A Survey of Lyapunov Techniques for Stochastic Differential Equations*

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Abstract

This note reviews Lyapunov-like theorems addressing qualitative properties of solutions to stochastic differential equations (in the sense of Itô). The qualitative properties we address are regularity, stochastic boundedness, stochastic stability, stability of moments as well as existence and properties of stationary solutions. We review the existing Lyapunov-like theorems which give conditions for a system to possess these properties; the conditions are given in terms of the existence of a auxiliary (Lyapunov) function.

We illustrate the theory with two examples and finally we point out some areas of current research within the field of stochastic Lyapunov theory.

1 Introduction

Stochastic differential equations are useful for modeling physical, technical, biological and economical dynamical systems in which significant uncertainty is present. Unfortunately, it is in general not possible to give explicit expressions for the solutions to stochastic differential equations and numerical solution is a cumbersome affair. It is therefore of great interest to be able to characterize at least qualitatively the behaviour of the solutions. Typical questions are

- Is there a unique solution and is it defined for all times?
- Is the solution bounded in a suitable stochastic sense?
- Is the solution stable in a suitable stochastic sense?

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- Does the solution approach a stationary (perhaps even ergodic) process? In this case, what properties does this limiting process have?

Indeed, in many cases we are more interested in these qualitative properties than in the exact form of the solutions, e.g. stability analysis of control systems.

As it is for deterministic systems, Lyapunov theory is a powerful tool for qualitative analysis of stochastic differential equations. A particular feature which makes the framework attractive is that it appeals to all our intuition regarding energy conserving systems. The mathematical machinery is supermartingale theory and ergodic theory of Markov processes. The greatest disadvantage of the Lyapunov approach is that no universal method has been given which enables you to find a Lyapunov function or determine that no such function exists.

Scope

This note covers the most important results within classical stochastic Lyapunov theory. We use the term *classical* to indicate that we work with Itô diffusions in Euclidean spaces and not the various modern extensions, and that we work with Lyapunov functions which are C^2 in the regions of interest. Furthermore we will treat only the *analysis* question (determine if a given system possesses a specified property) as opposed to the synthesis question (find a system in a given set which possesses a specified property). In section 9 we will give some hints as to approaches to synthesis.

Prerequisites of the reader

We assume that the reader is familiar with Lyapunov techniques for stability of autonomous deterministic ordinary differential equations as presented in e.g. [10]. Furthermore, we assume an elementary knowledge of continuous-time stochastic processes and Itô diffusions as presented in e.g. [16]. Stochastic Lyapunov theory makes heavy use of supermartingale theory; we have compiled the relevant results in an appendix.

Contributions

This note reviews existing literature and does not present new theory. Some of the statements are modified slightly from the form in which they can be found in existing literature. Some proofs differ from the ones found in the literature. Where this is the case, it is pointed out. Parts of the examples are original, in particular the example of section 8 is original. Propositions 13 and 14 cannot be found in the referenced litterature but follow practically immediately from statements therein.

Outline of the paper

The remainder of this note is organized as follows: In section 2 we present a simple one-dimensional example in order to motivate the following theory and in order to present the essence of the techniques in stochastic Lyapunov theory. In section 3 we give the mathematical preliminaries regarding stochastic differential equations. After this we give Lyapunov-type results for regularity and stochastic boundedness (section 4), stochastic stability (section 5), stability of moments (section 6) and existence of a stationary solution (section 7). Section 8 contains a scalar non-linear example in which we employ the techniques. Finally, in section 9 we give additional notes and bibliographical references.

Appendix A contains some reference material on martingales and supermartingales while appendix B contains a sequence of lemmas which illustrate the technique of stopping times in proofs of Lyapunov-type theorems; they lead up to a stochastic stability result.

2 The scalar example $dx_t = rx_t\, dt + \alpha x_t\, dB_t$

Consider the scalar stochastic process $x_t,\ t \in \mathbb{R}$ given by the first order Itô stochastic differential equation

$$dx_t = rx_t\, dt + \alpha x_t\, dB_t$$

which may loosely be written as

$$\dot{x} = rx + \alpha x w$$

where $w = dB/dt$ is white noise. Such a system is referred to as a *multiplicative noise* model or a *geometric Brownian motion* - it is also called a *homogeneous wide sense* linear stochastic differential equation. The explicit solution is [16, p. 61]

$$x_t = x_0 \exp((r - \frac{1}{2}\alpha^2)t + \alpha B_t)$$

With the solutions in hand we are able to characterize the qualitative behaviour of the process as $t \to \infty$ (see [16] - the argument relies on the law of the iterated logarithm):

- If $r - \alpha^2/2 > 0$ then $x \to \infty$ w.p. 1
- If $r - \alpha^2/2 < 0$ then $x \to 0$ w.p. 1
- If $r - \alpha^2/2 = 0$ then x fluctuates between arbitrary large and arbitrary small values w.p. 1

We emphasize that the stability boundary is not given by $r = 0$, and that sufficiently *large* noise intensities stabilize the system. A simulation of the system is shown in figure 1 using three choices of parameters and one realization of the noise.

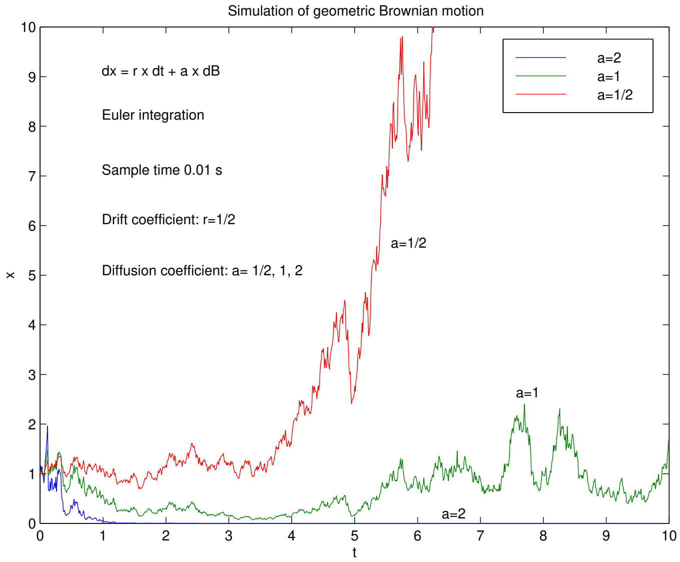

Figure 1: Simulation of geometric Brownian motion

Recall that the aim of Lyapunov theory is to be able to determine such qualitative behaviour of the solutions without knowing the solutions explicitly. We shall now do this. In [12, p. 55] the existence of stochastic Lyapunov functions of the form

