Dynamic models for fixed-income portfolio management under uncertainty

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Abstract

We develop multi-period dynamic models for fixed-income portfolio management under uncertainty, using multi-stage stochastic programming with recourse. The models integrate the prescriptive stochastic programs with descriptive Monte Carlo simulation models of the term structure of interest rates.

Extensive validation experiments are carried out to establish the effectiveness of the models in hedging against uncertainty, and to assess their performance vis-à-vis single-period models. An application to tracking the Salomon Brothers Mortgage Index is reported, with very encouraging results. Results that establish the efficacy of the models in hedging against out-of-sample scenarios are also reported for an application from money management. The multi-period models outperform classical models based on portfolio immunization and single-period models. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Portfolio management problems can be viewed as multi-period dynamic decision problems where transactions take place at discrete points in time.
At each point in time the manager has to assess the prevailing market conditions—such as prices and interest rates—and the composition of the existing portfolio. The manager also has to assess possible future movements of interest rates, prices, risk premia, and cashflows from the securities. This information is incorporated into a sequence of buying or selling actions, and short-term borrowing or lending decisions. At the next point in time the portfolio manager has at hand a seasoned portfolio and is faced with a new set of possible future movements. Transactions must now be executed that incorporate the new information. In this paper we develop, implement, and validate multi-period dynamic models for the management of portfolios of fixed-income securities. The models are *stochastic programs with recourse*. They are more flexible than currently widely used models, and in extensive testing using both market and simulated data they are shown to perform well.

There are two distinct mathematical formalisms that capture the essence of portfolio management problems for equities or fixed-income securities. In the equities world, portfolio management is based on the notion of *diversification* introduced by Markowitz (1952) in his seminal paper. Diversification is achieved by minimizing the variance of returns during a holding period, subject to constraints on the mean value of return. There is only one time interval under consideration, future transactions are not incorporated and this is a single-period myopic model.

For fixed-income securities the key portfolio management strategy has been that of *portfolio immunization*: portfolios are developed that are hedged against small changes from the current term structure of interest rates. Such models are, again, single-period and ignore future transactions. Furthermore, they ignore the truly stochastic nature of interest rates, and merely hedge against (small) changes from currently observed data. The idea of immunization dates back to Reddington (1952) and it has been used extensively since the mid-1970s. Recently we have seen a shift towards the use of Markowitz’s ideas of diversification in the management of fixed-income portfolios too, especially when dealing with complex securities with embedded options (like callable or putable bonds, mortgage-backed securities, insurance products, etc.). The general framework has been developed by Mulvey and Zenios (1994). Documented applications of this framework in industrial settings are due to Holmer (1994), Worzel et al. (1994), and Koskosides and Duarte (1997). The mean–variance model is not directly applicable to fixed-income portfolio management, since it assumes that returns are normally distributed during the holding period. Securities with embedded options have skewed distributions, and the cited references have developed models based on expected utility maximization or mean-absolute deviation minimization. These models were shown in the references cited above to be effective portfolio management tools and to perform better than portfolio immunization. Nevertheless, they remain single-period models.
Under some conditions, single-period models are good approximations to reality. It is known, Mossin (1968), that solving a sequence of single-period models based on expected utility maximization produces the same answer as solving a multi-period, dynamic model. This is true for certain forms of utility functions and assuming that (i) returns are temporally independent, (ii) there are no transaction costs or taxes, and (iii) there is no temporal infusion or withdrawal of cash. For fixed-income securities these assumptions are violated. First, for pathdependent instruments we do not have temporal independence of returns. For example, we know that the price of the security converges to par as it approaches maturity. Second, fixed-income securities generate cash (coupon payments, prepayments, lapse, exercise of call options) that cannot be reinvested in the same security, and cash infusion is, in a sense, imposed upon the investor. Third, fixed-income securities are frequently held in order to fund some liability stream (e.g., insurance or pension claims) and cash withdrawal is an important aspect of the problem. Finally, transaction costs could be high for certain types of securities (e.g., collateralized mortgage obligations).

These observations motivated a recent interest in the development of multi-period dynamic models for portfolio management problems using stochastic programming. A fundamental contribution was made by Bradley and Crane (1972); they proposed a multi-stage model for bond portfolio management but no results were reported on the efficacy of the model. Kusy and Ziemba (1986) developed a multi-stage stochastic programming model for Bank asset/liability management in the presence of uncertainty in deposits and withdrawals from accounts. Their results indicate the superiority of the stochastic programming model over a linear programming approximation based on the use of mean values. Mulvey and Vladimirou (1992) developed a two-stage stochastic programming model for managing portfolios of equities. This model was used to demonstrate that stochastic programming formulations yield better returns than single-period models when transaction costs are present (Mulvey, 1993). The problem of managing portfolios of mortgage-backed securities was formulated as a stochastic programming problem by Zenios (1993) who also suggested pricing models to generate the data – interest rate scenarios and returns – that are needed to implement this model. Hiller and Eckstein (1994) implemented a simplified version of this model without rebalancing decisions at future time periods. One of the most recent works in this area is due to Carino et al. (1994) who report on a successful application of stochastic programming models to asset/liability management problems for insurance firms. Golub et al. (1995) report on a successful application to money management. An expository article on the use of optimization models in portfolio management, that contains extensive discussions on the merits of stochastic programming formulations, is Hiller and Schaack (1990), and an advanced textbook treatment is given in Censor and Zenios (1997).

