Improving the convergence of a numerical method for the generalized Newtonian flow problem

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Abstract

In the paper we consider a boundary value problem describing the stationary motion of generalized Newton fluid in a bounded domain. We propose an iterative method for numerical solution of the considered problem and we give an estimate of it's convergence. To improve the rate of convergence we use variable iterative method parameters, and we estimate the convergence of the iterative method in this case.

1 Introduction

We consider a boundary value problem describing a stationary motion of a generalized Newton fluid with the viscosity function χ:

$$\begin{array}{l} -\mathrm{div}\chi(|\nabla \mathbf{u}|)\nabla \mathbf{u} + \nabla p = \mathbf{f} \\ \mathrm{div}\mathbf{u} = 0, \quad \mathbf{u}|_{\partial\Omega} = \mathbf{0} \end{array} \tag{1}$$

Here $\mathbf{u} = (u_1, u_2)$, $\chi(a) \geq 0$, $a \geq 0$, and $\chi(x) = 1 + k(x)$. It's essential for us, that the function $k(x)$ is unbounded near the point $x = 0$ and the function $k(x)x$ is an increasing function.

Discretization of the linear case of (1) (i.e., $k(x) = \mathrm{const}$) is described, for example, in [2], [9]. The proof of convergence of the solution to the linear problem is presented in [8]. In [12] approximation of (1) on a shifted grid is described.

For the linear case there exists a wide range of numerical methods for solving the obtained system of linear equations (see [1], [4], [13] and the bibliography there). For the nonlinear case we proposed in [5] an iterative method for the solution of (1), but this method needs to solve a nonlinear equation in every iteration. There exists a method, proposed in [11], which needs a Laplace operator inversion on every step only, but for the case of constant parameters we can't obtain the given problem solution with any accuracy. There is no estimate of the rate of convergence of this method.

In this paper we propose a two-level partially implicit iterative method with variable coefficients for the problem (1), which needs a Laplace operator inversion in every step only. We give the estimates of it's convergence. Also we give an explicit estimate for the rate of convergence of the same method with constant coefficients. Note, that in the last case the method converges up to some given accuracy only.

2 Statement of the problem

We consider a boundary value problem describing the stationary motion of a generalized Newton fluid with the viscosity function χ:

$$\begin{aligned} -\mathrm{div}\chi(|\nabla \mathbf{u}|)\nabla \mathbf{u} + \nabla p = \mathbf{f} \\ \mathrm{div}\mathbf{u} = 0, \quad \mathbf{u}|_{\partial\Omega} = \mathbf{0} \end{aligned} \tag{2}$$

Again we assume the domain Ω to be bounded and Liptshitz. Here $\mathbf{u} = (u_1, u_2)$ belongs to $\mathbf{H}_0^1 \equiv \{\mathbf{u} \in \mathbf{W}_2^1(\Omega)^2 : \mathbf{u}|_{\partial\Omega} = 0\}$, p belongs to $\mathbf{L}_2/\mathbf{R} \equiv \{p \in \mathbf{L}_2(\Omega)^2 : (p, 1) = 0\}$, $\mathbf{f} \in \mathbf{H}^{-1}$, where $\mathbf{H}^{-1}$ is the dual space to $\mathbf{H}_0^1$. The functions $\chi(a) \geq 0$, $a \geq 0$, and $\chi(x) = k_0 + k(x)$ are usually subject to the following constraints:

$$\begin{aligned} &(\mathbf{a}) \quad s(a) = k(a)a \text{ is an increasing function for } a > 0; \\ &(\mathbf{b}) \quad s(a) \to 0 \text{ as } a \to 0; \\ &(\mathbf{c}) \quad k(a) \text{ is a decreasing function for } a > 0; \\ &(\mathbf{d}) \quad k_0 > 0. \end{aligned} \tag{3}$$

Note, that the following LBB inequality holds (see [1], [6])

$$\|p\| \leq C_0 \sup_{w \in \mathbf{H}_0^1} \frac{(p, \mathrm{div}\mathbf{w})}{\|\mathbf{w}\|_1} \tag{4}$$

Under these conditions the problem has a unique weak solution (see [10]).

Remark 1. In contrast to the Navier-Stokes equations, where domination of nonlinear terms is assotiated with *high* velocity gradients, this problem causes main difficulties in the case of *low* velocity gradients (the function $k(a)$ is unbounded for $a \to 0$, in general).

Remark 2. We note a general choice (not the only possible, of course) of $k(a) = a^{-\alpha}$ with $\alpha \in [0, 1)$, satisfying all the conditions required.

