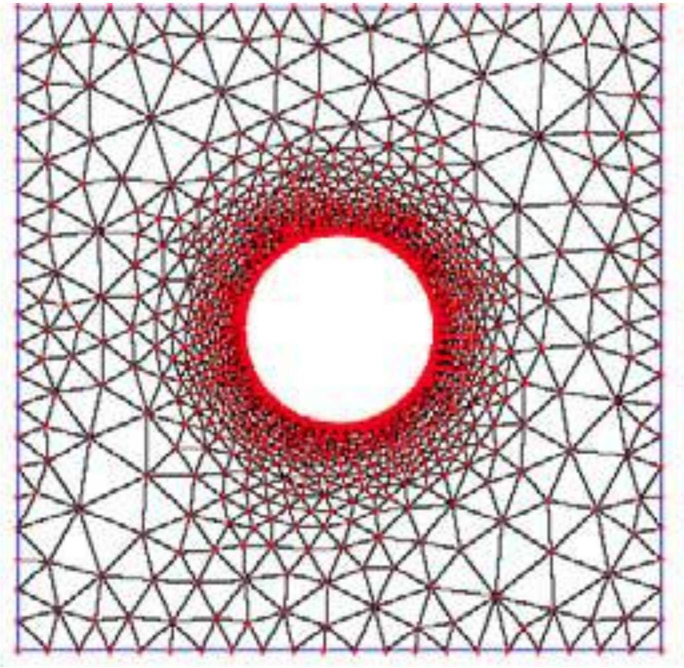
If M is a domain of the Euclidean plane $M \subset \mathbb{R}^2$, the Laplace-Beltrami operator reduces to the well known Laplace operator:

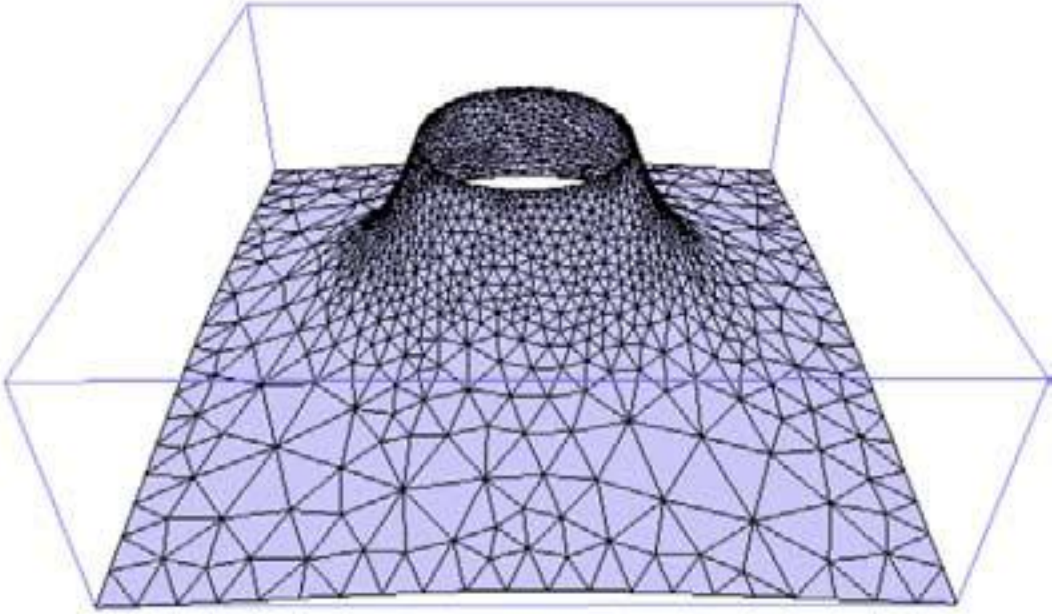
$$\Delta f = \frac{\partial^2 f}{(\partial x)^2} + \frac{\partial^2 f}{(\partial y)^2}$$

How to solve this on some shape?

- 1. Discretize geometry (elements)
  - here triangle mesh
- 2. Discretize function space (basis or form functions)
  - select basis functions on mesh
  - here linear hat functions
- 3. Transform the Differential Equation (Variational Formulation)
  - multiply equation by arbitrary test functions
  - integrate over domain
  - try to replace higher order derivatives with lower order

Geometry Discretization





Hat Functions

- Function values defined at vertices: u_i
- Extend piecewise linear function by choosing basis of linear hat functions F_i (value 1 at vertex i and zero at others): $U = \sum u_i F_i$

Inner Product

- Inner product of two functions U and H:

$$(U,H)_{L_2(M)} := \int_M UH \, d\sigma$$

- Norm of U:

$$\| U \|_{L_2(M)} := \left(\int_M |U|^2 \, d\sigma \right)^{\frac{1}{2}} = \sqrt{(U,U)_{L_2(M)}}$$

- Volume (Area in 2D):

$$Area_M = \int_M 1 \, d\sigma = (1,1)_{L_2(M)}$$

Integral of single function

- For piecewise linear functions $U = \sum u_i F_i$

$$\int_M U \; d\sigma = \sum u_i \int_M F_i \; d\sigma = (1, ..., 1) \; D \; \vec{u}$$

$$\text{where } D = \text{diag}\left(\int_M F_i \; d\sigma\right) \text{ and } \vec{u} = (u_0, u_1, ..., u_n)^T.$$

- Interestingly: the elements of D are simply the area of all triangles at a vertex divided by 3 -> Desbrun mass!