In this paper we formulate a multi-stage stochastic programming model for problems in fixed-income portfolio management. We assume a binomial lattice
model of the evolution of interest rates (although any stochastic process for generating interest rates can be incorporated in the model). We then show how uncertainty can be resolved at different states of the binomial lattice, and how this information can be incorporated in the portfolio manager’s decisions. This model integrates descriptive interest rate models from the finance-economic literature, with prescriptive models for portfolio management from the management science literature. This is the scope of Section 2.

One of the key contributions of this paper is in implementing the model in different settings and providing extensive empirical evidence on its efficacy. Empirical studies on the superiority of stochastic programming formulations over other, simpler, models were at the writing of this paper virtually non-existent in the literature. It is presently unclear if the difficulties of developing stochastic programming models are compensated by improved performance. This paper shows superior efficacy of such models.

Our model is applied to two problems from fixed-income portfolio management: (i) The problem of tracking the Salomon Brothers Mortgage Index (Section 4), and (ii) a problem of money management with mortgage-backed securities (Section 5). In the former application we use market data over the period January 1989–December 1991 to validate the model, and compare it with a single-period model based on expected utility maximization. In the later application we conduct simulated experiments to demonstrate the efficacy of the model in achieving its stated goals even with out-of-sample data. We also carry out simulated dynamic games whereby the various competing models are used repeatedly over a rolling planning horizon, and they are re-run as new

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![Diagram](image.png)

Fig. 1. The evolution of scenarios on a binomial lattice of interest rates and the structure of the portfolio investment decisions.
information becomes available. Such games give a more accurate view of the performance of the models as opposed to the simulations whereby a model is used only at the beginning of the planning horizon, and its recommendations are followed until the end of the horizon. Results show, unambiguously in our opinion, that multi-stage models outperform significantly the single-period models that are currently in widespread use. While Mossin’s results encourage the use of myopic models in an idealized world, it is shown here that the absence of the idealized conditions could make a substantial difference in model performance.

We point out that other methods also exist that handle inter-temporal aspects of portfolio problems, such as the stochastic control method recently investigated by Brennan, et al. (1997), the scenario immunization of Dembo (1993), the vector forecasting strategies of Oester (1991), and robust optimization of Mulvey et al. (1995). We do not stop to review all these methods here, nor do we compare them with stochastic programming. Indeed, it is a topic worthy of investigation to determine the relative merits of these methods and assess these merits against the computational complexity of each.

2. Mathematical formulations

The model specifies a sequence of investment decisions at discrete points in time. Decisions are made at the beginning of each time period. The portfolio manager starts with a given portfolio and with a set of anticipated scenarios about future states of the economy. This information is incorporated into an investment decision. At the beginning of the next time period the manager has at hand a seasoned portfolio. The precise composition of the portfolio depends on the transactions at the previous decision point, and on the scenario realized in the interim. Another set of investment decisions are now made, and they incorporate the current status of the portfolio and new information about future scenarios.

We assume that scenarios unfold on a binomial lattice model of interest rates (see, e.g. Black et al., 1990). Such models are prevalent in the finance literature and provide key input data to our formulation. However, the model can be formulated using other models for scenario generation. We develop a general $T$-stage model, with decisions made at time instances $t = 0, 1, 2, \ldots, T$. At $t = 0$ all information is assumed known. Scenarios then unfold between periods $t$ and $t + 1$, and this information must be reflected in the portfolio decisions made at period $t$. Note that a sequence of scenarios at $t = 0, 1, 2, \ldots, T$, are linked together to form a path, and portfolio decisions are pathdependent. This feature adds substantial realism to the model but it also increases computational complexity. The value of the portfolio at the end of the planning horizon $T$ is evaluated under different scenarios. The objective is to determine the sequence of portfolio decisions that maximize some expected utility of terminal wealth under the various scenarios. A simple lattice for a 3-stage problem is illustrated in Fig. 1.
At instance $t_0$ two scenarios are anticipated, and by instance $t_1$ this uncertainty is resolved. Denote these scenarios by $s^0_0$ and $s^1_0$. At $t_1$ two more scenarios are anticipated, $s^0_1$ and $s^1_1$. A complete path is denoted by a tuple of scenarios. For example there are four paths from $t_0$ to $t_2$ denoted by the pairs $(s^0_0, s^0_1), (s^0_0, s^1_1), (s^1_0, s^0_1), (s^1_0, s^1_1)$.

2.1. The multi-stage stochastic programming model

The model is a nonlinear program maximizing the expected value of the manager’s utility function. Expectations are computed over the set of postulated scenarios. The value of the utility function is obtained from the solution of a multi-stage program, which is structured in such a way that decisions at every time period influence future decisions. Decisions also depend on the path of realized scenarios. This is the general framework of stochastic programming problems with recourse – see, e.g. Wets (1974) or Censor and Zenios (1997) – within which the portfolio model is developed.