In [11], [12] several iterative methods for solving this problem were proposed. In particular, consider the partially implicit iterative algorithm:

$$\begin{cases} \mathbf{B}\mathbf{v}_\tau - k_0\Delta\hat{\mathbf{v}} - \Delta_k\mathbf{v} + \nabla\hat{q} = \mathbf{f} \\ \beta\tau q_\tau + \mathrm{div}\hat{\mathbf{v}} = 0 \end{cases} \tag{5}$$

where $\mathbf{v} = \mathbf{v}^n$, $\hat{\mathbf{v}} = \mathbf{v}^{n+1}, \mathbf{v}_\tau = \dfrac{\hat{\mathbf{v}} - \mathbf{v}}{\tau}$, and $\Delta_\chi\mathbf{u} = \mathrm{div}\chi(|\nabla\mathbf{u}|)\nabla\mathbf{u}$; β, τ are the iteration parameters. The preconditioner $\mathbf{B}$: $\mathbf{H}_0^1 \to \mathbf{H}^{-1}$ is some symmetric positive definite operator that will be defined later.

To derive an explicit formula for the iterative process (5) we transform it to the form

$$\begin{cases} \mathbf{C}\hat{\mathbf{v}} = \tau\mathbf{f} + \mathbf{B}\mathbf{v} + \tau\Delta_k\mathbf{v} - \tau\nabla q \\ \hat{q} = q - \beta^{-1}\mathrm{div}\hat{\mathbf{v}} \end{cases} \tag{6}$$

where $\mathbf{C} = \mathbf{B} - \tau k_0\Delta - \frac{\tau}{\beta}\nabla\mathrm{div}$.

In [11] the following theorem about the convergence of iterative process (5) is proved

Theorem 1. *Let $\partial\Omega$, $\mathbf{f}$ and k satisfy necessary for further considerations conditions (see [11]) and the following relations are valid for some positive λ_0, λ_1*

$$-\lambda_0\Delta \le \mathbf{B} \le -\lambda_1\Delta.$$

Then for any $\varepsilon > 0$ there exist positive constants τ and β such that the error of the iterative process (5) $\delta^n = \sqrt{\|\mathbf{y}^n\|^2 + \beta\tau\|r^n\|^2}$ tends to zero as $n \to \infty$ with the rate of a geometric progression, while $\delta^n > \varepsilon$. Here $\mathbf{y}^n$, r^n are the errors of velocity and pressure respectively.

Numerical experiments confirm this theorem. We will give some estimates on τ and β.

3 Estimate of convergence in the case of constant iteration parameters

Let us introduce the errors functions of iterative process $\mathbf{y} = \mathbf{v} - \mathbf{u}$ and $r = q - p$. From (5) we obtain the following relations

$$\begin{cases} \mathbf{B}\mathbf{y}_\tau - k_0\Delta\hat{\mathbf{y}} - \Delta_k\mathbf{v} + \Delta_k\mathbf{u} + \nabla\hat{r} = \mathbf{0} \\ \beta\tau r_\tau + \mathrm{div}\hat{\mathbf{y}} = 0 \end{cases} \tag{7}$$

Let us multiply the first equation from (7) by $\tau\hat{\mathbf{y}}$ and the second one by $\hat{r}$, then we have

$$\begin{cases} (\mathbf{B}\hat{\mathbf{y}},\hat{\mathbf{y}}) - (\mathbf{B}\mathbf{y},\hat{\mathbf{y}}) + \tau k_0(\nabla\hat{\mathbf{y}},\nabla\hat{\mathbf{y}}) \\ \quad +\tau(k(|\nabla\mathbf{v}|)\nabla\mathbf{v} - k(|\nabla\mathbf{u}|)\nabla\mathbf{u}, \nabla\hat{\mathbf{v}} - \nabla\mathbf{u}) \\ \quad +\beta\tau\|\hat{r}\|^2 - \beta\tau(r,\hat{r}) = 0 \\ \beta(\hat{r},\hat{r}) - \beta(r,\hat{r}) + (\mathrm{div}\hat{\mathbf{y}},\hat{r}) = 0 \end{cases} \tag{8}$$

In the same way as in [3] we obtain from the first equation of (7) the estimate

$$C_0^{-1}\|\hat{r}\| \leq \|\mathbf{y}_\tau\|_B + k_0\|\hat{\mathbf{y}}\|_1 + \|k(|\nabla\mathbf{v}|)\nabla\mathbf{v} - k(|\nabla\mathbf{u}|)\nabla\mathbf{u}\|$$

or

$$\|\hat{r}\|^2 \leq 3C_0^2(\|\mathbf{y}_\tau\|_B^2 + k_0^2\|\hat{\mathbf{y}}\|_1^2 + \|k(|\nabla\hat{\mathbf{v}}|)\nabla\hat{\mathbf{v}} - k(|\nabla\mathbf{u}|)\nabla\mathbf{u}\|^2) \tag{9}$$