$$d_i = \frac{\text{area}_i}{3}$$

Inner Product

- Inner Product of functions U and H

$$(U,H)_{L_2} := \int_M UH \; d\sigma = \vec{u}^T B \vec{h}$$

$$\text{with } \vec{u} = (u_0, u_1, ..., u_n)^T \text{ and } \vec{h} = (h_0, h_1, ..., h_n)^T$$

- and B a positive definite symmetric sparse matrix:

$$b_{ij} = \int_M F_i F_j \; d\sigma$$

- What happens when lumping (summing rows onto diagonal):

$$\sum_j b_{ij} = \sum_j \int_M F_i F_j \; d\sigma = \int_M F_i \underbrace{\sum_j F_j}_{1} \; d\sigma = d_i = \frac{\text{area}_i}{3}$$

Variational Formulation of Laplace Eigenvalue Problem

The computation can be done with FEM (any dimension):

Multiply Helmholtz equation with test functions φ, then integrate and apply Greens Formula:

$$\begin{aligned} & \varphi \Delta f && = -\lambda \varphi f \\ \Leftrightarrow \quad & \iint \varphi \Delta f \, d\sigma && = -\lambda \iint \varphi f \, d\sigma \\ \Leftrightarrow \quad & \iint Df \, G^{-1} \, (D\varphi)^T \, d\sigma && = \lambda \iint \varphi f \, d\sigma \end{aligned}$$

(with $Df = (\partial_1 f, \partial_2 f, ...)$, $d\sigma = W \, du \, dv$ being the surface element in the 2D case or the volume element $d\sigma = W \, du \, dv \, dw$ in the 3D case).

Form Functions

Approximating $f \approx \sum U_l F_l$ (where F_l form functions):

$$\begin{aligned} \iint \sum_l U_l (DF_l)\, G^{-1}\, (DF_m)^T\, d\sigma &= \lambda \iint \sum_l U_l F_l F_m\, d\sigma \\ \Leftrightarrow \quad \sum_l U_l \iint (DF_l)\, G^{-1}\, (DF_m)^T\, d\sigma &= \lambda \sum_l U_l \iint F_l F_m\, d\sigma \end{aligned}$$

$$\text{yields:} \quad AU = \lambda BU$$

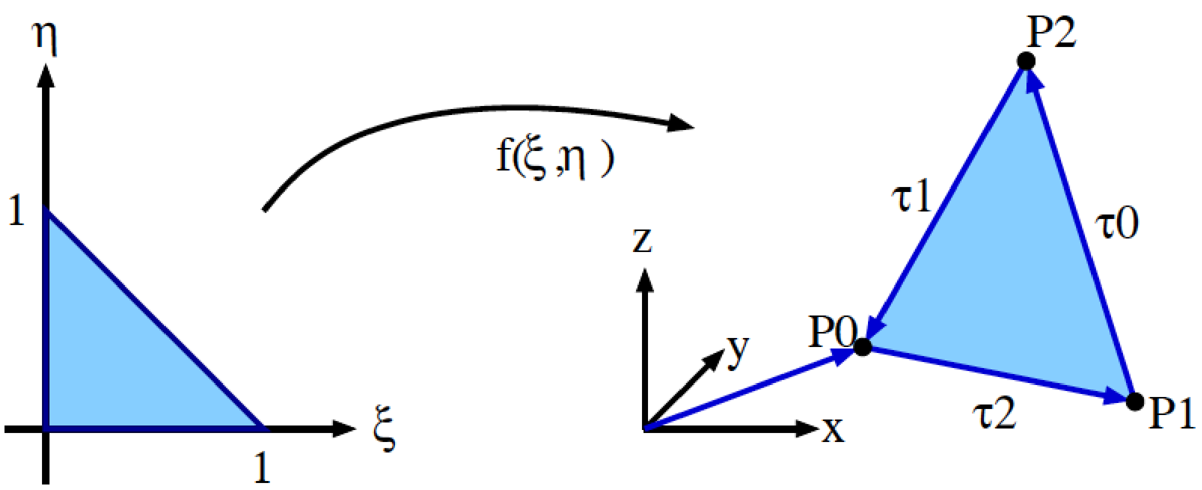
with the matrices (sparse, symmetric, positiv semi-definit):

$$\begin{aligned} A &= (a_{lm}) &&:= \left(\iint (DF_l)\, G^{-1}\, (DF_m)^T d\sigma \right), \\ B &= (b_{lm}) &&:= \left(\iint F_l F_m d\sigma \right). \end{aligned}$$

Solve with Lanczos Method from ARPACK

We used up to cubic form functions $\rightarrow$ fast convergence!

Triangle Meshes (piecewise flat)



$$T(\xi, \eta) = P_0 + \xi\tau_2 - \eta\tau_1$$

$$\text{with} \quad T_\xi = \frac{\partial T}{\partial_\xi} = \tau_2 \quad \text{and} \quad T_\eta = \frac{\partial T}{\partial_\eta} = -\tau_1$$

when setting $\tau_i := P_{(i+2)\%2} - P_{(i+1)\%2}$
(with the modulo operator %).