2.2. Notation

We develop the model using cashflow accounting equations and inventory balance equations for each security category. Investment decisions are in dollars of face value. Define first parameters of the model:

$S_t$: set of scenarios anticipated at time $t$, for $t = 0, 1, \ldots, T$. We use $s_t$ to index scenarios from the set $S_t$. Let $l_t$ denote paths of scenarios that are resolved (i.e., all information becomes known) until period $t$, where $t = 1, 2, \ldots, T$. Paths are denoted by $l_t = (s_0, \ldots, s_t, \ldots, s_{t-1})$ where $s_t \in S_t$. With each path we associate a probability $\pi_{l_t}$, and let $P_t$ denote the set of all paths that can be constructed by combining scenarios from the scenario sets $S_0, S_1, \ldots, S_{t-1}$. Note that paths are not defined for $t = 0$ since all information is assumed known at this time instance.

$J$: set of available securities, with cardinality $m$.

$c_0$: riskless asset available at $t = 0$.

$(b_0)_j^m$: vector denoting the composition of the initial portfolio.

$(\xi_0)_j^m$: vectors of bid and ask prices, respectively, at $t = 0$. These prices are known with certainty.

$(\epsilon_t(l_t))_j^n$ for all $l_t \in P_t$ and $t = 1, 2, \ldots, T$: vectors of bid and ask prices, respectively, realized at $t$. These prices depend on the path of scenarios followed from 0 to $t$.

$(a_t(l_t, s_t))_j^n$ for $t = 0, 1, \ldots, T$: vectors of amortization factors during the interval from $t$ to $t+1$. The amortization factors indicate the fraction of outstanding face value of the securities at the end of the interval, and capture the effects of the embedded options, like prepayments, lapse, calls, etc. For example, a corporate security that is called during the interval has an amortization
factor of 0, and an uncalled bond a factor of 1. A mortgage security that experiences a 10% prepayment and that pays, through scheduled payments, an additional 5% of the outstanding loan has an amortization factor of 0.85. For path-dependent instruments these factors depend on the path followed up to \( t \), and are conditioned on the scenario to be realized during the interval from \( t \) to \( t + 1 \). For simple instruments (e.g., straight bonds) that are not path-dependent the index \( l \) is superfluous. At \( t = 0 \) we define \( z_d(l, s) = z_0(s_0) \), where \( z_0(s_0) \) is given conditioned only on scenarios anticipated at \( t = 0 \), but not on any path since all information prior to \( t = 0 \) is assumed known.

\[
(k_d(l, s))_{j=1}^m \quad \text{for } t = 0, 1, \ldots, T: \text{vectors of cash accrual factors during the interval from } t \text{ to } t + 1. \text{ These factors indicate cash generated per unit face value of the security due to scheduled payments and exercise of the embedded options, accounting for accrued interest. For example, a corporate security that is called at the beginning of a 1-year interval, in a 10% interest rate environment, will have a cash accrual factor of 1.10. These factors depend on the path followed up to } t \text{, and are conditioned on the scenario to be realized during the interval from } t \text{ to } t + 1. \text{ At } t = 0 \text{ we define } k_0(l, s) = k_0(s_0), \text{ where } k_0(s_0) \text{ is given conditioned only on the scenarios anticipated at } t = 0, \text{ but not on any path since all information prior to } t = 0 \text{ is assumed known.}
\]

\[
\rho_d(l, s) \quad \text{for } t = 0, 1, \ldots, T: \text{short term riskless reinvestment rates during the interval from } t \text{ to } t + 1. \text{ These rates depend on the path followed up to } t, \text{ and are conditioned on the scenario to be realized during the interval from } t \text{ to } t + 1. \text{ At } t = 0 \text{ we define } \rho_0(l, s) = \rho_0(s_0), \text{ where } \rho_0(s_0) \text{ is given conditioned on the scenarios anticipated at } t = 0, \text{ but not on any path since all information prior to } t = 0 \text{ is assumed known.}
\]

\[L_d(l_i): \text{liability payments at } t. \text{ Liabilities may depend on the path.}\]

Now define decision variables. We have four distinct decisions at each point in time: how much of each security to buy, sell, or hold in the portfolio, and how much to invest in the riskless asset.

2.2.1. First stage variables at \( t = 0 \)

We define the variables for decisions made at the beginning of the planning horizon, when all current information is assumed completely specified.

\[
(x_0)_{j=1}^m: \text{vector denoting the face value bought of each security.}
\]

\[
(y_0)_{j=1}^m: \text{vector denoting the face value sold of each security.}
\]

\[
(z_0)_{j=1}^m: \text{vector denoting the face value held in the portfolio.}
\]

\[v_0: \text{amount invested in the riskless asset.}\]

2.2.2. Time-staged variables

We now define variables for decision made at some future point in time \( t = 1, 2, \ldots, T \). These decisions are conditioned on the path followed up to time
instance $t$.

1. $\{x_t(l_t)\}_{j=1}^m$: vector denoting the face values bought of each security.
2. $\{y_t(l_t)\}_{j=1}^m$: vector denoting the face values sold of each security.
3. $\{z_t(l_t)\}_{j=1}^m$: vector denoting the face values held in the portfolio.

$v_t(l_t)$: amount invested in the riskless asset.