From the first equation of (8) it follows

$$\begin{aligned} &\|\hat{\mathbf{y}}\|_B^2 + \|\hat{\mathbf{y}} - \mathbf{y}\|_B^2 + 2\tau k_0\|\hat{\mathbf{y}}\|_1^2 \\ &+2\tau(k(|\nabla\mathbf{v}|)\nabla\mathbf{v} - k(|\nabla\mathbf{u}|)\nabla\mathbf{u}, \nabla\hat{\mathbf{v}} - \nabla\mathbf{u}) \\ &+\beta\tau\|\hat{r}\|^2 + \beta\tau\|\hat{r} - r\|^2 = \|\mathbf{y}\|_B^2 + \beta\tau\|r\|^2 \end{aligned} \tag{10}$$

If we transform (10) to the form

$$\begin{aligned} &\|\hat{\mathbf{y}}\|_B^2 + \|\hat{\mathbf{y}} - \mathbf{y}\|_B^2 + 2\tau k_0\|\hat{\mathbf{y}}\|_1^2 \\ &+2\tau(k(|\nabla\mathbf{v}|)\nabla\mathbf{v} - k(|\nabla\mathbf{u}|)\nabla\mathbf{u}, \nabla\mathbf{v} - \nabla\mathbf{u}) \\ &+2\tau(k(|\nabla\mathbf{v}|)\nabla\mathbf{v} - k(|\nabla\mathbf{u}|)\nabla\mathbf{u}, \nabla\hat{\mathbf{v}} - \nabla\mathbf{v}) \\ &+\beta\tau\|\hat{r}\|^2 + \beta\tau\|\hat{r} - r\|^2 = \|\mathbf{y}\|_B^2 + \beta\tau\|r\|^2 \end{aligned}$$

then according ε-inequality we obtain the relation:

$$\begin{gathered}
\|\hat{\mathbf{y}}\|_B^2 + \|\hat{\mathbf{y}} - \mathbf{y}\|_B^2 + 2\tau k_0 \|\hat{\mathbf{y}}\|_1^2 \\
+2\tau(k(|\nabla\mathbf{v}|)\nabla\mathbf{v} - k(|\nabla\mathbf{u}|)\nabla\mathbf{u}, \nabla\mathbf{v} - \nabla\mathbf{u}) \\
-\tau\varepsilon\|k(|\nabla\mathbf{v}|)\nabla\mathbf{v} - k(|\nabla\mathbf{u}|)\nabla\mathbf{u}\|^2 - \tau/\varepsilon\|\hat{\mathbf{v}} - \mathbf{v}\|_1^2 \\
+\beta\tau\|\hat{r}\|^2 + \beta\tau\|\hat{r} - r\|^2 \le \|\mathbf{y}\|_B^2 + \beta\tau\|r\|^2
\end{gathered} \tag{11}$$

Let us multiply inequality (9) by $\mu\tau^2$ and add the result to (11). Then we have

$$\begin{gathered}
\|\hat{\mathbf{y}}\|_B^2 + (2\tau k_0 - 3C_0^2 k_0^2 \tau^2 \mu)\|\hat{\mathbf{y}}\|_1^2 + \left(1 - 3C_0^2\mu - \tfrac{\tau}{\varepsilon\lambda_0}\right)\|\hat{\mathbf{y}} - \mathbf{y}\|_B^2 \\
+2\tau\Big[\xi(\nabla\mathbf{v} - \nabla\mathbf{u}, \nabla\mathbf{v} - \nabla\mathbf{u}) + (k(|\nabla\mathbf{v}|)\nabla\mathbf{v} - k(|\nabla\mathbf{u}|)\nabla\mathbf{u}, \\
(1 - (\tfrac{3}{2}C_0^2\mu\tau + \tfrac{\varepsilon}{2})k(|\nabla\mathbf{v}|))\nabla\mathbf{v} - (1 - (\tfrac{3}{2}C_0^2\mu\tau + \tfrac{\varepsilon}{2})k(|\nabla\mathbf{u}|))\nabla\mathbf{u})\Big] \\
+(\beta\tau + \tau^2\mu)\|\hat{r}\|^2 + \beta\tau\|\hat{r} - r\|^2 \le \|\mathbf{y}\|_B^2 + 2\tau\xi\|\mathbf{y}\|_1^2 + \beta\tau\|r\|^2
\end{gathered} \tag{12}$$

where μ, ξ are any positive constants, λ_0 is introduced in the Theorem 1.

To continue the proof we need the following lemma that was proved in [11].

Lemma 1. *There exists a $C_1 > 0$ such that for any $\gamma > 0$ and arbitrary $u, v \in \mathbf{U}$ the following statement holds:*
if $\|u - v\| \ge \epsilon$, $0 < \hat{\mu} \le \mu_\epsilon = \dfrac{C_1\gamma}{k^2(\epsilon)}$, *then*

$$\gamma(u - v, u - v) + (k(|u|)u - k(|v|)v, (1 - \hat{\mu}k(|u|))u - (1 - \hat{\mu}k(|v|))v) \ \ge\ 0$$

By means of the same proof technique as in [11] we can give a more exact estimate. Moreover, it's possible to prove that the estimate from the following lemma is exact to within a multiplicative constant.