$$V(x) = |x|^p, \quad p > 0$$

is considered. By an application of Itô's lemma [16] it is found[1] that $V(x_t)$ is a well-defined stochastic process which satisfies the stochastic differential equation[2]

$$dV = p(r + (p-1)\alpha^2/2)V\ dt + p\alpha V\ dB$$

In particular, assume $r - \alpha^2/2 < 0$ and choose $p > 0$ such that

$$\beta := r + (p-1)\alpha^2/2 \leq 0$$

which gives the inequality for $T > t$

$$\mathcal{E}^t V(x_T) = V(x_t) + \mathcal{E}^t \int_t^T p\beta V(x_s)\ ds \leq V(x_t)$$

[1]At this point we ignore the technical difficulty that $V(x)$ is not necessarily C^2 and hence Itô's lemma cannot be applied. With a careful treatment of the absorbing point $x = 0$ the argument can be made rigorous, see [8, rem. 2, p. 163]. For an alternative, refer to appendix B.

[2]This SDE is also wide sense linear and hence we can write up the solution explicitly. There is no advantage in doing it, though. The argument does not use the explicit solution.

Here the expectation $\mathcal{E}^t\cdot$ is conditioned on x_t. The inequality follows from $p\beta$ being non-positive and $V(x_s)$ being non-negative. It follows that $V(x_t)$ is a non-negative *supermartingale* - i.e. has decreasing expectation - which gives the following probability bound (see appendix A):

$$P(\sup_{s\geq t} V(x_s)\geq R)\leq \frac{V(x_t)}{R}\quad \text{for any } R>0$$

This inequality forms the definition of stochastic stability of the zero solution, see section 5. It is the generalization of (deterministic) Lyapunov stability: Assume that we specify a probability $\epsilon>0$ and a radius $R>0$ and then ask: Can we, by proper choice of the initial condition, guarantee that $|x|^p$ remains smaller than than R with a probability greater than $1-\epsilon$? The answer is affirmative: Yes, a sufficient condition on the initial condition is

$$V(x_t)\leq R\epsilon$$

Actually, we can show stronger results. Assume again that $r-\alpha^2/2<0$ and let $p,\lambda>0$ be such that

$$r+(p-1)\alpha^2/2<-\lambda/p<0$$

Define the Lyapunov function

$$W(x,t)=|x|^p e^{\lambda t}$$

Using Itô's lemma again we readily find that W is a supermartingale which shows that $\mathcal{E}|x|^p$ converges exponentially to 0. This is a stability property called *exponential p-stability* of the zero solution, see section 6.

Finally, it is possible to show[3] that $V(x_t)$ and $W(x_t)$ converge w.p. 1 to some stochastic variables V_∞ and W_∞. It is easy to see that these variables must equal 0 w.p. 1 which implies that x_t converges to 0 exponentially fast w.p. 1.

We conclude this example with a couple of summarizing remarks:

1. Whereas deterministic Lyapunov theorems relies on $V(x_t)$ being non-increasing, the stochastic equivalent is that $V(x_t)$ is a supermartingale, i.e. has non-increasing expectation.
2. Under suitable hypothesis, it is necessary and sufficient for stochastic stability that it can be proved with a Lyapunov function (in this example of the form $|x|^p,\ p>0$).
3. There may be a significant conservatism in demanding the stochastic Lyapunov function to be differentiable everywhere.
4. The system of this example is stable in probability if and only if some moment $\mathcal{E}^x|x_t|^p$ with $p>0$ converges to 0 as $t\to\infty$. We shall see that for general systems the 'if' implication holds.

[3]using the super-martingale convergence result (corollary 19) of appendix A.

5. For this scalar case $V(x_t)$ may very well be a supermartingale even if x_t is not - take $r > 0$ and $r - \alpha^2/2 < 0$. The converse situation is also possible.

6. At the stability margin, i.e. for $r - \alpha^2/2 = 0$, the system is not stochastically stable in contrast to many deterministic systems.

3 Preliminaries: On Stochastic Differential Equations

This note concerns the qualitative theory of the autonomous stochastic processes given by the stochastic differential equation (SDE)

$$dx_t = f(x_t)dt + g(x_t)dB_t \tag{1}$$

where the state x_t belongs to Euclidean space $\mathbb{X} = \mathbb{R}^n$ and B_t is standard m-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with respect to the filtration $\mathcal{F}_t$.

See [6, 7, 19, 16] for general theory of stochastic differential equations. In the following we summarize a few necessary notions.

With a *solution* to the SDE we understand a stochastic process $\{x_t;\ t \geq 0\}$ which satisfy the following regularity conditions: (1) The *initial condition* x_0 is $\mathcal{F}_0$-measurable and independent of the increments $B_t - B_s$ for $t \geq s \geq 0$. (2) x_t satisfies (1), i.e.

$$x_t = x_0 + \int_0^t f(x_s)\, ds + \int_0^t g(x_s)\, dB_s$$

for all times $t \geq 0$ and almost all realizations $\omega \in \Omega$. Here, the last integral on the right hand side is an Itô integral.

We will use the notation x_t and $x(t)$ interchangeably. If we want to emphasize the dependence on the initial condition $x_0 = x(0)$ we will write $x(t, x_0)$. If J_t is a functional of $x_s,\ 0 \leq s \leq t$ then $\mathcal{E}^x J_t$ denotes expectation of J_t where x_s solves the SDE with initial condition $x_0 = x$ w.p. 1.

We will say that the SDE (1) is *non-degenerate* on $\Omega \subset \mathbb{X}$ if $m \geq n$ and

$$\underline{\sigma}(g(x)g'(x)) > k(x) \quad \text{for } x \in \Omega$$

where $\underline{\sigma}$ denotes smallest singular value (equivalently, smallest eigenvalue) and where $k(x) > 0$ for $x \in \Omega$. If $\Omega = \mathbb{X}\backslash\{0\}$ then we will just say that the SDE is non-degenerate.