2.3. Model formulation

The constraints of the model express cashflow accounting for the riskless asset, and inventory balance for each security at each time period.

2.3.1. First-stage constraints

At the first stage (i.e., at time $t = 0$) all prices are known with certainty. The cashflow accounting equation specifies that the original endowment in the riskless asset, plus any proceeds from liquidating part of the existing portfolio, equals the amount invested in the purchase of new securities plus the amount invested in the riskless asset

$$ c_0 + \sum_{j=1}^m \xi_{0j} y_{0j} = \sum_{j=1}^m \xi_{0j} x_{0j} + v_0. $$

(1)

For each security $j \in J$ in the portfolio we have an inventory balance constraint

$$ b_{0j} + x_{0j} = y_{0j} + z_{0j}. $$

(2)

2.3.2. Time-staged constraints

Decisions made at any time period $t$, after $t = 0$, depend on the path $l_t$ and are conditioned on the scenarios $S_t$ anticipated at $t$. Hence, at each time instance $t$ we have one constraint for each path in $P_t$ and each scenario in $S_t$. These decisions also depend on the investment decisions made at previous periods.

Cashflow accounting constraints the amount invested in the purchase of new securities and the riskless asset to be equal to the income generated from the existing portfolio during the holding period, plus any cash generated from sales after liability payments are made. There is one constraint for each $l_t \in P_t$ and each $s_t \in S_t$:

$$ \rho_{t-1}(l_{t-1}, s_{t-1})v_{t-1}(l_{t-1}) + \sum_{j=1}^m k_{(t-1)j}(l_{t-1}, s_{t-1})z_{(t-1)j}(l_{t-1}) $$

$$ + \sum_{j=1}^m \xi_{tj}(l_t)y_{tj}(l_t) = v_t(l_t) + \sum_{j=1}^m \xi_{tj}(l_t)x_{tj}(l_t) + L_t(l_t). $$

(3)

We note that when $t = 1$ the subscript $t - 1$ refers to the first stage, whereby all information is assumed known, and by definition we have $v_{t-1}(l_{t-1}) = v_0$ and $z_{(t-1)j}(l_{t-1}) = z_{0j}$ for all $j = 1, 2, \ldots, m$. 

Inventory balance equations constrain the amount of each security sold or remaining in the portfolio to be equal to the outstanding amount of face value at the end of the previous period, plus any amount purchased at the beginning of the current period. There is one constraint for each security \( j \in J \) and for each \( l_t \in P_t \) and each \( s_t \in S_t \):

\[
z_{(t-1)j}(l_{t-1}, s_{t-1})z_{(t-1)j}(l_{t-1}) + x_{ij}(l_t) = y_{ij}(l_t) + z_{ij}(l_t).
\]

(4)

### 2.3.3. Calculation of terminal wealth

After the last portfolio decisions are made at time instance \( T \) and the anticipated scenarios \( S_T \) are observed we may calculate the terminal wealth of our investor. This wealth depends on the path \( l_T \) and the scenario \( s_T \), on the composition of the portfolio and the value of the securities at \( T \), and on accrued cashflows. All securities in the portfolio are marked-to-market, in agreement with recent FASB rules that require reporting market values of a portfolio, in addition to book values. The terminal wealth calculation is given by

\[
WT = W(z_T(l_T, s_T)) = \rho_T(l_T, s_T)w_T(l_T) + \sum_{j=1}^{m} \xi_T(l_T)z_T(l_T).
\]

(5)

### 2.3.4. Objective function

The objective function maximizes the expected utility of terminal wealth. The choice of maximizing a utility function of terminal wealth is based on capital growth theory, whereby investors seek to maximize the growth of their capital over a long planning horizon. Since our basic optimization model is a multi-period one, it is consistent to assume that the investors wish to follow a growth optimal strategy over the long run. In our numerical investigations we employ the logarithmic utility function which leads almost surely – under some assumptions – to investment policies with more capital in the long run than any other policy. See Hakansson and Ziemba (1995) for a discussion of capital growth theory and the properties of the related utility optimization models. The objective function is written as

\[
\text{Maximize } \sum_{l_{r+1} \in P_{r+1}} \pi_{l_{r+1}} \mathcal{U}(WT),
\]

(6)

where \( \pi_t \) is the probability for the \( l_t \)th path, \( WT \) is the terminal wealth as calculated in Eq. (5) and \( \mathcal{U}(\cdot) \) denotes the investor’s utility function.

### 2.4. A Monte Carlo procedure for scenario generation

In order to implement the model we need a procedure to generate scenarios for the parameters, i.e., bid/ask prices, amortization factors and cash accrual
factors. These scenarios are driven by the term-structure model. They also incorporate the risk premia inherent in each type of fixed-income security. The model parameters are consistent within each fixed-income sector, but are also consistent across different sectors.

We discuss the estimation of price scenarios using a binomial lattice. (Any other stochastic process for generating interest rates could be used.) Amortization and cash accrual factors – conditioned on paths \( l \), drawn from the lattice – are computed using a cashflow generation model appropriate for each type of security. At the end of each path \( l \), we also need to calculate the bid and ask prices of the securities. These prices are conditioned on the path and they reflect the market expectations on future scenarios that are anticipated after \( t \).