Lemma 2. *For any $\gamma > 0$ and arbitrary $u, v \in \mathbf{U}$ the following estimate holds:*
if $\|u - v\| \ge \epsilon$ *and* $0 < \hat{\mu} \le \mu_\epsilon = \frac{1}{2\,k(\delta)}$, *where* $k(\delta)\delta^2 = \frac{\gamma\epsilon^2}{2|\Omega|}$, *then*

$$\begin{gathered}
\gamma(u - v, u - v) + \\
+ (k(|u|)u - k(|v|)v, (1 - \hat{\mu}k(|u|))u - (1 - \hat{\mu}k(|v|))v) \ \ge\ 0.
\end{gathered}$$

For $k(x) = k_1 x^{-\alpha}$ from Lemma 1 it follows $\mu_\epsilon = \dfrac{C_1\gamma\epsilon^{2\alpha}}{k_1}$, and from Lemma 2 we obtain $\mu_\epsilon = 0.5\left(\dfrac{\gamma}{2|\Omega|k_1}\right)^{\frac{\alpha}{2-\alpha}}\epsilon^{\frac{2\alpha}{2-\alpha}}$.

The following lemma gives a more exact estimate for μ_ε than the two previous ones, but it needs a greater number of arithmetic operations to realize it on a computer and it's not so useful later.

Lemma 3. *For any $\gamma > 0$ and arbitrary $u, v \in \mathbf{U}$ the following statement holds:*
if $\|u - v\| \geq \epsilon$ and $0 < \hat{\mu} \leq \mu_\epsilon = \frac{1}{2\,k(\delta)}$, where $\delta = \max\Big\{\delta > 0 : k(\delta)\delta^2|\Omega|_\delta \leq \frac{\gamma\epsilon^2}{2}\Big\}$, then

$$\begin{aligned}&\gamma(u-v,u-v)+\\ &+(k(|u|)u-k(|v|)v,(1-\hat{\mu}k(|u|))u-(1-\hat{\mu}k(|v|))v)\ \geq\ 0.\end{aligned}$$

Here $|\Omega|_\delta$ is the size of subdomain Ω, where $|u| \leq \delta$.

Let us denote the dependence μ_ϵ on ϵ, obtained from Lemma 3, as $\mu_\epsilon = \varphi_{\gamma,u}(\epsilon)$. Note, that the function $\varphi_{\gamma,0}(\epsilon)$ is the corresponding function obtained from the Lemma 2.

For simplicity let us assume $\mathbf{B} = -\Delta + \frac{\tau}{\beta}\nabla\mathrm{div}$. Then

$$\mathbf{C} = (1+\tau k_0)\Delta, \quad \lambda_0 = 1-\tau/\beta, \quad \lambda_1 = 1 \tag{13}$$

and the condition $\tau < \beta$ should be satisfied for positive definiteness of $\mathbf{B}$.

From Lemma 3 it follows that for positiveness of the value in square brackets of (12) it is sufficient that

$$\frac{3}{2}C_0^2\mu\tau + \frac{\varepsilon}{2} \leq \varphi_{\xi,|\nabla v|}(\epsilon) \tag{14}$$

where $\varphi_{\xi,|\nabla v|}$ is an increasing function, $\varphi_{\xi,|\nabla v|}(\epsilon) \to 0$ as $\epsilon \to 0$.

Let us suppose that

$$1 - 3C_0^2\mu - \frac{\tau}{\varepsilon\lambda_0} \geq 0 \tag{15}$$

then we obtain from (12):

$$
\begin{aligned}
&\|\hat{\mathbf{y}}\|_B^2 + 2\tau k_0(1 - \tfrac{3}{2}C_0^2 k_0\tau\mu)\|\hat{\mathbf{y}}\|_1^2 \\
&+2\tau\Big[\xi(\nabla\mathbf{v} - \nabla\mathbf{u}, \nabla\mathbf{v} - \nabla\mathbf{u}) + (k(|\nabla\mathbf{v}|)\nabla\mathbf{v} - k(|\nabla\mathbf{u}|)\nabla\mathbf{u}, \\
&(1 - (\tfrac{3}{2}C_0^2\mu\tau + \tfrac{\varepsilon}{2})k(|\nabla\mathbf{v}|))\nabla\mathbf{v} - (1 - (\tfrac{3}{2}C_0^2\mu\tau + \tfrac{\varepsilon}{2})k(|\nabla\mathbf{u}|))\nabla\mathbf{u})\Big] \\
&+(\beta\tau + \tau^2\mu)\|\hat{r}\|^2 + \beta\tau\|\hat{r} - r\|^2 \le \|\mathbf{y}\|_B^2 + 2\tau\xi\|\mathbf{y}\|_1^2 + \beta\tau\|r\|^2
\end{aligned}
\tag{16}
$$