We will put the restriction on the data for each x_0 there corresponds a unique solution x_t which is a continuous strong Markov process.[4] See section 9 for a remark on the sufficient local Lipschitz-like conditions on the data.

[4] on $\mathrm{X} \cup \{\infty\}$ since we do not wish to exclude finite escape times a priori

Associated with this stochastic differential equation we define the differential generator L which maps C^2 functions $V : \mathbb{X} \to \mathbb{R}$ to C^0 functions $LV : \mathbb{X} \to \mathbb{R}$ given by

$$LV(x) = V_x(x)f(x) + \frac{1}{2}\mathrm{tr} g'(x)V_{xx}(x)g(x)$$

The importance of this operator follows from Itô's lemma:

$$dV(x_t) = LV(x_t)\ dt + V_x(x_t)g(x_t)\ dB_t$$

If V furthermore has compact support then

$$AV(x) := \lim_{t \downarrow 0} \mathcal{E}^x \frac{V(x_t) - V(x)}{t} = LV(x)$$

The operator A is denoted the infinitesimal generator associated with the SDE. See [16] for further comments and generalizations.

A function $V : \mathbb{X} \to \mathbb{R}$ is *proper* if it satisfies

$$a(|x|) \leq V(x) \leq b(|x|)$$

for some strictly increasing functions a and b which satisfy $a(0) = b(0) = 0$ and $a(\xi) \to \infty$ for $\xi \to \infty$.

A *Lyapunov* function is a proper continuous function which is C^2 on $\mathbb{X}\backslash\{0\}$.

A set $D \subset \mathbb{X}$ is said to be*invariant* under the system dynamics if (it is Borel and)

$$P^x(x_t \in D) = 1 \text{ for } x \in D,\ t \geq 0$$

If D is an invariant set then the set $U \subset D$ is said to be *inessential* if (it is Borel and)

$$\forall x \in D : \quad P^x(x_t \in U) \to 0 \text{ as } t \to \infty$$

Conversely, the set $U \subset D$ is said to be *recurrent* in D if

$$\forall x \in D :\ P^x(\exists t > 0 :\ x_t \in U) = 1$$

If in addition

$$\sup_{x \in K} \mathcal{E}^x \inf\{t \geq 0 :\ x_t \in U\} < \infty$$

for any compact set $K \subset D$, then we say that U is *positive-reccurent*.

A *domain* is a simply connected Borel set with non-empty interior.

4 Finite escape times and boundedness

Since local Lipschitz conditions on the data f and g only provide *local* existence and uniqueness it is natural to ask when one can guarantee that almost all sample paths are well defined for $t \geq 0$. In this case we say that x is *regular* or that finite escape times occur w.p. 0. A Lyapunov-type criterion is the following slight generalization of [6, p. 131]:

Theorem 1: Let there exist a proper C^2 function V and numbers $K > 0$, $c > 0$ and $\epsilon \geq 0$ such that for $|x| > K$ we have $LV \leq cV + \epsilon$. Then w.p. 1 the sample paths do not converge to ∞ in finite time. □

Proof: The intuition behind the condition is - as in the deterministic case - that V cannot grow faster than exponentially outside the region $|x| > K$ and since V is proper this induces bounds on how fast $|x|$ can grow. We give an argument similar to the ones in [6, p. 131] and [8, p. 85].

By continuity there exists a $\delta \geq 0$ such that $LV \leq cV + \delta$ for all x. Define

$$\tilde{V}_t = e^{-ct}(V(x_t) + \delta/c)$$

then Itô's lemma gives

$$\mathcal{E}\tilde{V}_t \leq \tilde{V}_0$$

which says that $\tilde{V}_t$ is a non-negative supermartingale. It follows that

$$P\{\sup_{0 \leq t \leq T} \tilde{V}_t \geq N\} \leq \frac{V(x_0) + \delta/c}{N}$$

In particular, for any finite T the probability that x_t escapes to infinity before T is 0. ■

Example 2: Assume that f and g are affine in x and let $P > 0$ be a positive definite matrix. Then with $V(x) = x'Px$ there exists numbers $\epsilon > 0$ and $c > 0$ such that for each $K > 0$ the conditions of the theorem are met. Hence such systems define regular processes. Notice that the system satisfies a global Lipschitz condition. □

Once finite escape times have been ruled out the next question to ask is whether the solutions are bounded. For a deterministic system boundedness means [10] that for each solution there exists some bounded set in which the solution remains. There are two ways to generalize this to stochastic systems: We say that the system is *stochastically bounded* or *bounded in probability* if

$$\forall \epsilon > 0, x : \ \exists \beta > 0 : \ \sup_{t \geq 0} P^x(|x_t| > \beta) < \epsilon$$

A sufficient condition for this is that for each initial condition x there exists $\alpha > 0$ and $K > 0$ s.t.

$$\forall t > 0 : \ \mathcal{E}^x |x_t|^\alpha < K$$

which follows directly from Markov's inequality.

We quote a stronger result from [6, p. 129]:

Theorem 3: Let there exist a proper C^2 function V and a number $K > 0$ such that for $|x| > K$ we have $LV \leq 0$. Let $\tau = \tau_K$ be the first exit time from $\{|x| > K\}$ for the solution x_t where $|x_0| > K$. Then for each $\epsilon > 0$ there exists a $\beta_\epsilon > 0$ such that

$$P^x\left(\sup_{0\le t\le\tau} |x_t| \le \beta_\epsilon\right) > 1-\epsilon$$

and we say that the system is stochastically sample path bounded. □

Proof: See [6, p. 129]. We briefly sketch the proof: We choose $N > K$ and stop the process (see section B) when it leaves the region $K < |x| < N$. Then V applied to the stopped process is a supermartingale. When we let $N \to \infty$ we notice that w.p. 1 the process stops for $|x| = K$ since the process is regular according to the previous theorem. The supermartingale inequality then gives the conclusion using the bounds on V. See the appendices for super-martingale theory and the technique of stopping. ■

Example 4: Consider the system $dx_t = Ax_t\, dt + G\, dB_t$ where A is asymptotically stable. Then there exists a matrix $P > 0$ and a number $K > 0$ such that with $V(x) = x'Px$ the conditions are met. Notice that

$$P^x(\sup_{t\ge 0}|x_t| \le \beta) = 0$$

for any initial condition x and bound β, provided $G \ne 0$. □

5 Stochastic stability of the zero solution

In this section and we will assume that $f(0) = 0$ and $g(0) = 0$ such that $x_t \equiv 0$ is a solution of the SDE. See section 5.1 for a comment on the situation when this assumption does not hold.