The calculation of one sample price scenario is illustrated in Fig. 2. Let \( \tau \) indicate the specific time instance when a portfolio decision is made, and

![Fig. 2. The calculation of a sample price scenario using a binomial lattice. A path from \( t = 0 \) to \( t = \tau \) passing through the state \( \sigma \) belongs to the set \( P_{\tau} \) used in the stochastic programming model. The shown sub-lattice describes the set of scenarios \( \Omega_{\sigma} \) used to estimate the prices at the end of the path in \( P_{\tau} \).]
assume an interest rate path \( l \) during the interval \((0, \tau)\) passes through the \( \sigma \)th state of the binomial lattice. We need to calculate the price \( P_{\tau}^l \) that will be observed given the interest rate path \( l \). This price is given by the expected present value of the cashflows \( C_t \) generated by the security from \( t = \tau \) until maturity \( T \). Expectations are computed over the scenarios emanating from the \( \sigma \)th state of the lattice, denoted by \( \Omega_{\sigma} = \{1, 2, \ldots, S_{\sigma}\} \), and the security cashflows are conditioned on scenarios from this set. Specifically we have

\[
P_{\tau}^l = \frac{1}{S_{\sigma}} \sum_{s=1}^{S_{\sigma}} \sum_{t=\tau+1}^{T} \prod_{i=\tau+1}^{t} \left(1 + \rho r_{i}^{s}\right)^{-1} C_{i}^{s}.
\]

Here, \( \rho \) is the risk-adjusted premium, a factor that incorporates the risk premia due to defaults, illiquidity, prepayments, lapse and other risks that are relevant to the fixed-income security; see Babbel and Zenios (1992). We point out that the cashflow values \( C_t \) may also incorporate other forms of risks that are present in the specific security that is priced, and the discount rates \( r_t \) may incorporate other risks than the volatility of the risk-free (i.e., treasury) rates. For example, Ben-Dov et al. (1992) discuss the calculation of risk-adjusted premia for mortgage securities accounting for prepayment risk; Assay, et al. (1993) discuss the calculation of risk premia for single-premium deferred annuities accounting for the lapse behavior; the calculation of risk-adjusted premia for callable bonds accounting for call and credit risk, has been discussed in Consiglio and Zenios (1997). Finally, we point out that the risk premia may change over time, in which case such temporal behavior would need to be modeled. At the present time there is no approach we know of capable of estimating time-dependent risk premia; see the discussion in Babbel and Zenios (1992).

There are two nested Monte Carlo simulation procedures in our model. One simulation generates the scenarios of interest rates \( S_t \) and creates the paths \( P_t \) that are used in the portfolio optimization model; the second procedure generates prices as described above, for each point \( t \) along the paths \( P_t \). In general we sample a few dozen scenarios to form the sets \( S_t \) for the portfolio optimization model, while several hundred scenarios are sampled to form the sets \( \Omega_{\sigma} \) for the pricing calculations.

3. Computational issues

We now touch upon computational issues in order to illustrate the efficiency with which state-of-the-art technology – hardware and numerical software – can solve the developed models. We look both at the scenario generation procedures and at the large-scale stochastic programming models. The Monte Carlo procedures for scenario generation are compute intensive. The evaluation of the price equation (7) requires the generation of a large set of
scenarios, projection of cashflows for each security type under each scenario, and expected present value calculations. This simulation has to be repeated for each path that is included in the optimization model, and at each time instance when a portfolio decision is made. For example, on a DECstation 5000 it takes approximately 9 min to compute a scenario of prices for a mortgage-backed security during its 360-month term. With the use of massively parallel procedures on a Connection Machine CM–2a the same computation is executed in 3–4 s, (Hutchinson and Zenios, 1991). The simulation procedures also parallelize naturally on networks of workstations or other parallel machines, see Censor and Zenios (1997), (p. 10) and almost linear speedups are achieved (Fig. 3). The calculation of all input data required for the implementation of the model takes several hours on a single DECstation 5000, but is executed within a few minutes.

Fig. 3. The performance of the Monte Carlo simulation procedure for pricing the fixed-income securities improves almost linearly on a variety of computer platforms with different parallel architectures, with either shared memory or distributed memory.
either on the CM-2 or the network of workstations. In general the scenario
generation procedures do not pose a major computational challenge.