To satisfy (15) we can choose ε and μ by the following way

$$
\varepsilon = \frac{2\tau}{\lambda_0}, \quad \mu = \frac{1}{6C_0^2} \tag{17}
$$

Thus the condition (14) can be transformed to the form

$$
\frac{\tau}{4} + \frac{\tau}{\lambda_0} = \tau\left(\frac{1}{4} + \frac{1}{\lambda_0}\right) \le \varphi_{\xi,|\nabla v|}(\epsilon) \tag{18}
$$

We obtain, that if the inequalities (17), (18) are hold, then from (16) it follows the estimate

$$
\begin{aligned}
&\|\hat{\mathbf{y}}\|_B^2 + 2\tau k_0(1 - \tfrac{1}{4}k_0\tau)\|\hat{\mathbf{y}}\|_1^2 \\
&+(\beta\tau + \tfrac{\tau^2}{6C_0^2})\|\hat{r}\|^2 + \beta\tau\|\hat{r} - r\|^2 \le \|\mathbf{y}\|_B^2 + 2\tau\xi\|\mathbf{y}\|_1^2 + \beta\tau\|r\|^2
\end{aligned}
\tag{19}
$$

If we additionally suppose that

$$
k_0\tau < 2 \tag{20}
$$

then we obtain from (19)

$$
\begin{aligned}
&\|\hat{\mathbf{y}}\|_B^2 + \tau k_0\|\hat{\mathbf{y}}\|_1^2 \\
&+(\beta\tau + \tfrac{\tau^2}{6C_0^2})\|\hat{r}\|^2 + \beta\tau\|\hat{r} - r\|^2 \le \|\mathbf{y}\|_B^2 + 2\tau\xi\|\mathbf{y}\|_1^2 + \beta\tau\|r\|^2
\end{aligned}
$$

or

$$
\begin{aligned}
&\|\hat{\mathbf{y}}\|_B^2 + 2\tau\xi\|\hat{\mathbf{y}}\|_1^2 + \tau(k_0 - 2\xi)\|\hat{\mathbf{y}}\|_1^2 \\
&+(\beta\tau + \tfrac{\tau^2}{6C_0^2})\|\hat{r}\|^2 + \beta\tau\|\hat{r} - r\|^2 \le \|\mathbf{y}\|_B^2 + 2\tau\xi\|\mathbf{y}\|_1^2 + \beta\tau\|r\|^2
\end{aligned}
$$

It follows from the last inequality, that we can choose $\xi = k_0/4$. Thus we obtain

$$
\begin{aligned}
&\|\hat{\mathbf{y}}\|_B^2 + \tfrac{1}{2}\tau k_0\|\hat{\mathbf{y}}\|_1^2 + \tfrac{\tau k_0}{2}\|\hat{\mathbf{y}}\|_1^2 \\
&+(\beta\tau + \tfrac{\tau^2}{6C_0^2})\|\hat{r}\|^2 + \beta\tau\|\hat{r} - r\|^2 \le \|\mathbf{y}\|_B^2 + \tfrac{1}{2}\tau k_0\|\mathbf{y}\|_1^2 + \beta\tau\|r\|^2
\end{aligned}
\tag{21}
$$

and according to the assumptions on $\mathbf{B}$, (21) is transformed to the form

$$\begin{aligned}&\left(\tfrac{\tau k_0/2}{1+\tau k_0/2}+1\right)(\|\hat{\mathbf{y}}\|_B^2+\tfrac{1}{2}\tau k_0\|\hat{\mathbf{y}}\|_1^2)\\&+(\beta\tau+\tfrac{\tau^2}{6C_0^2})\|\hat{r}\|^2+\beta\tau\|\hat{r}-r\|^2\le\|\mathbf{y}\|_B^2+\tfrac{1}{2}\tau k_0\|\mathbf{y}\|_1^2+\beta\tau\|r\|^2\end{aligned}\tag{22}$$

From the last inequality and (20) it follows that the rate of convergence of iterative method (5) can be estimated by

$$q=\left[1+\tau\min\left(\frac{k_0}{4},\frac{1}{\beta 6C_0^2}\right)\right]^{-1}\tag{23}$$

Let us consider now the case of constant parameters τ and β. From (22), (23) it follows