Metric Values on Triangle Meshes

Metric values of this parametrization:

$$G = \begin{pmatrix} (\tau_2)^2 & -\tau_1\tau_2 \\ -\tau_1\tau_2 & (\tau_1)^2 \end{pmatrix}$$

$$W = \sqrt{det(G)} = \sqrt{(\tau_1)^2(\tau_2)^2 - (\tau_1\tau_2)^2} = \| \tau_1 \times \tau_2 \|$$

$$G^{-1} = \frac{1}{W^2} \begin{pmatrix} (\tau_1)^2 & \tau_1\tau_2 \\ \tau_1\tau_2 & (\tau_2)^2 \end{pmatrix}$$

All these values are constant for the entire triangle

Plugging it into the Variational Eq.

$$(a_{lm}) \quad += \iint \left(\sum_{j,k} (\partial_j F_l)(\partial_k F_m) g^{jk}\right) W \, d\xi d\eta$$

$$= \iint \left[(\tau_1)^2 \partial_\xi F_l \, \partial_\xi F_m + (\tau_2)^2 \partial_\eta F_l \, \partial_\eta F_m + \tau_1 \tau_2 (\partial_\xi F_l \, \partial_\eta F_m + \partial_\eta F_l \, \partial_\xi F_m) \right] \frac{1}{W} \, d\xi d\eta$$

$$= \frac{1}{\|\tau_1 \times \tau_2\|} \Big[(\tau_1)^2 \iint \partial_\xi F_l \, \partial_\xi F_m \, d\xi d\eta + (\tau_2)^2 \iint \partial_\eta F_l \, \partial_\eta F_m \, d\xi d\eta + \tau_1 \tau_2 \iint (\partial_\xi F_l \, \partial_\eta F_m + \partial_\eta F_l \, \partial_\xi F_m) \, d\xi d\eta \Big]$$

$$(b_{lm}) \quad += \iint F_l F_m W \, d\xi d\eta = \| \tau_1 \times \tau_2 \| \iint F_l F_m \, d\xi d\eta$$

with the integral boundaries: $\int_0^1 \int_0^{1-\eta}$

Linear Form Functions

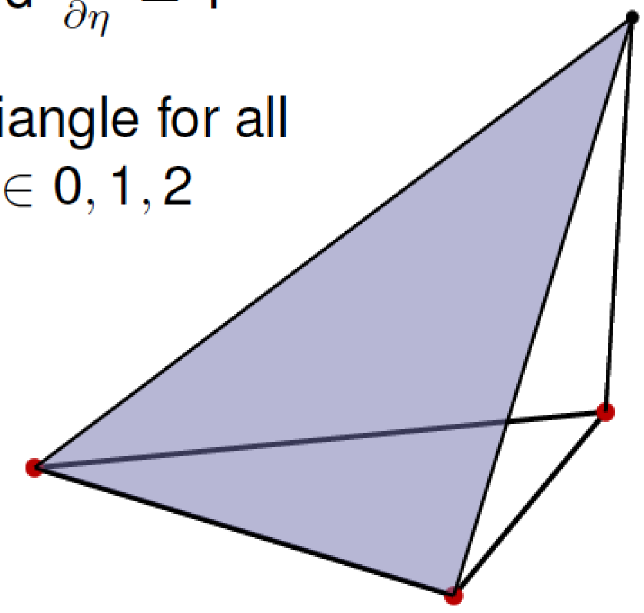
When using the linear formfunctions on a triangle:

$$F_0(\xi,\eta) = 1-\xi-\eta \quad \text{with } \frac{\partial F_l}{\partial \xi} = -1 \quad \text{and } \frac{\partial F_l}{\partial \eta} = -1$$

$$F_1(\xi,\eta) = \xi \quad \text{with } \frac{\partial F_l}{\partial \xi} = 1 \quad \text{and } \frac{\partial F_l}{\partial \eta} = 0$$

$$F_2(\xi,\eta) = \eta \quad \text{with } \frac{\partial F_l}{\partial \xi} = 0 \quad \text{and } \frac{\partial F_l}{\partial \eta} = 1$$

We can compute the integrals over the unit triangle for all the possible combinations of local indices $i, j \in 0, 1, 2$



All combinations:

$$\left(\iint F_i F_j \, d\xi d\eta\right) = \begin{pmatrix} \frac{1}{12} & \frac{1}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{12} \end{pmatrix}$$

$$\left(\iint \partial_\xi F_i \, \partial_\xi F_j \, d\xi d\eta\right) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left(\iint \partial_\eta F_i \, \partial_\eta F_j \, d\xi d\eta\right) = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\left(\iint \partial_\xi F_i \, \partial_\eta F_j + \partial_\eta F_i \, \partial_\xi F_j \, d\xi d\eta\right) = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Linear FEM

We can thus simply compute the local contributions to the corresponding entries in the A and B matrices for the linear case of a triangle T:

$$B' = (b'_{ij}) = \frac{area(T)}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$A' = (a'_{ij}) = \frac{1}{4area(T)} \begin{pmatrix} (\tau_0)^2 & \tau_0\tau_1 & \tau_0\tau_2 \\ \tau_0\tau_1 & (\tau_1)^2 & \tau_1\tau_2 \\ \tau_0\tau_2 & \tau_1\tau_2 & (\tau_2)^2 \end{pmatrix}$$

These symmetric 3x3 matrices A' and B' are called the element (stiffness and mass) matrices.