The solution of large-scale multi-stage stochastic programming problems still
remains a computational challenge. In particular, if we develop a multi-stage
model with a decision point at every time instance when an instrument yields
income we have a very large number of stages and exponential growth of the size
of the problem and computational time. Hence, it is essential to limit the number
of stages; this can be done by having more short-term stages early on in the
decision making process, and fewer stages with longer intervals later on. For
example, Carino et al. (1994) and Nielsen and Zenios (1996) do not go beyond the
6-stage model, since by doing so there would be no discernible effect on the
first-stage decisions, which are the decisions actually implemented. Even so, the
problems remain large scale, but substantial progress has been made in recent
years in solving them with the design of novel algorithms (especially using interior
point methods) and with the use of parallel computations. Even if the computa-
tional requirements grow exponentially the solution time can improve linearly
with the number of processors or using specialized matrix factorization techniques
in interior point methods; for a small number of stages the exponential growth
does not overtake the linear improvements. We cite some recent results from the
literature to make the point that large-scale stochastic programs with a few stages
(six or so) and several thousand scenarios can be solved rather routinely. Czyzyk
et al. (1993) report linear growth in solution time of an interior point algorithm
with the size of the problem, when using a specialized matrix factorization of the
augmented system in an interior point algorithm; Nielsen and Zenios (1996)
report that ‘the massively parallel algorithm took about 4 times longer whenever
the number of scenarios was quadrupled, whereas OSL (our note: a linear
programming simplex-based software) took 14–16 times longer’; Jessup et al.
(1994) report – for the range of problems they solved with up to 130K scenarios
– that the exponential growth of the problem size does not overcome the linear
improvements realized with the special-purpose matrix factorization techniques
on a parallel computer; similar results are reported in Berger et al. (1994) with the
use of a decomposition algorithm. Censor and Zenios (1997) (Chapter 15) discuss
extensively the findings of recent literature in solving large-scale stochastic pro-
grams. We conclude by noting that the empirical results of this paper were
obtained by solving the large-scale models on a DECstation 5000; no detailed
solution timing reports have been compiled but solution times would range from
several minutes (20min or so) to a few hours (3 h) depending on the model.

4. Model validation: tracking a fixed-income index

We develop a two-stage stochastic programming model for tracking the
Salomon Brothers Mortgage Index. This index is commonly used as the
benchmark against which the performance of mortgage portfolios is measured. It captures the return of approximately 300,000 mortgage pools with a market value of more than $1 trillion. The index is composed of 118–144 representative generic securities.

The two-stage model assumes a holding period of two months, and determines the composition of the portfolio at $t = 0$ (i.e., $x_0, y_0, z_0$) taking into account potential portfolio rebalancing decisions at $t = 1$ month and the return of the portfolio at the end of the 2-month holding period. Decisions are made based on scenarios of holding period returns that are computed using the Monte Carlo simulation described above. Cashflows are obtained using the mortgage prepayment model of Kang and Zenios (1992); for a discussion on the accuracy of this model see Golub and Pohlman (1993). The objective function is to maximize the expected utility of excess return at the end of the two-month holding period, using a logarithmic utility function. Excess return is defined as the ratio of portfolio return to index return.

The precise formulation of the indexation model is based on the general model given in Section 2, with the following modifications. No liabilities $L_t$ are present and there is no investment in the riskless asset, i.e., variable $v_t$ is eliminated from the indexation model. All cash generated by the portfolio—coupon payments, prepayment, etc.—are invested in the short-term riskless rate, and the value of this investment is embedded in the calculation of holding period returns for each security. All other variables and parameters are as defined in Section 2.2 and the constraints are those in Sections Section 2.3.1 and Section 2.3.2. In addition all variables are constrained to be nonnegative, so that no short sales or borrowing are allowed.

The objective function maximizes the expected utility of excess return of the portfolio over the index, and is formulated here. The initial value of the portfolio is given as $V_0 = v_0 + \sum_{j=1}^{m} b_{0j} \xi_{0j}$. The terminal value of the portfolio at period $T$ is conditioned on the path $l_T$ and the scenario $s_T$, and is given by

$$V_T^{(l_T, s_T)} = \sum_{j=1}^{m} z_T(l_T) \xi_T(l_T)(1 + R_j(s_T)), \quad (8)$$

where $R_j(s_T)$ is the return of the $j$th security during the $T$-th period, under scenario $s_T$. Then the return of the portfolio is given by

$$R_p^{(l_T, s_T)} = \frac{V_T^{(l_T, s_T)}}{V_0} - 1. \quad (9)$$

If we let $I^{(l_T, s_T)}$ denote the return of the index during the planning period $T$ under the paths indicated in the superscript, then the terminal wealth $W_T$ of the indexation model is defined as the excess return of the portfolio over the index. It is given by $W_T = R_p^{(l_T, s_T)} / I^{(l_T, s_T)}$, and this expression is used in the evaluation of
the objective function (6), where \( \mathcal{U}(\cdot) \) is taken to be the natural logarithmic function.

The model was backtested using data for the period January 1989–December 1991. Historical information is available for the starting and ending price of each security in the index during this period. Monthly term structure data are also available. Our testing proceeds as follows. The term structure of 1 January 1989, is used to fit a binomial lattice and calculate holding period returns until 1 February and 1 March 1989. The optimization model is then solved to select a portfolio that is kept for a month. After one month (i.e., at the end of January) the binomial lattice is recalibrated using the term structure of 1 February 1989, holding period returns are calculated until 1 March and 1 April 1989, and the process is repeated. For any security added or dropped from the portfolio a transaction fee of 1/16th basis points is charged. The return of the portfolio at the end of each month is calculated based on \textit{ex post} returns that are available from the Salomon data. We emphasize that simulation date are used to build the model and select the portfolio, but observed market prices are used to evaluate the performance of the portfolios.