Theorem 2. *Let $\partial\Omega$, $\mathbf{f}$ and k satisfy necessary for further considerations conditions (see [11]), $\mathbf{B}=-\Delta+\frac{\tau}{\beta}\nabla\mathrm{div}$, $\tau<\beta$ and the relations (17), (20) are fulfilled. Then the error of iterative process (5) $\delta^n=\sqrt{\|\mathbf{y}^n\|_B^2+\frac{1}{2}\tau k_0\|\mathbf{y}^n\|_1^2+\beta\tau\|r^n\|^2}$ tends to zero as $n\to\infty$ with the rate of geometric progression*

$$q=\left[1+\tau\min\left(\frac{k_0}{4},\frac{1}{\beta 6c_0^2}\right)\right]^{-1}=[1+\tau C_2]^{-1}$$

while $\varphi_{k_0/4,0}(\|\mathbf{y}\|_1)\ge\frac{2\tau}{\lambda_0}$. Here $\mathbf{y}^n$, r^n are the errors of velocity and pressure respectively.

From this theorem we obtain explicit formulas for iterative parameters τ and β for the case when we want to achieve the given accuracy.

4 Estimate of convergence in the case of variable iteration parameters

It is possible to improve the rate of convergence by choosing variable parameters. Note that the iterative parameters can not be varied in an arbitrary way because the definition of error: $\sqrt{\|\mathbf{y}\|_B^2+\frac{1}{2}\tau k_0\|\mathbf{y}\|_1^2+\beta\tau\|r\|^2}$ includes both of them. Therefore, in the case of $\beta/\tau=\mathrm{const}$ the operator $\mathbf{B}$ does not change during iterative process, so in this case we can vary parameters β and τ, insuring $\beta\tau$ (or, as follows from the previous condition, only τ) to be a decreasing function of iterative step number.

In [12] we obtained the following estimate on a function $s(x) = k(x)x$, where function k is satisfied (3):

$$\|k(|\nabla \mathbf{u}|)\nabla \mathbf{u} - k(|\nabla \mathbf{v}|)\nabla \mathbf{v}\| \le 8s(\|\mathbf{u}-\mathbf{v}\|_1) \tag{24}$$

By means of (24) we obtain from (12)

$$\begin{aligned}&\|\hat{\mathbf{y}}\|_B^2 + (2\tau k_0 - 3C_0^2 k_0^2 \tau^2 \mu)\|\hat{\mathbf{y}}\|_1^2 + \left(1 - 3C_0^2\mu - \tfrac{\tau}{\varepsilon\lambda_0}\right)\|\hat{\mathbf{y}} - \mathbf{y}\|_B^2\\ &+(\beta\tau + \tau^2\mu)\|\hat{r}\|^2 + \beta\tau\|\hat{r} - r\|^2\\ &\le \|\mathbf{y}\|_B^2 + \tau^2(3C_0^2\mu + \varepsilon/\tau)64s^2(\|\mathbf{y}\|_1) + \beta\tau\|r\|^2\end{aligned}$$

According to (15), (17), (20), from the last inequality it follows

$$\begin{aligned}&\|\hat{\mathbf{y}}\|_B^2 + \tau k_0\|\hat{\mathbf{y}}\|_1^2 + \left(\beta\tau + \tfrac{\tau^2}{6C_0^2}\right)\|\hat{r}\|^2\\ &\le \|\mathbf{y}\|_B^2 + \tau^2(1 + 4/\lambda_0)32s^2(\|\mathbf{y}\|_1) + \beta\tau\|r\|^2\end{aligned}$$

and by means (13) we obtain

$$\begin{aligned}&(1+\tau k_0)\|\hat{\mathbf{y}}\|_B^2 + \left(\beta\tau + \tfrac{\tau^2}{6C_0^2}\right)\|\hat{r}\|^2\\ &\le \|\mathbf{y}\|_B^2 + \tau^2(1 + 4/\lambda_0)32s^2(\|\mathbf{y}\|_B)/\lambda_0^2 + \beta\tau\|r\|^2\end{aligned} \tag{25}$$

Let us take $\beta = \rho\tau$, with some $\rho > 1$. Then from (25), (13) we obtain

$$\begin{aligned}&(1+\tau k_0)\|\hat{\mathbf{y}}\|_B^2 + \rho\tau^2\left(1 + \tfrac{1}{6\rho C_0^2}\right)\|\hat{r}\|^2\\ &\le \|\mathbf{y}\|_B^2 + \tau^2\tfrac{5\rho-1}{\rho-1}32s^2(\|\mathbf{y}\|_B)/\lambda_0^2 + \rho\tau^2\|r\|^2\\ &\le \|\mathbf{y}\|_B^2 + \tau^2\tfrac{\rho^2(5\rho-1)}{(\rho-1)^3}32s^2(\sqrt{\|\mathbf{y}\|_B^2 + \rho\tau^2\|r\|^2}) + \rho\tau^2\|r\|^2\end{aligned} \tag{27}$$

Assume, that

$$\tau \le \frac{k_0(\rho-1)^3}{64\rho^2(5\rho-1)}k^{-2}\left(\sqrt{\|\mathbf{y}\|_B^2 + \rho\tau^2\|r\|^2}\right) \tag{28}$$

then, if we propose, that

$$\tau \le \frac{1}{k_0 6\rho C_0^2} \tag{29}$$

then from (27) and (28) it follows

$$(1+\tau k_0)\left(\|\hat{\mathbf{y}}\|_B^2 + \rho\tau^2\|\hat{r}\|^2\right) \le (1 + \tau k_0/2)\left(\|\mathbf{y}\|_B^2 + \rho\tau^2\|r\|^2\right) \tag{30}$$

Thus, we can formulate the following theorem.