Even in the deterministic case one has to distinct carefully between various notions of stability [9, 10]. This is true even more so in the stochastic case. We shall concentrate on *stochastic stability*.

Definition 5: We say that the solution $x_t \equiv 0$ is *stochastically stable* if

$$\lim_{x\to 0} P^x\{\sup_{t\ge 0}|x_t| > \epsilon\} = 0$$

holds for any $\epsilon > 0$. □

To illustrate the definition, let $\epsilon > 0$ be given. Then the subset $(t \ge 0, |x| < \epsilon)$ defines a cylindrical permitted region in the (t, x)-space. For each x_0 there is a probability

$$p(x_0) = P\{\sup_{t\ge 0}|x(t, x_0)| > \epsilon\}$$

that the solution $x(t, x_0)$ ever leaves this permitted region. For $x_0 = 0$ this probability is obviously 0. The definition says that this probability $p(x_0)$ should be continuous at $x_0 = 0$: a small deviation from equilibrium implies a small probability of escape.

A stronger notion of stability is *stochastic asymptotic stability*:

Definition 6: We say that the zero solution $x_t \equiv 0$ of (1) is *stochastically asymptotically stable* if it is stable in probability and if furthermore

$$\lim_{x\to 0} P^x(\lim_{t\to\infty} x_t = 0) = 1$$

If $P^x(x_t \to 0 \text{ as } t \to \infty) = 1$ for *any* x then we say that the zero solution is *globally* asymptotically stable in probability. □

We quote the following theorem from [8, p. 164]:

Theorem 7: Assume that there exists a Lyapunov function V defined on a neighbourhood D of $x = 0$ such that

$$LV(x) \leq 0$$

for $x \in D\backslash\{0\}$; then the equilibrium solution $x_t \equiv 0$ of the stochastic differential equation (1) is stochastically stable. □

Proof: We sketch the proof in [8]. Find a positive number v such that $V(x) \leq v$ implies $x \in D$ and stop the process when and if $V(x_t) \geq v$. For the stopped process, x_t^v, it follows from the Itô formula that $V(x_t^v)$ is a supermartingale. The supermartingale inequality (equation (3) in appendix A) gives the desired probability bound. For details on the use of stopped processes and for the detail that V needs not be C^2 at $x = 0$, refer to appendix B. ■

It turns out [8] that the condition is not only sufficient but also almost necessary:

Theorem 8: Assume that the system (1) is stochastically stable and non-degenerate. Then there exists a neighbourhood D of $x = 0$ and a function V which is C^2 on D such that $V(x) > 0$ and $LV(x) = 0$ for $x \in D\backslash\{0\}$. □

Proof: We omit the proof; see [8, p. 165]. In the proof the function

$$V(x) = P^x\{\sup_t |x_t| \geq r\}$$

is shown to possess the required properties for r small enough. ■

In stochastic stability analysis, stability almost implies asymptotical stability as was the case in the example in section 2. In fact we have the following result also due to Has'minskiĭ:

Theorem 9: Let the assumptions of theorem 7 hold and assume in addition there exists a neighbourhood Ω of $\{0\}$ such that at least *one* of the following holds:

1. The derivative $LV(x)$ is negative definite on $\Omega\backslash\{0\}$, or
2. the stochastical differential equation (1) is non-degenerate on $\Omega\backslash\{0\}$.

Then the zero solution is stochastically asymptotically stable. □

Proof: Our proof differs from that of Has'minskiĭ since he states the theorem as a corollary to a more general result. It follows from supermartingale convergence that $V(x_t^v)$ converges w.p. 1 for each initial condition x; denote the limit $V_\infty(x)$. We wish to show that the hypotheses imply that $P(V_\infty(x) = 0) \to 1$ as $x \to 0$. Notice that stochastic stability implies that $P(V_\infty(x) = v) \to 0$ as $x \to 0$. Hence it suffices to show that for any initial condition x we have $V_\infty(x) \in \{0, v\}$ w.p. 1.

To this end we consider a compact set D of the form $D = \{x | \ a \leq V(x) \leq v\}$ where $a > 0$ and where v is chosen sufficiently small so that $LV < 0$ (respectively, $\underline{\sigma}(gg') > 0$) on D.

First consider the case $LV < 0$. Let the initial condition satisfy $x \in D$ and let τ be the time of exit of D, i.e.

$$\tau = \inf\{t : \ x_t \notin D\}$$

Let $\tau(t) = \min\{\tau, t\}$. Then Dynkin's formula gives

$$\mathcal{E}^x\{V(x_{\tau(t)})\} = \leq V(x) + \mathcal{E}^x \int_0^\tau LV(x_s)\, ds \leq V(x) - \mathcal{E}^x \tau(t)$$

Since V is non-negative it follows that $\mathcal{E}^x \tau(t) \leq V(x)$. Since this bound is independent of t we see that $\mathcal{E}^x \tau \leq V(x)$ which implies that the process exits D w.p. 1 for any $x \in D$. Now let a tend to 0. It follows that almost all sample paths of x_t^v either converge to 0 or get absorbed at $\{x | V(x) = v\}$. The conclusion follows.

For the non-degenerate case it is well-known [8] that the process will almost surely in finite time escape a given bounded domain in which the diffusion is bounded away from zero. The result follows as above. ■

5.1 Example: The system $dx_t = Ax_t\, dt + G\, dB_t$

The above theorems do not apply immediately to the system

$$dx_t = Ax_t\, dt + G\, dB_t$$

for the simple reason that $x_t \equiv 0$ is not a solution to this stochastic differential equation. First, let us show how to tackle this problem with a standard trick which is well-known [10] from the deterministic situations.

Let y_t be a solution to the above SDE corresponding to the initial condition $y(0) = y_0$; we shall call y_t the *nominal* solution and are interested in the stability of this solution. Let ξ_t be any other solution, called the *perturbed* solution. Define the difference or the *perturbation*:

$$\tilde{x}_t := \xi_t - y_t$$

then clearly $\tilde{x}_t$ satisfies the degenerate SDE[5]

$$d\tilde{x}_t = d\xi_t - dy_t = A\tilde{x}_t \; dt$$

We define the nominal solution y_t to be stable (in any specified sense) if the zero solution $\tilde{x}_t \equiv 0$ to this ODE is stable (in the specified sense).