The stochastic programming model is compared to the single-period model developed by Worzel et al. (1994) for Metropolitan Life Insurance. Worzel et al. (1994) describe two models in their paper; our comparisons are made with their utility maximization model which also uses a logarithmic utility function. During the backtesting period the Salomon index realized an (annualized) return of 14.05%, the single-period model realized a return of 14.18%, while the two-stage stochastic program realized a return of 15.10%. Fig. 4 shows the

![Graph showing tracking error for single-period and two-stage models.](image)

Fig. 4. Tracking the Salomon Brothers Mortgage Index: tracking error for the single-period and the two-stage models.
tracking error of the two-stage stochastic programming model and the tracking error of the single-period model. We observe that the multi-stage model tracks the index very closely and outperforms the single-period model by a large margin. The returns of both the single-period and multi-period models have lower variance than the return of the index during the testing period.

5. Model validation: money management in the mortgage market

Next we develop multi-stage models for the following problem; see Golub et al. (1995). A money management firm wishes to offer a financial product that ‘promises’ a target return during the holding period. The target is set at above the comparable treasury rates, and it is surmised that with careful investments in fixed-income securities this target will be achieved without exposing investors to the risks of the underlying securities. The precise model is obtained from the model of Section 2 by specifying the liability $L_T$ at the end of the holding period to be equal to the prespecified target. We built a 3-stage model, with portfolio rebalancing decisions at the middle of the planning horizon and at the end of the horizon. All variables and parameters are as defined earlier, and terminal wealth is any excess return realized over and above the promised target. In addition all variables are constrained to be nonnegative, so that no short sales or borrowing are allowed.

Our specific problem is to fund a product with a target return of 7.72%, the return on the 3-year Treasury at 1 April 1991, by investing in mortgage-backed securities. We develop a single-period model, a three-stage stochastic program and a duration matched (i.e., immunized) portfolio for this problem. The single-period model is an expected utility maximization model using a logarithmic utility function. Interest rate scenarios for the model were generated using a binomial lattice from the term structure of 1 April 1991 (Fig. 5) and constant volatility of 20%.

We perform two sets of experiments in order to evaluate the effectiveness of the models. First, we test the model using out-of-sample data, i.e., using interest rate scenarios that are not included in the scenario set of the optimization model. We expect the selected portfolios to perform well even under eventualities that were not explicitly accounted for in the optimization model and the first experiment tests this hypothesis. The second test is based on the execution of simulated dynamic games. The models are compared against each other when used to actively manage a portfolio (in a simulated environment) over a 3-year horizon.

5.1. Model performance using out-of-sample data

Both the single-period and the multi-period models are built based on the interest rate scenarios shown in Fig. 6 (call this set $S1$). The multi-period model
Fig. 5. The term structure of 1 April 1991.

Fig. 6. Sample scenario set $S_1$ used to build the models.
allows for portfolio rebalancing at the 18th month. Both models will perform well if the data used to build the model are realized in practice. But the efficacy of a model hinges on its ability to hedge against eventualities that were not explicitly accounted for. The returns of the portfolios selected by the two models are therefore evaluated based on the set \( S_2 \) of 100 randomly generated scenarios shown in Fig. 7, which is different from the sample scenario \( S_1 \) used to build the model.

Fig. 8 illustrates the performance of both portfolios, based on the sample scenarios in \( S_1 \) and using out-of-sample scenarios in \( S_2 \). We observe that both models achieve the target return. They do so for the out-of-sample data as well. The recommendations of the model are robust with respect to changes in the input data. We mention that the conclusions of this experiment are also supported by the more recent work of Dupacova et al. (1997) who use contamination techniques for the postoptimality analysis of bond portfolio optimization models.

Table 1 provides summary statistics of the return profiles of the three portfolio management strategies (single-period, multi-period and immunization) under all scenarios in both sets \( S_1 \) and \( S_2 \). The three-stage model produces a portfolio with a higher mean and higher standard deviation than the single-period model. In order to develop a measure of performance that incorporates
Fig. 8. Performance of the single-period model (2s), the multi-stage stochastic programming model (3s), and the immunized portfolio (Im) using both the sample scenarios $S_1$ and the out-of-sample scenarios $S_2$.

Table 1
Summary statistics of the return profiles of three portfolio management strategies under the scenarios of both sets $S_1$ and $S_2$

<table>
<thead>
<tr>
<th></th>
<th>Single-period</th>
<th>Multi-period</th>
<th>Immunization</th>
</tr>
</thead>
<tbody>
<tr>
<td>min</td>
<td>7.76</td>
<td>8.03</td>
<td>3.15</td>
</tr>
<tr>
<td>max</td>
<td>9.21</td>
<td>10.38</td>
<td>10.10</td>
</tr>
<tr>
<td>mean</td>
<td>8.27</td>
<td>8.85</td>
<td>8.86</td>
</tr>
<tr>
<td>std. dev.</td>
<td>0.25</td>
<td>0.53</td>
<td>0.84</td>
</tr>
</tbody>
</table>

the distribution of returns of each portfolio we use a utility function to calculate certainty equivalent excess wealth. This is the wealth that will be accepted by an investor as being indistinguishable from a portfolio with a given probability distribution of wealth. Two portfolios with different distributions of wealth can be compared based on their certainty equivalent excess wealth: the one with the higher certainty equivalent excess wealth is preferable. Using the logarithmic utility function we calculate a certainty equivalent excess wealth of $2.81$ for the three-stage model, compared to a certainty equivalent excess wealth of $2.72$ for the single-period model. Both models are effective in meeting the target, but the
Fig. 9. Sets of random scenarios used as input to the portfolio optimization models during one run of the dynamic games, at time instances $t = 0, 6, 12, 18, 24, 30$. Horizontal axis gives month of planning horizon; vertical axis gives interest rate.
Fig. 10. The interest rate scenarios used for 15 simulated runs of the dynamic games.
multi-stage model has better certainty equivalent excess wealth than the single-period model.