Linear FEM and Mesh Laplace

So the contribution of each triangle T to the matrix A are

$$a'_{ij} = \tau_i \tau_j / (4\ area(T))$$

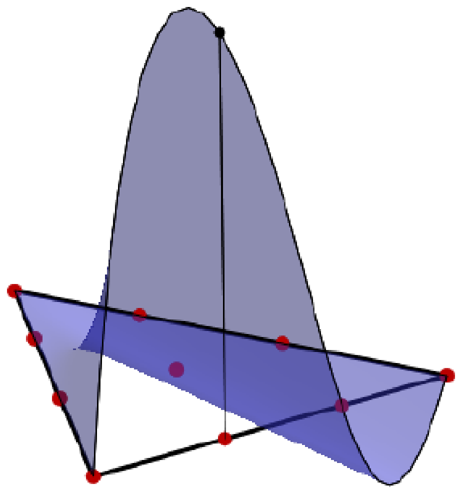
Since every edge has two triangles, the sum is equivalent to the well known cotangent weights (see Pinkall and Polthier 1993).
By lumping the mass matrix B:

$$D = (d_{ii}) = \sum_j b_{ij} = Area_i / 3$$

where $Area_i$ 1-Star Area around vertex i.
This is in fact the mesh Laplace suggested by Desbrun et.al. 1999.

Higher Order

For better results higher order approximations are recommended.



- Cubic functions : 10 degrees of reedom
- fixed by values at 10 nodes over triangle
- two new nodes along each edge, one in barycenter
- then using cubic formfunctions
- similarly for tetrahedra

Higher Order

Theorem (Convergence)

For decreasing mesh size h and order p form functions: Eigenvalues converge with order

$$O(2p)$$

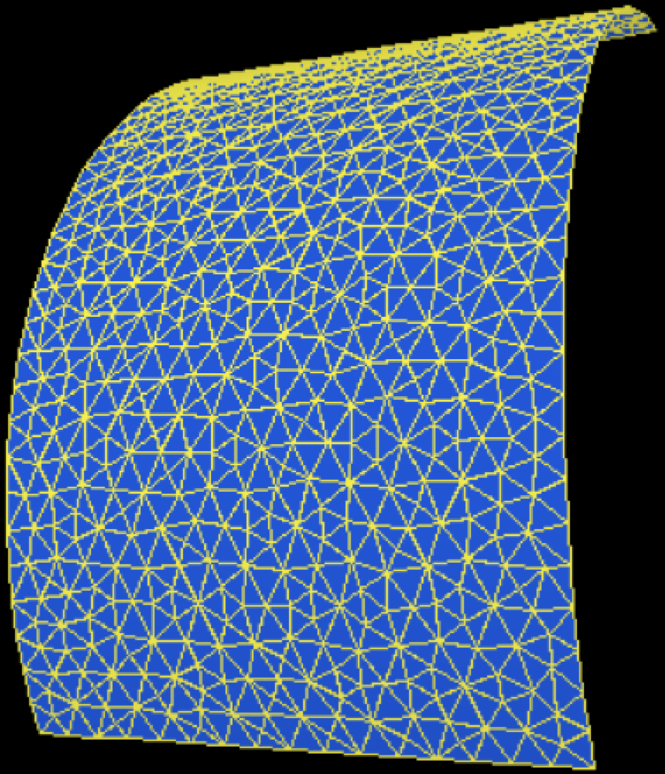
and Eigenfunctions with order

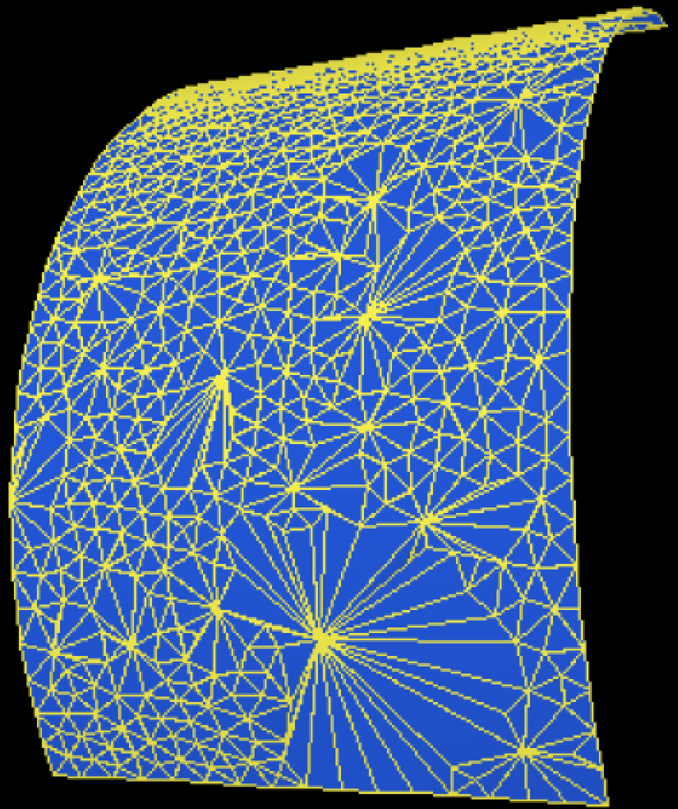
$$O(p+1)$$

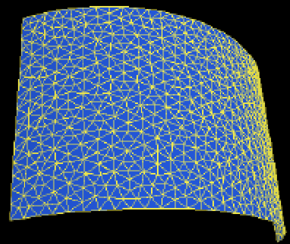
in the L_2 norm .

$\Rightarrow$ Always prefer higher order FEM approximations over mesh refinement.