We see that e.g. y_t is stochastically stable if and only if A is Lyapunov stable; i.e, no eigenvalues in the open right half plane and all eigenvalues on the imaginary axis have same geometric and algebraic multiplicity.

For example, in the scalar case with $A = 0$ and $G = 1$, the solution $y_t = B_t$ is stochastically stable although the solution itself is unbounded w.p. 1. A deterministic formulation of the same phenomenon is that an integrator is Lyapunov stable but not input/output stable. For instance, for the system $\dot{x} = 1$ all solutions are Lyapunov stable but unbounded.

6 Stability of moments

Let $p > 0$. We say that the zero solution of the SDE (1) is

1. p-stable if $\lim_{x\to 0} \sup_{t\geq 0} \mathcal{E}^x |x_t|^p = 0$
2. asymptotically p-stable if it is p-stable and $\lim_{t\to\infty} \mathcal{E}^x |x_t|^p = 0$
3. exponentially p-stable if there exists positive constants C and λ such that

$$\mathcal{E}^x |x_t|^p \leq C|x|^p \exp(-\lambda t)$$

In the last definition λ is a bound on the stochastic Lyapunov exponent of the system.

Notice that these definitions correspond completely to usual concepts of stability for the *deterministic* system given by $y(t;x) = \mathcal{E}^x |x_t|^p$. Also, it is easy to see using Chebyshev's inequality that p-stability implies a weak notion of stability; namely

$$\forall \epsilon : \lim_{x\to 0} \sup_{t\geq 0} P^x\{|x_t| > \epsilon\} = 0 \tag{2}$$

See [7, p. 311]; a related statement for instability is given in [8, p. 189]. See also section 9 regarding terminology.

For the case of exponential p-stability we have the following theorem[6] due to Nevel'son and Has'minskiĭ, see [8, p. 186]:

[5] In general, the perturbations will satisfy an SDE which depends on the nominal solution y_t. In wide sense linear cases such as in section 2 the dependence on y disappears which implies that all solutions are stable if one solution is stable. In the narrow sense linear case such as this the SDE degenerates to an ODE.

[6] The result is also stated on p. 156 in [6] but in a form which is useless for $p < 2$ since V is required to be C^2 everywhere.

Theorem 10: A sufficient condition for the system (1) to be exponentially p-stable, $p > 0$, is that there exists a Lyapunov function V such that

$$k_1|x|^p \leq V(x) \leq k_2|x|^p \text{ and } LV(x) \leq -k_3|x|^p$$

where k_i are positive constants. If f and g are C^2 and have bounded derivatives up to second order, then this condition is also necessary. □

Has'minskiĭ also shows [8, p. 190] that the existence of such a Lyapunov function also implies that the solutions are almost surely exponentially stable in a certain sense.

Proof: We sketch the proof of [8]. To see sufficiency, notice that by theorem 1 the process is regular. Then apply Itô's formula and the bounds on V to get

$$\frac{d}{dt}\mathcal{E}^x V(x_t) \leq -\frac{k_3}{k_2}\mathcal{E}^x V(x_t)$$

which implies that $\mathcal{E}^x V(x_t)$ is exponentially decreasing. Again the bounds on V implies that $\mathcal{E}^x|x_t|^p$ is exponentially decreasing.

Necessity is shown using the function

$$V(x) = \int_0^T \mathcal{E}^x|x_s|^p \, ds$$

where T must be chosen sufficiently large. ■

7 Stationary solutions

In many situations the question of stability is not as interesting as the question whether there exists a steady-state, whether it is unique, and what properties it has. These questions are the subject of this section.

Recall that given a probability measure μ on $\mathbb{X}$ we say that a stochastic variable ξ taking values in $\mathbb{X}$ is distributed according to μ if for all Borel sets $B \subset \mathbb{X}$ we have $P(x \in B) = \mu(B)$. Now let the initial condition x_0 be distributed according to μ. If all $t \geq 0$ we then have that $x_t = x(t, x_0)$ is distributed according to μ then we say that the measure μ is invariant under the system dynamics.

If an invariant measure has a smooth density p, then it satisfies Kolmogorov's stationary forward equation (the stationary Fokker-Planck equation), see [16].

We quote the following theorem from [8, p. 90]

Theorem 11: Let V be a non-negative C^2 function such that $LV(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$. Assume that there exists a regular solution to the SDE (1). Then there exists a stationary solution to the SDE. □

Remark 12: If V in addition is proper, then regularity is guaranteed by theorem 1. $\Box$

A particular situation when this is useful is when the system under consideration can be viewed as a stable system perturbed by additive noise:

Proposition 13: Let V be a Lyapunov function which satisfies the assumptions of theorem 10 to show exponential stability in the mean square of the zero solution of the system (1). Assume furthermore that V_{xx} is bounded; then the system

$$dx_t = f(x_t)\,dt + [g(x_t) + h(x_t)]\,dB_t$$

admits a stationary solution for any bounded h. $\triangle$

If V is C^2 and μ is an invariant distribution such that $\mu(V) := \int V d\mu < \infty$, then $\mu(LV) = 0$. From this we get the simple but very useful proposition

Proposition 14: With the same assumptions as in theorem 11, let $f : \mathbb{X} \to \mathbb{R}$ be such that

$$LV(x) + f(x) \leq 0$$

for all $x \in \mathbb{X}$. Then

$$\mu(f) = \mathcal{E}^x\{f(x_t)\} \leq 0$$

provided $t > 0$ and x is distributed according to the invariant measure μ. $\triangle$

What makes the proposition useful is that the Lyapunov function V both guarantees existence of an invariant distribution as well as provides a performance bound.

