Exercises.

Define the quadratic canonical form of a quadratic pseudo-boolean function to be

$$L + \sum_{1 \le i < jn} a_{ij} \bar{x}_i x_j$$

where L is linear with non-negative coefficients on x_i or $\bar{x}_i$ and not both x_i and $\bar{x}_i$ occur for any i.

Define cubic canonical form to be

$$L + \sum_{1 \le i < jn} a_{ij} \bar{x}_i x_j + \sum_{1 \le i < j < k \le n} a_{ijk} u_i u_j u_k$$

where the quadratic part is in quadratic canonical form, $a_{ijk} < 0$ and $u_i u_j u_k = x_i x_j x_k$ or $\bar{x}_i \bar{x}_j \bar{x}_k$.

1. Write the following pseudo-boolean function as a
   (a) posiform
   (b) polynomial
   (c) posiform with constant term equal to $\min_{\mathbf{x}} f(\mathbf{x})$.
   (d) posiform in cubic canonical form.

   Hence, write down a graph representation of the function. Is the function submodular?

$$f(\mathbf{x}) \equiv \begin{bmatrix} x_1 & x_2 & x_3 & \text{val} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & -5 \\ 0 & 1 & 1 & 6 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 1 & -7 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & -4 \end{bmatrix}$$

2. If it is not submodular, then change some signs of the quadratic terms in the cubic canonical form so that it is submodular. Then by hand find a maximal flow, and hence the values x_i that minimize the function.

3. **Using graph techniques**, simplify the following expression by putting it in the quadratic polynomial form.

$$1 + 5x_1 + 6x_2 - 3\bar{x}_3 + 2x_1x_2 - 3\bar{x}_1\bar{x}_2 - 4x_1x_3 + 2x_2\bar{x}_3 + 5\bar{x}_2\bar{x}_3$$

   Do this by modifiying a graph representation of the function via a flow.