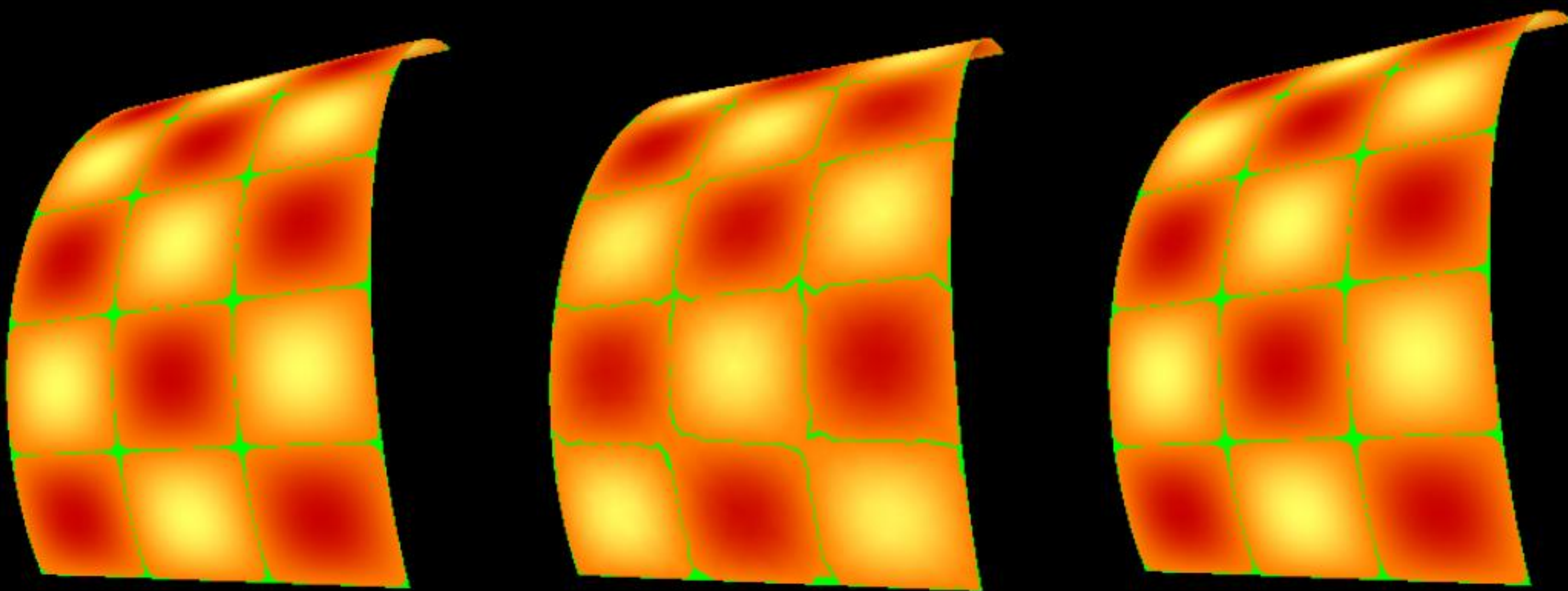
Uniform and Non-Uniform Mesh







Uniform Mesh (Efunc 23):



real linear cubic

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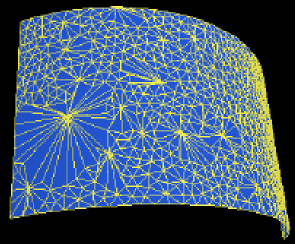
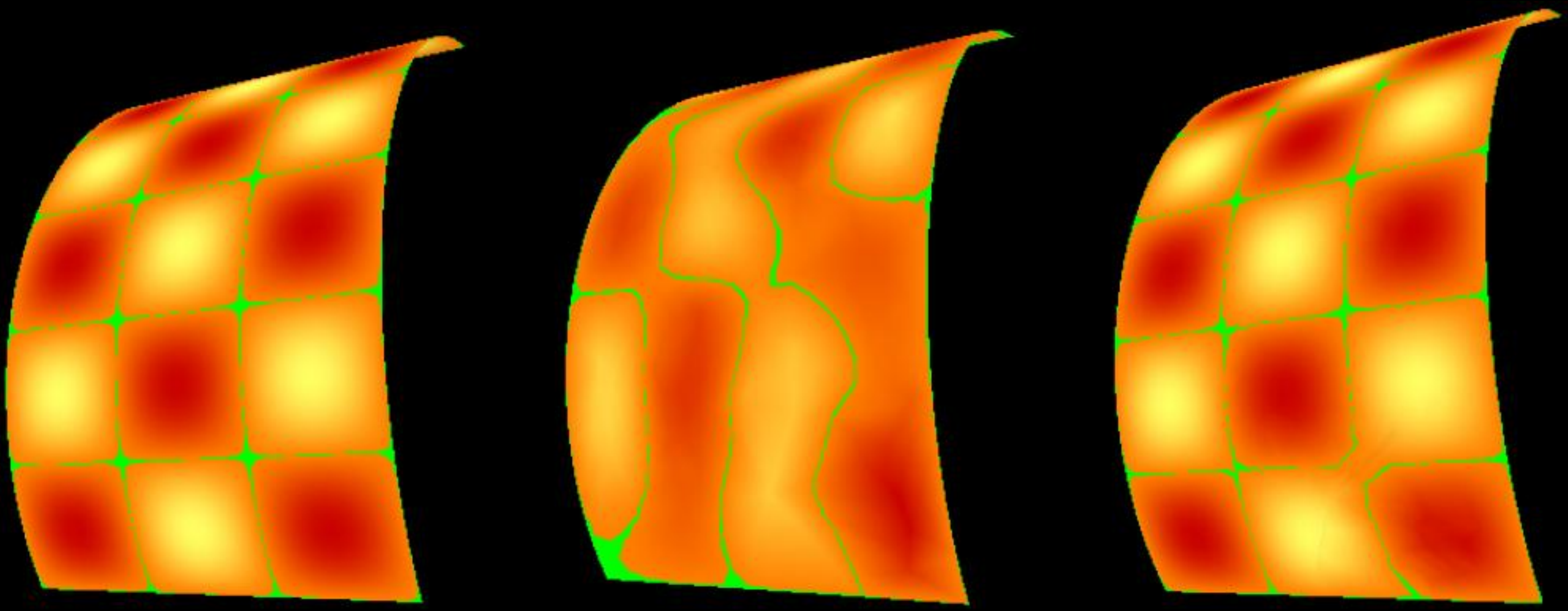
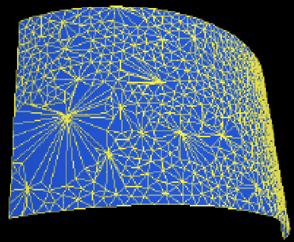