Example 15: Consider the narrow sense linear SDE $dx_t = Ax_t\,dt + G\,dB_t$ where A is asymptotically stable. Let $P > 0$ be such that $PA + A'P + C'C < 0$. Define $f(x) = x'C'Cx - \text{tr}G'PG$. It follows that a steady-state distribution exists and satisfies

$$\mathcal{V}(Cx) \leq \text{tr}G'PG$$

where $\mathcal{V}(Cx)$ denotes the steady-state variance of Cx_t. $\Box$

In many situations it is possible to show that the invariant measure is unique and that the distribution of x_t will converge towards the invariant one. In fact, we have the following result due to Has'minskiĭ [8, p. 134]:

Theorem 16: Let D be an invariant domain of the system (1) and let $U \subset D$ be a bounded positive-recurrent domain. Assume in addition that $gg' > 0$ on some neighbourhood of U. Then there exists a unique invariant distribution μ such that $\mu(D) = 1$. Furthermore, if $x \in D$ w.p. 1 and $A \subset D$ then

$$P^x(x_t \in A) \to \mu(A) \text{ as } t \to \infty, \text{ w.p. } 1\ .$$

Finally, let $A \subset U$ have non-empty interior, then $\mu(A) > 0$. $\Box$

It is possible to show positive-recurrence using Lyapunov techniques; see [8, p. 98] or the example in the following section.

8 An example

Consider the system

$$dx_t = -\sin x_t \; dt + \cos x_t \; dB_t$$

We make the following comments regarding the structure of this system: The *deterministic* part $dx = \sin x \; dt$ admits equillibria at $x = n\pi$ for $n \in \mathbb{N}$, and these equillibria are stable for n even and unstable for n odd. The diffusion term $\cos x \; dB$ is maximum at the equillibria and vanishes in the middle of two neighbouring equillibria. These boundaries can hence only be passed in one direction, e.g. from $\pi/2 < x < 3\pi/2$ to $-\pi/2 < x < \pi/2$. It is hence fairly easy to convince oneself that the sets $B_n = [\pi/2 + 2n\pi, 3\pi/2 + 2n\pi]$ are *inessential*. Conversely, the sets $A_n = (-\pi/2 + 2n\pi, \pi/2 + 2n\pi)$ are *invariant*.

Let us now show these two properties with Lyapunov-type arguments. First we wish to show that if the system starts in some B_n then it will exit this set w.p. 1 and with finite expected time, i.e. the set $\cup_n A_n$ is positive-recurrent. Consider the Lyapunov function

$$V(x) = 2 - 2\cos x$$

Notice that if $x \in B_n$ then $\cos x \leq 0$ which implies

$$LV(x) = -2\sin^2 x + \cos^3 x \leq -1$$

Let τ be the time of exit from B_n. It follows from Dynkin's formula that

$$\mathcal{E}^x V(x_\tau) \leq V(x) - \mathcal{E}\tau$$

and hence that $\mathcal{E}\tau \leq V(x)$; in particular τ is finite w.p. 1.

Second we wish to show that the sets A_n are invariant. Consider the Lyapunov function

$$V(x) = \frac{1}{\cos x}$$

which is positive for $x \in A_n$ and has the expected time derivative

$$LV(x) = -\frac{1}{2}\cos x + 2 + \frac{1}{\cos x} - \frac{1}{\cos^2 x}$$

In particular, as x approaches the boundary of A_n from the interior we have $V(x) \to \infty$ and $LV(x) \to -\infty$. It follows from a minor modification of theorems 1 and 3 that the state stays in A_n for all times $t \geq 0$ w.p. 1 and that it is stochastically sample path bounded away from the boundary of A_n.

Using the same Lyapunov function, a minor modification of theorem 11 yields that for each n there exists a stationary distribution which is zero outside A_n.

It is possible to solve the stationary Fokker-Planck equation (forward Kolmogorov equation) explicitly using a general formula for scalar diffusions [6, p. 148]. This yields densities of the form

$$p_n(x) = c \cdot \frac{1}{\cos^2 x} \exp\left(-\frac{2}{\cos x}\right), \text{ for } x - 2n\pi \in (-\pi/2, \pi/2) \text{ and } 0 \text{ else.}$$

where c is a normalizing constant which ensures that the integral of p_n equals 1. Any positive convex combination of these densities also forms a possible stationary distribution.

Using proposition 14 with the Lyapunov function $V(x) = 2 - 2\cos x$ and the performance output $f(x) = 3\sin^2 x - 1$ we find that

$$LV(x) + f(x) = -(1 - \cos x)\cos^2 x \leq 0$$

which implies the performance bound

$$\mathcal{E} \sin^2 x \leq \frac{1}{3}$$

where expectation is with respect to the invariant distribution. A numerical computation using the actual invariant distribution shows that $\mathcal{E} \sin^2 x \approx 0.31$.

Finally, it is possible to show that any solution with initial condition $x \in A_n$ is stochastically asymptotically stable. This can be done using the Lyapunov function on $A_n \times A_n$

$$V(x, y) = (x - y)(\tan x - \tan y)$$

With a symbolic package such as `Maple` - or sufficient paper, pencils, and patience - it can be shown that $V(x_t, y_t)$ is a non-negative supermartingale when x_t and y_t solve the SDE; in fact $V(x_t, y_t)$ converges to 0 w.p. 1. Notice that this fact is not at all trivial.

9 Notes and references

Early contributions

The stability of stochastic differential equations in the sense of Itô was first investigated in a sequence of papers published in the first half of the sixties; primarily by Has'minskiĭ in the Soviet Union and by Kushner in the United States. Notice that this is not long after Lyapunov theory was introduced to the Western control community in [9]. Both authors compiled their results in monographs in the second half of the sixties, see [12] and [8] (the Russian edition of Has'minskiĭ's book appeared in 1969 while English-speaking readers had to wait for a translation until 1980).

More advanced topics in stability theory such as invariance theorems and converse theorems were studied already in the sixties by Kushner and Has'minskiĭ. The 1980 translation of Has'minskiĭ's 1969 book [8] is still the standard reference in English-speaking literature. More recent textbooks such as [6] contain very little development beyond [8] on the topic of stability although some proof techniques have been streamlined.

A nice review of the classical results with special emphasis on stability of moments and absolute stability (in a stochastic sense analogous to that of Lur'e and Popov) can be found in [13].

Other processes than Itô diffusions

Already the works of Kushner [12] used a set-up which did not require that the underlying stochastic differential equation was of Itô's type. In stead he focused on the infinitesimal generator A. Apart from Itô equations he gave special attention to Poisson differential equations.[7] A particularity with the generator A associated with Poisson differential equations is that it cannot be represented by any *differential* operator L.

A more radical generalization is presented in [14] where the Itô integral is replaced with a stochastic integral which needs not be linear and which needs not be driven by a martingale.