Theorem 3. *Let the standard conditions for smoothness of $\partial\Omega$, $\mathbf{f}$, k are hold (see [11]), $\mathbf{B} = -\Delta + \frac{\tau}{\beta}\nabla \mathrm{div}$, $\beta = \rho\tau$, $\rho > 1$ and the relations (17), (20), (29) are fulfilled. Then the following statements are valid:*

1) The error of iterative process (5) $\delta^n = \sqrt{\|\mathbf{y}^n\|_B^2 + \beta\tau\|r^n\|^2}$ tends to zero as $n \to \infty$ with the rate of geometric progression $q = \left[1 + \tau\frac{k_0}{2+\tau k_2}\right]^{-1} \leq \left[1 + \tau\frac{k_0}{4}\right]^{-1}$ while τ satisfy the condition (28).

2) Let we have two approximations to the solutions $(\mathbf{v}^1, q^1)$ and $(\mathbf{v}^2, q^2)$ and the errors $(\mathbf{y}^1, r^1)$ and $(\mathbf{y}^2, r^2)$ respectively. After applying of one iteration of the method (5) with the described parameters τ, β we obtain the errors $(\hat{\mathbf{y}}^1, \hat{r}^1)$ and $(\hat{\mathbf{y}}^2, \hat{r}^2)$ respectively. The following property holds for these functions: if $\|\mathbf{y}^1\|_B^2 + \beta\tau\|r^1\|^2 \geq \|\mathbf{y}^2\|_B^2 + \beta\tau\|r^2\|^2$, then $\|\hat{\mathbf{y}}^1\|_B^2 + \beta\tau\|\hat{r}^1\|^2 \geq \|\hat{\mathbf{y}}^2\|_B^2 + \beta\tau\|\hat{r}^2\|^2$.

From the last theorem it follows that to build an iterative process with variable parameters we need the error estimate in the first iterative method step only. We write down the algorithm of choosing τ and β in the following way.

Algorithm.

Set $\beta = \rho\tau$ in the each step, where $\rho > 1$ is a constant.
Choose initial $\tau_0 = \min\left(\frac{1}{6k_0\rho C_0^2}, \frac{2}{k_0}\right)$.
Set iteration number $n = 1$.
Assume we have some initial estimate $\sqrt{\|\mathbf{y}_0\|_B^2 + \beta\tau\|r_0\|^2} \leq \varepsilon_0$.
while $\tau_0 \leq \frac{k_0(\rho-1)^3}{64\rho^2(5\rho-1)}k^{-2}(\varepsilon_{n-1})$
  Make one step of iterative process (2.4) with $\tau_n = \tau_0$.
  Put $\varepsilon_n = \varepsilon_{n-1} * q_0$, where $q_0 = \left(1 + \tau_0\frac{k_0}{4}\right)^{-1}$.
  $n = n + 1$
end of while
while *($\varepsilon_{n-1} > err_{min}$ AND residual of method is greater than err_{min})*
  Make one step of iterative process (5) with $\tau_n = \frac{k_0(\rho-1)^3}{64\rho^2(5\rho-1)}k^{-2}(\varepsilon_{n-1})$
  Put $\varepsilon_n = \varepsilon_{n-1} * q_n$, where $q_n = \left(1 + \tau_n\frac{k_0}{4}\right)^{-1}$
  $n = n + 1$
end of while

Note, that $\beta\tau$ is a decreasing function of n and $\beta/\tau = \rho$ is a constant, hence $\|\mathbf{y}_{n+1}\|_B^2 + \beta_{n+1}\tau_{n+1}\|r_{n+1}\|^2 \leq \|\mathbf{y}_{n+1}\|_B^2 + \beta_n\tau_n\|r_{n+1}\|^2$. It means, that we can use variable norm in all arguments.

Thus, we can formulate the theorem about the convergence of the iterative method (5).

Theorem 4. *The iterative method (5) realized in the* **Algorithm** *converges to the solution of the problem (2)—(3). For pressure-velocity errors there exists estimate on the rate of the method convergence:*
$\|\mathbf{y}_n\|_B^2 + \beta\tau\|r_n\|^2 \le \varepsilon_n$, *where* ε_n *is defined in the* **Algorithm**.