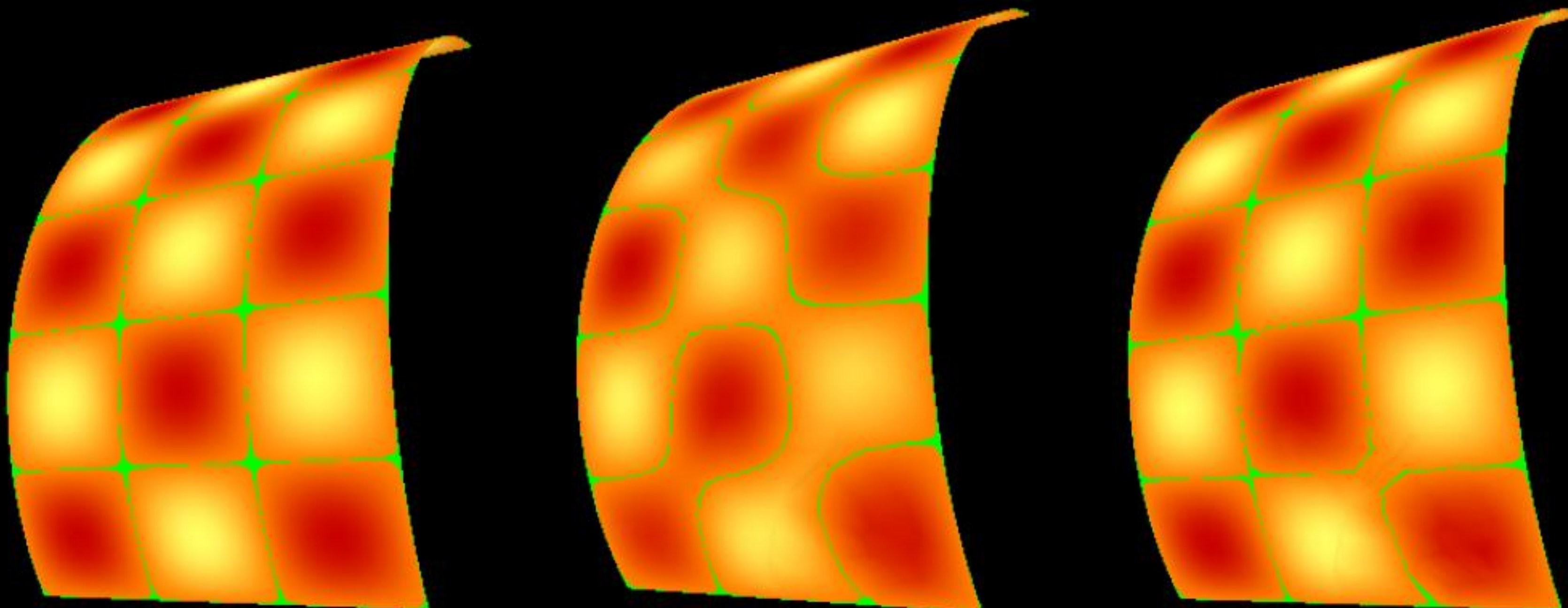
Non-Uniform Mesh (Efunc 23):