Terminology of stochastic stability

We have concentrated on the concept of stochastic stability using the terminology of Gard [6] and Gihman and Skorohod [7]. Kushner [12] uses the term *stability with probability one* for the same concept while numerous authors, e.g. [4], follow Has'minskiĭ [8] who called it *stability in probability*. To make the confusion complete, Gihman and Skorohod [7] use the term *stability in probability* for the property of equation (2) on page 12.

Existence and Uniqueness of solutions

The most well known theorem for existence and uniqueness of solutions to initial value problems assumes a *global* Lipschitz condition on the data as well as a *global* linear growth condition, see e.g. [16]. These conditions are needed to make a Picard iteration converge. Roughly speaking, the Lipschitz condition guarantees uniqueness while the linear growth condition rules out finite escape times. For time-invariant systems the Lipschitz condition implies the linear growth condition. The global Lipschitz condition can be weakened to a *local* Lipschitz condition which is a great advantage since global Lipschitz is very restrictive in applications. This relaxation is due to Gihman, see [7],

[7] where B_t in stead of being Brownian motion is a step process, where steps occur according to a Poisson process and where a regularity condition is imposed on the distribution of the step heights in order to guarantee that B_t is a well-behaved martingale

and a convenient statement of the result can be found in [6] as theorem 3.4, p. 76. If both the Lipschitz and linear growth conditions hold locally only, then theorem 1 gives a sufficient condition for existence and uniqueness. Has'minskiĭ has shown that this condition is also necessary, provided the SDE is non-degenerate.

Lyapunov techniques can be used to guarantee uniqueness of solutions even when the local Lipschitz conditions do not hold. See [6, p. 152].

See [14] for further recent results on existence and uniqueness of SDEs.

Solving stochastic differential equations

The main point of qualitative theory of SDEs as presented in this note is that it enables you to make statements about the solutions without actually finding these solutions. This is important because it is often impossible to find closed-form analytical solutions and very cumbersome to find numerical approximations. Even when explicit solutions are available it may not be advantageous to make use of them.

A nice introduction to the solution of SDE's (in analytical as well as numerical sense) can be found in chapters 4 and 7 in [6].

Generating families of Lyapunov functions

Let V be a Lyapunov function for the *deterministic* system $dx = f(x)\ dt$ such that $LV(x) = V_x(x)f(x) \leq 0$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $\phi(0) = 0$ and $\phi'(v) > 0$ for $v > 0$. Then $W(x) := \phi(V(x))$ is another Lyapunov function such that $LW(x) \leq 0$.

This does not hold for stochastic systems as seen from the example 2. However, if ϕ in addition is concave, i.e. $\phi''(v) \leq 0$ for $v > 0$, then $LW(x) \leq 0$.

Proceeding along this line we get the following result: Assume that V satisfies the hypothesis of theorem 10 to show exponential p-stability. Let $\phi(v) = |v|^\delta$ where $0 < \delta < 1$. Then with $W(x) = \phi(V(x))$ it is possible to show exponential δq-stability of the system. We have thus shown that if a system is exponentially p-stable (and satisfies the regularity conditions of theorem 10) then it is exponentially q-stable for $0 < q < p$.

Recent developments in control applications

There has been made some attempts to extend the results of the 1980's and early 1990's in *deterministic* non-linear control to a stochastic setting. In particular, Florchinger addresses in [4, 5] the *control Lyapunov function* approach of Sontag and others and extends it to stochastic problems. In [5] he also provides a recursive technique remniscent of deterministic backstepping in the sense of [11, 18]. Another contribution in this direction is [17].

The example of section 2 shows that it is possible to stabilize an unstable system $dx = r\ dt$ with $r > 0$ by adding a white noise term $\alpha x\ dB$ where α is large enough. This controversial approach to stabilization has been discussed intensely. A survey as well as a recent contribution to the discussion can be found in [15] where the result is generalized to nonlinear systems which satisfy a linear bound; furthermore it is shown that stable systems can be destabilized by adding mulitplicative noise provided the dimension of the state space is at least 2.

Construction of Lyapunov functions

In order to make the theory surveyed in this paper operational one needs a fail-safe tool for finding Lyapunov functions or determining that no such function exists. For the wide sense linear case methods exist, see [8], and if one considers mean square stability of such systems linear matrix inequalities can be employed, see [2, 3].

In [1] an analysis of a 2 degrees-of-freedom vibration isolation system is presented. Sufficient analytical conditions are given and as well as a non-conservative condition which must be verified numerically by simulation of a sub-system.

Other approaches to stability analysis

A quite different approach to stability analysis of a given stohastic system is to compute the *stochastic Lyapunov exponent*

$$\lim_{t\to\infty} \log \frac{|x_t|}{|x_0|}$$

If the limit exists and is negative, then the zero solution is exponentially stable. To show that the limit exists and is independent of x_0 makes use of ergodic theory and typically requires that the system is non-degenerate. In certain almost-linear cases the exponent can be computed analytically: Has'minskiĭ did this for linear systems [8]; in [15] the method is extended to non-linear but linearly bounded systems and in [20] a similar result is shown for non-linear systems which are homogenous in the sense that $f(k\ x) = k\ f(x)$ and $g(k\ x) = k\ g(x)$ for any k and x. In other cases the exponent can be computed numerically by simulating a sub-system, see [1] for an example.

Relaxing the non-degeneracy conditions

The non-degeneracy condition $gg' > 0$ appears in many of the results: Theorems 8, 9 and 16. This condition has the implication that the process x_t is *recurrent* relative to some domain. One would expect that a weaker *controllability* condition would do. A relaxation involving a PDE condition can be found on [8, p. 132]. Conversely, in theorem 11 the condition $LV(x) \to -\infty$ as $x \to \infty$ could probably be relaxed to some observability condition.

A Supermartingale theory

The following definition is quoted from [16]:

Definition 17: [Filtration, martingale] A filtration on the probability space $(\Omega, \mathcal{F})$ is a family $\{\mathcal{M}_t\}_{t\geq 0}$ of σ-algebras $\mathcal{M}_t \subset \mathcal{F}$ such that

$$0 \leq s < t \Rightarrow \mathcal{M}_s \subset \mathcal{M}_t$$

i.e. $\{\mathcal{M}_t\}$ is increasing. An n-dimensional process $\{M_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ is a martingale w.r.t. the filtration $\{\mathcal{M}_t\}$ and with respect to P if

(i) M_t is $\mathcal{M}_t$-measurable for all t,

(ii) $\mathcal{E}\{|M_t|\} < \infty$ for all t, and

(iii) $\mathcal{E}\{M_s|\mathcal{M}_t\} = M_t$ for all $s \geq t$.