As a benchmark for comparison we also develop an immunized portfolio that is duration matched against the target liability. Durations are computed by shifting the term structure of Fig. 5 by ± 50 basis points. The return of the immunized portfolio does not achieve the target return of 7.72% for some of the scenarios in sets $S_1$ and $S_2$ (Fig. 8). Both the single-period and the multi-period models are more effective than portfolio immunization.

5.2. Dynamic games with portfolio management strategies

We conduct next simulated dynamic games whereby a single-period and a multi-period model are used to manage a portfolio over a 3-year horizon. A dynamic game proceeds as follows. At the beginning of the time horizon both models are run using as input a set of interest rate scenarios such as those illustrated in Fig. 9 ($t = 0$). The time clock is then advanced by 6 months, and an interest rate scenario is randomly generated for the time interval $[0, 6]$. The performance of both portfolios for this random scenario during the interval is recorded. This models are then rerun to generate new portfolios at $t = 6$, using as input a set of interest rate scenarios, such as those illustrated in Fig. 9. The time clock is again advanced by 6 months, and the procedure is repeated until the end of the 3-year period, and the total (annualized) return of the portfolio is tabulated.

Table 2
Annualized return over a 3-year horizon for two portfolios that are actively managed using single-period and multi-stage models, under 15 randomly generated interest rate scenarios

<table>
<thead>
<tr>
<th>Scenario (From Fig. 10)</th>
<th>Single-period model</th>
<th>Multi-stage model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.28</td>
<td>7.74</td>
</tr>
<tr>
<td>2</td>
<td>11.63</td>
<td>8.93</td>
</tr>
<tr>
<td>3</td>
<td>8.23</td>
<td>8.03</td>
</tr>
<tr>
<td>4</td>
<td>8.11</td>
<td>8.28</td>
</tr>
<tr>
<td>5</td>
<td>8.65</td>
<td>9.63</td>
</tr>
<tr>
<td>6</td>
<td>8.49</td>
<td>9.63</td>
</tr>
<tr>
<td>7</td>
<td>8.41</td>
<td>8.79</td>
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<td>8</td>
<td>8.69</td>
<td>8.93</td>
</tr>
<tr>
<td>9</td>
<td>8.11</td>
<td>8.64</td>
</tr>
<tr>
<td>10</td>
<td>8.45</td>
<td>8.96</td>
</tr>
<tr>
<td>11</td>
<td>8.17</td>
<td>8.48</td>
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<tr>
<td>13</td>
<td>8.55</td>
<td>9.38</td>
</tr>
<tr>
<td>14</td>
<td>8.30</td>
<td>8.52</td>
</tr>
<tr>
<td>15</td>
<td>8.28</td>
<td>8.73</td>
</tr>
</tbody>
</table>
Table 3
Summary statistics of the return profiles of two portfolios that are actively managed using single-period and multi-period models over a 3-year period, under 15 simulations of the dynamic games

<table>
<thead>
<tr>
<th></th>
<th>Single-period</th>
<th>Multi-period</th>
</tr>
</thead>
<tbody>
<tr>
<td>min</td>
<td>8.11</td>
<td>7.74</td>
</tr>
<tr>
<td>max</td>
<td>11.61</td>
<td>9.78</td>
</tr>
<tr>
<td>mean</td>
<td>8.60</td>
<td>8.83</td>
</tr>
<tr>
<td>std. dev.</td>
<td>0.86</td>
<td>0.59</td>
</tr>
</tbody>
</table>

portfolio is recorded. This completes one simulated run of a dynamic game. The game is repeated, starting at \( t = 0 \) and running again the model every 6 months using different scenarios. The results of this section were obtained by repeating the game 15 times for the interest rate scenarios illustrated in Fig. 10.

Table 2 summarizes annualized returns under the 15 scenarios and Table 3 reports statistics of these returns. We observe that both models achieve the target return under all scenarios. The three-stage model outperforms the single-period model for 12 out of the 15 scenarios. Computing the certainty equivalent excess wealth of both portfolios we get \( $2.24 \) for the three-stage model, compared to \( $2.00 \) for the single-period model.

6. Conclusions

We have developed multi-stage, dynamic models for fixed-income portfolio management, using stochastic programming with recourse. Such models are extremely versatile; they incorporate transaction costs, cash infusions or withdrawals, and do not depend on assumptions for temporal independence or normality of returns. They extend substantially single-period models. One of the key conclusions is that the developed models could perform very well in practice. Backtesting results have been very encouraging. The models also generate recommendations that are robust with respect to changes in the input scenarios. These recommendations were shown to perform better than the portfolios generated by single-period models.

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References