real linear cubic



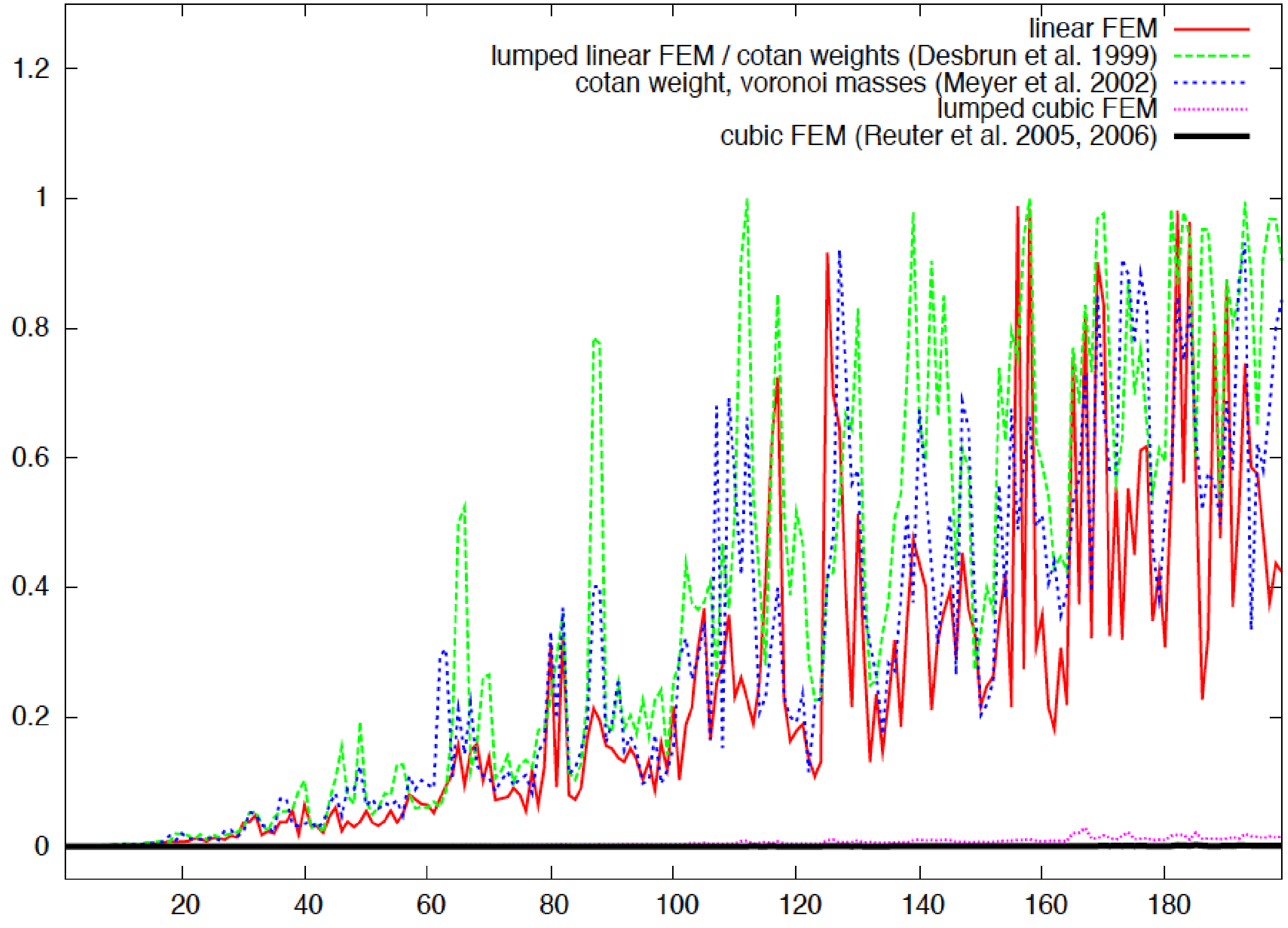
Non-Uniform Mesh same DOF (Efunc 23):