□

For example, Brownian motion is a martingale with respect to its own filtration [16, p. 28].

If the equality in (iii) is replaced with $\leq$ ($\geq$), then we say that M_t is a supermartingale (submartingale).

We have Doob's martingale inequality [16, p. 28] (see [14, p. 4] for an alternative formulation):

Theorem 18: Let M_t be a martingale with continuous sample paths, a.s. Let $p \geq 1$, $T \geq 0$ and $\lambda > 0$, then

$$P(\sup_{0\leq t\leq T} |M_t| \geq \lambda) \leq \frac{1}{\lambda^p} \cdot \mathcal{E}\{|M_T|^p\}$$

□

which generalizes Markov's inequality. For a non-negative supermartingale this implies

$$P(\sup_{0\leq t\leq T} M_t \geq \lambda) \leq \frac{M_0^p}{\lambda^p} \tag{3}$$

Supermartingales have nice convergence properties established by Doob:

Theorem 19: Let N_t be a right continuous supermartingale such that

$$\sup_{t>0} \mathcal{E}\{N_t^-\} < \infty$$

where $N_t^- = \max(-N_t, 0)$. Then there exists a stochastic variable N such that $N_t \to N$ w.p. 1. furthermore, $\mathcal{E}N^- < \infty$. □

Notice that non-negative supermartingales automatically satisfy the hypothesis.

B On the use of stopping times

Lyapunov theorems work by establishing supermartingale properties of Lyapunov functions $V(x_t)$. A problem in this is that the chain rule of stochastic calculus, namely Itô's formula, cannot always be applied to V because V will not always be C^2 everywhere - see the example in section 2. In this appendix we go through a sequence of lemmas in order to demonstrate how this technical difficulty can be circumvented by a systematic use of *stopping times* or *Markov times*. Recall that random variable τ is a stopping time (w.r.t. an underlying filtration $\mathcal{F}_t$) if

$$\forall t: \ \{\omega | \ \tau(\omega) < t\} \in \mathcal{F}_t$$

i.e, at each time t we can tell whether or not the event $\tau < t$ has occured using only the information in $\mathcal{F}_t$.

Lemma 20: Let V be a non-negative continuous function $\mathbb{X} \to \mathbb{R}$. Let D be the preimage of some compact interval $I = [a, b]$, i.e. $D = \{x | \ a \le V(x) \le b\}$. Assume that D is bounded, and that V is C^2 on D and satisfies $LV \le 0$ on D. Let the diffusion x_t start in $x_0 = x$ and define the stopping time τ_D and the stopped process x_t^D by

$$\tau_D = \inf\{s > 0: \ x_s \notin D\} \ \text{ and } \ x_t^D = x_{t \wedge \tau_D}$$

Then the inequality

$$\mathcal{E}^x V(x_t^D) \le V(x)$$

holds for all $t > 0$ and $x \in D$. Furthermore, the probability that x_s exits through $V(x) = b$ is bounded by

$$P\{\sup_{s\ge 0} V(x_s) \ge b\} \le \frac{V(x)}{b}$$

□

Proof: By assumptions D is compact, hence we can modify V outside D such that V is C^2 everywhere and has compact support. Define the stopping time $\tau = \tau_D \wedge t$. Then $\mathcal{E}^x \tau < \infty$ and we may apply Dynkin's formula [16]

$$\mathcal{E}^x V(x_\tau) = V(x) + \mathcal{E}^x \int_0^\tau LV(x_s) ds$$

to find that $\mathcal{E}^x V(x_\tau) \leq V(x)$ since $LV(x_s) \leq 0$ for all $0 < s < \tau$. Now use that $x_\tau = x_{\tau_D \wedge t} = x_t^D$.

To see the probability bound, notice that $V(x_t^D)$ is a non-negative supermartingale w.r.t. the filtration $\mathcal{F}_t$ and that $\sup_{s\geq 0} V(x_s) \geq b$ happens if and only if $V(x_t^D) = b$ for some $t > 0$. ■

Lemma 21: Let the assumptions of the previous lemma hold except that the interval I is open and bounded, $I = (a, b)$. Then the same conclusions hold. □

Proof: Let $I_n = [a_n, b_n]$ be a sequence of increasing closed intervals which converge to I. For each I_n the previous theorem applies which gives that

$$\mathcal{E}^x V(x_t^n) \leq V(x) \quad \text{where} \quad \tau_n = \inf\{s > 0 : x_s \notin D_n\} \quad \text{and} \quad x_t^n = x_{t\wedge\tau_n}$$

Now notice that $t \wedge \tau_n$ converges to $t \wedge \tau_D$ w.p. 1. It follows from continuity that $V(x_t^n)$ converges to $V(x_t^D)$ w.p. 1 and hence also in law, in particular

$$\mathcal{E}^x V(x_t^n) \to \mathcal{E}^x V(x_t^D)$$

from which the conclusion follows. To see the probability bound notice that

$$P\{\sup_{s\geq 0} V(x_s) \geq b\} \leq P\{\sup_{s\geq 0} V(x_s) \geq b_n\} \leq \frac{V(x)}{b_n}$$

since this holds for all $b_n < b$ the conclusion follows. ■

Lemma 22: Let the assumption of the previous lemma hold except that the interval I is unbounded, i.e. $I = (a, \infty)$ or $I = [a, \infty)$, and that D is not necessarily bounded. Assume furthermore that V is proper. Then then the process x_t is regular and $\mathcal{E}^x V(x_t^D) \leq V(x)$ as before. □

Proof: Let $I_n = [a + 1/n, n]$. Then for each I_n lemma 20 applies. In particular the probability that x_t escapes any bounded set in finite time is smaller than $V(x)/n$ for any n and hence zero. So the process is regular. Using the same notation as in the previous proof, it follows that $\mathcal{E}^x V(x_t^n) \leq V(x)$ and $\tau_n \to \tau_D$ w.p. 1. The conclusion follows as in the previous proof. ■

See [14, p. 24–34] for a number of related results.

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