Thus, we obtain the estimate $\|\mathbf{y}_n\|_B^2 \le \varepsilon_n$, but the estimate of $\|r_n\|$ is not satisfactory: $\|\tau_n r_n\| \le \varepsilon_n$. Let us consider the behaviour of the sequence $\{\varepsilon_n/\tau_n\}$. From the **Algorithm** we know, that

$$\frac{\varepsilon_{n+1}}{\tau_{n+1}} = \frac{\varepsilon_n}{\tau_n}\frac{\tau_n}{\tau_{n+1}}\left(1+\frac{C_4}{k^2(\sqrt{\varepsilon_n})}\right)^{-1} \tag{31}$$

where $C_4 = \frac{k_0^2(\rho-1)^3}{256\rho^2(5\rho-1)}$, and from the **Algorithm** by means elementary transformations we obtain

$$\begin{aligned}\frac{\tau_n}{\tau_{n+1}}\left(1+\frac{C_4}{k^2(\sqrt{\varepsilon_n})}\right)^{-1} &= \frac{k^2\left(\sqrt{\frac{\varepsilon_n}{1+\frac{C_4}{k^2(\sqrt{\varepsilon_n})}}}\right)}{k^2(\sqrt{\varepsilon_n})+C_4} \\ &= \frac{s^2\left(\frac{\sqrt{\varepsilon_n}}{\sqrt{1+\frac{C_4}{k^2(\sqrt{\varepsilon_n})}}}\right)}{s^2(\sqrt{\varepsilon_n})} \le 1\end{aligned} \tag{32}$$

where $s(x) = k(x)x$ is an increasing function. From the last inequality it follows that $\|r_n\| \le \varepsilon_0$ for any n.

For simplicity we assume, that $s(x) \in \mathrm{C}^2(0,\infty)$ and the following estimate holds[1]

$$\frac{s'(x)x}{s(x)} \le C_5. \tag{33}$$

Then from decreasing of function $s'(x)$ we obtain

$$s\left(\frac{\sqrt{\varepsilon_n}}{\sqrt{1+\frac{C_4}{k^2(\sqrt{\varepsilon_n})}}}\right) \le s(\sqrt{\varepsilon_n}) - \left(\sqrt{\varepsilon_n} - \frac{\sqrt{\varepsilon_n}}{\sqrt{1+\frac{C_4}{k^2(\sqrt{\varepsilon_n})}}}\right)\frac{\partial s}{\partial x}(\sqrt{\varepsilon_n}) \tag{34}$$

[1]For $k(x) = x^\alpha$ $\frac{s'(x)x}{s(x)} = \mathrm{const}.$

and from (32), (33), (34) we obtain the estimate for sufficiently small ε_n

$$\begin{aligned} \frac{\tau_n}{\tau_{n+1}}\left(1+\frac{C_4}{k^2(\sqrt{\varepsilon_n})}\right)^{-1} &\leq \left(1-C_5\frac{C_4}{k^2(\sqrt{\varepsilon_n})+C_4}\right)^2 \\ &\leq 1-\frac{C_6}{k^2(\sqrt{\varepsilon_n})} \end{aligned} \tag{35}$$

Thus, from (20), (35) and from the Theorem 4 we obtain

Theorem 5. *The iterative method (5) realized in the* **Algorithm** *converges to the solution of the problem (2)—(3). For sufficiently small pressure-velocity errors (see the second part of the* **Algorithm***) there exists estimates on the rate of the method convergence:*

1) $\|\mathbf{y}_n\|_B^2 \leq \varepsilon_0\varepsilon_n$, *where* $\frac{\varepsilon_{n+1}}{\varepsilon_n} \leq \left(1+\frac{C_4}{k^2(\sqrt{\varepsilon_n})}\right)^{-1}$, *where* C_4 *is defined in (31).*

2) $\|r_n\|$ *is bounded and if we additionally suppose, that the estimate (33) holds, then* $\|r_n\|^2 \leq \gamma_0\gamma_n$, *where* $\frac{\gamma_{n+1}}{\gamma_n} \leq 1-\frac{C_8}{k^2(\sqrt{\varepsilon_n})}$, *with the same* ε_n.

5 Conclusion

We have given explicit estimates of the convergence of the iterative method (5) for the case of constant coefficients. In this case the iterative method generally does not converge, but we gave estimates of coefficients for which the iterative method computes the solution with the given accuracy.

We proposed to use method (5) with variable coefficients and gave explicit formulas for the coefficients. In this case the iterative method converges and we obtained the estimate of it's convergence. It's very interesting, that the method (5) with variable coefficients has the same rate of convergence as the fully implicit method with constant coefficients, proposed in [5]; but this method needs a nonlinear operator inversion in every iteration.

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