real linear cubic

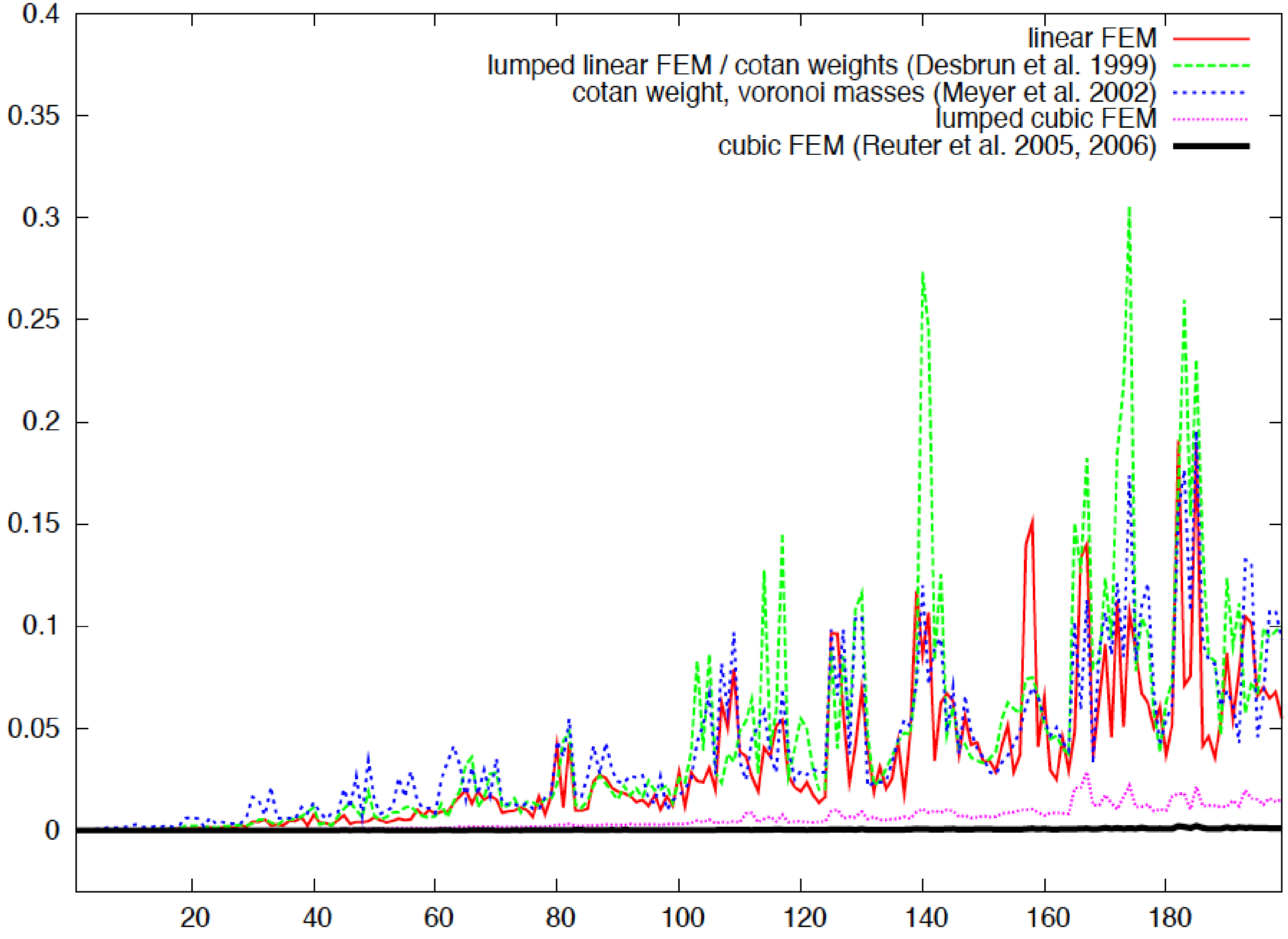
Comparison Eigenfunctions

Rectangle - Uniform Mesh - first 200 Eigenfunctions:

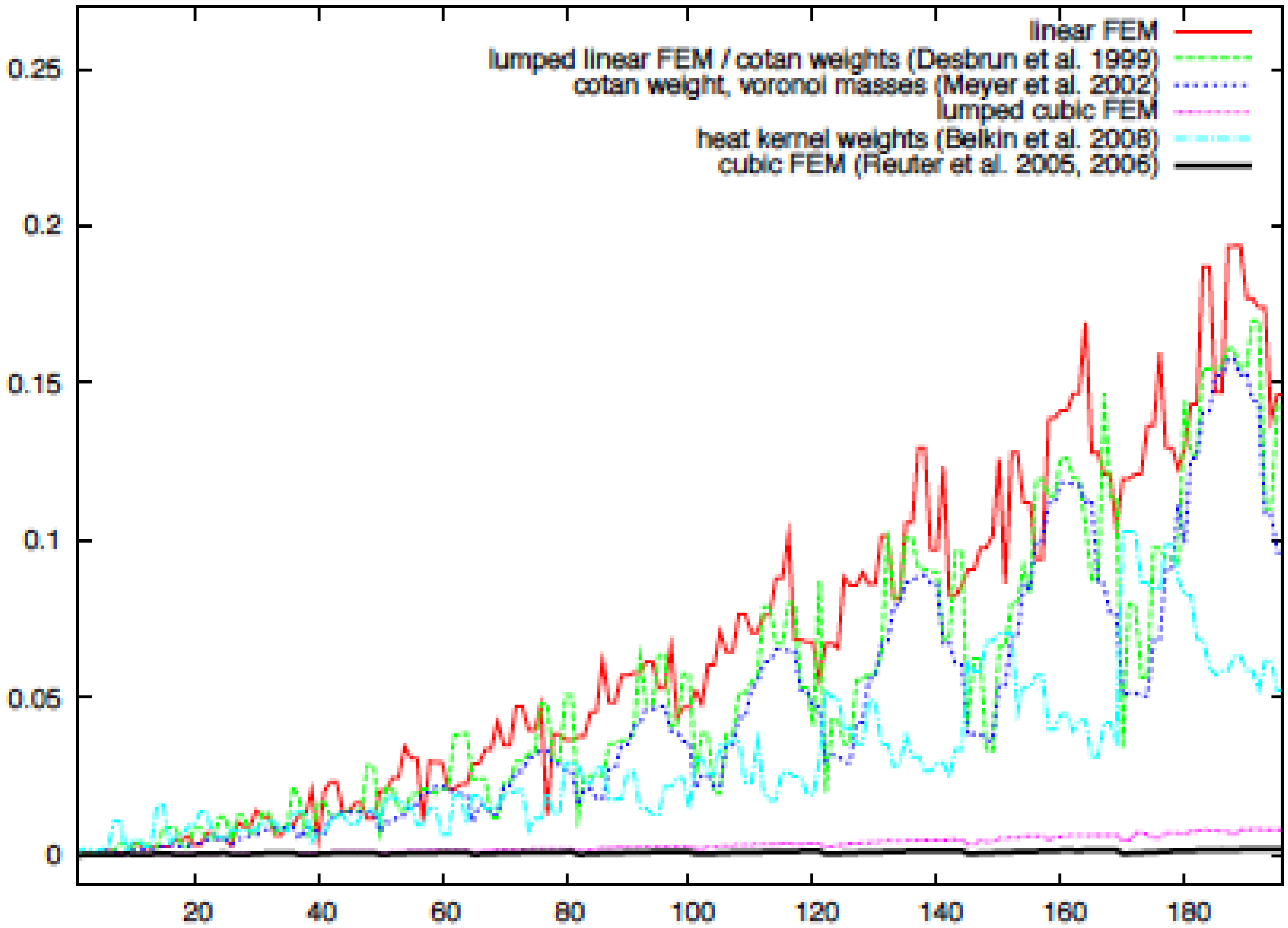


Comparison Eigenfunctions

Rectangle - Uniform Mesh (same DOF as cubic) - 200 EF:



Comparison on the sphere



Sphere – same DOF