On Multi-Dissipative Perturbations in Linear Systems

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Abstract

We consider the question of robust stability of a linear time invariant plant subject to dynamic perturbations, which are dissipative in the sense of Willems with respect to *several* quadratic supply rates. For instance, parasitic dynamics are often both frequency weighted small gain *and* passive, and including this in the robustness analysis results in conditions that are less conservative than for instance the small gain criterion.

We reduce several robustness analysis questions to Linear Matrix Inequalities: Robust stability, robust stability in presence of parameter uncertainty, robust $\mathcal{H}_2$ performance, and finally robust performance in presence of disturbances with finite signal-to-noise ratio in the sense of Skelton.

The corresponding problem of static state feedback controller design reduces to non-convex matrix inequalities, which only in special cases can be transformed to convex ones. We suggest heuristic coordinate descent type algorithms to solve these problems.

Keywords

Linear Systems, Robustness, Dissipativity, Performance

1 Introduction

It is popular to deal with dynamic perturbations in control systems using the framework of dissipativity, originally introduced in [12]. Roughly speaking, dissipative systems are unable to produce a specified quantity, and (under certain regularity conditions) dissipativity is preserved under interconnections and implies Lyapunov stability. Although the notion of dissipativity is quite general, most attention has been given to the two special cases of passivity and bounded $\mathcal{L}_2$ gain.

In this paper we examine perturbations which are dissipative with respect to *several* supply rates. One example is parasitic dynamics, which (with proper choice of inputs and outputs) are both passive and small gain. Another example is block diagonal perturbations which inherit the dissipativity properties of each of the diagonal elements - for instance μ-type situations.

For systems consisting of a nominal part in feed-back with a multi-dissipative perturbation, stability and various performance properties are implied by linear matrix inequalities which describes some dissipativity property of the nominal part. The connection between dissipativity for LQ systems and LMIs was noted already in [12] and has received much interest during the last few years due to efficient numerical algorithms for solving LMI problems [1, 3].

The paper is organized as follows: In section 3, we recap some definitions and results from dissipativity theory [12]. In section 4 we derive our first result which concerns robust stability of a system containing contains a multi-dissipative perturbation, and in section 5 we present a simple example which should illustrate the relevance of the result. In section 6 we extend the robust stability analysis to systems with uncertain parameters. In section 7 we give conditions for robust $\mathcal{H}_2$ performance. In section 8 we address the Finite Signal-to-Noise Ratio model of Skelton [7] and give sufficient conditions for robust performance. All conditions up to now are in terms of linear matrix inequalities which can be verified directly numerically with publically available packages such as [3]. In section 9 we discuss a heuristic method for solving the bi-linear matrix inequalities which appear in the design of state feedback controllers. Finally section 10 offers some concluding remarks and points at a few open problems.

Some of the results presented here are published in the conference paper [9].

2 Notation

$\mathbb{R}_+$ denotes the non-negative reals. When A is a subset of a linear space, $\mathbf{Co}(A)$ denotes the convex hull of A. The symbol $\mathbf{\Delta}$ denotes a set of dynamic (perturbation) systems with state-space representations.

3 Preliminaries

We deal with dynamic systems Σ defined by ordinary differential equations in state-space:

$$\Sigma : \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bw(t) \\ z(t) & = & Cx(t) + Dw(t) \end{array} \tag{1}$$

Here, the system has input $w(t) \in \mathbb{W}$, output $z(t) \in \mathbb{Z}$ and state $x(t) \in \mathbb{X}$, and these spaces are linear finite-dimensional spaces defined over $\mathbb{R}$.

Our notion of dissipativity is identical to the original one of Willems [12]:

Definition 1: A system Σ is said to be *dissipative* with respect to the supply rate $s : \mathbb{W} \times \mathbb{Z} \to \mathbb{R}$ if there exist a *storage* function $V : \mathbb{X} \to \mathbb{R}_+$ such that for all time intervals $[0, T]$, all initial conditions x_0 and all inputs $w \in \mathcal{L}_2([0, T])$ the state trajctories are well defined on $[0, T]$ and the *dissipation* inequality

$$V(x(T)) \leq V(x(0)) + \int_0^T s(w(t), z(t))\ dt \tag{2}$$

holds. □

For linear time invariant systems with quadratic supply rates (i.e, when $s(w, z) = (w' \ \ z')Q(w' \ \ z')')$, there is no loss of generality [12] in assuming V to be quadratic ($V(x) = x'Px$ with $P = P' \geq 0$), in which case the Dissipation Inequality may be written in differential form $\dot{V}(x(t)) \leq s(w(t), z(t))$ or equivalently [12]

$$\forall x \in \mathbb{X},\ w \in \mathbb{W} : (x' \ \ w')\Phi \begin{pmatrix} x \\ w \end{pmatrix} \leq 0$$

where Φ is shorthand for

$$\Phi = \begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} - \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q \begin{bmatrix} O & I \\ C & D \end{bmatrix}$$

Notice that this is a linear matrix inequality (LMI) in P [1]. There exist software packages which enable a numerical search for a P which satisfies this LMI [3].

We will find the following stronger notion of dissipativity useful:

Definition 2: The system Σ is said to be *strongly dissipative* with respect to the supply rate s, if for some $\epsilon > 0$ it is dissipative with respect to the supply rate $s(w, z) - \epsilon \|z\|^2$. Let V be a corresponding storage function, then V is said to be a *strong storage function* for Σ w.r.t. s. □

This definition fits together with our notion of stability: Whereas the stability results of Willems [12] aim at showing Lyapunov stabilty of an interconnection of given systems, we are interested in robustness of interconnections of nominal plants and unknown perturbations:

Definition 3: Let a nominal system Σ be connected in feedback with some (unknown) perturbation $\Delta \in \mathbf{\Delta}$ as in figure 1 (a). The configuration $(\Sigma, \mathbf{\Delta})$ is said to be *robustly stable*, if for any perturbation $\Delta \in \mathbf{\Delta}$ and any initial conditions x_0 and ξ_0 (of Σ and Δ, respectively), the signals $w(\cdot)$ and $z(\cdot)$ are well defined on $[0, \infty)$ and $z \in \mathcal{L}_2$. □

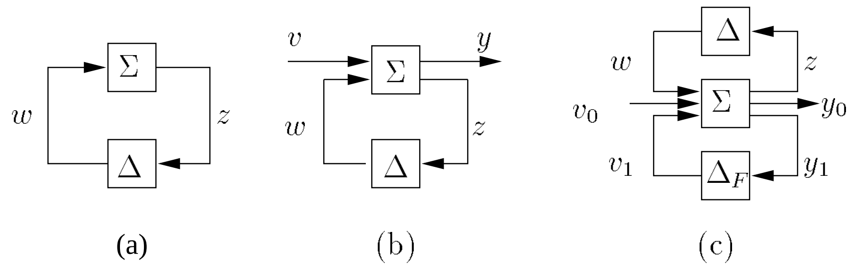

Figure 1: The three problems considered: (a) Robust stability. (b) Robust $\mathcal{H}_2$ performance. (c) FSN performance.

Remark 4: Notice that the signal z serves two purposes: It is the output to the perturbation and it is also the error signal which we wish to guarantee is bounded in $\mathcal{L}_2$. If for example we wish to establish that $x \to 0$ as $t \to \infty$, we could include x and w in z (notice that $w \in \mathcal{L}_2$ and $x \in \mathcal{L}_2$ implies $x \to 0$). □

With this definitions we may pose a stability condition which is slightly different from the one in [12]:

Lemma 5: Assume that every $\Delta \in \boldsymbol{\Delta}$ is linear time invariant and dissipative w.r.t. $-s$ and that Σ is strongly dissipative with respect to the supply rate s. Then the feed-back configuration $(\Sigma,\ \boldsymbol{\Delta})$ is robustly stable. $\square$

Remark 6: If Σ is dissipative w.r.t. s but not strongly so, and if the interconnection is well posed (which in this case is not guaranteed by the dissipativity), and if the storage functions are (quadratic and) positive definite, then we have Lyapnuov stability [12], but z need not be in $\mathcal{L}_2$. $\square$

Proof: To show well-posedness is a standard exercise: The dissipation inequalities for Σ and Δ are

$$\begin{aligned} V(x(T)) &\leq V(x(0)) + \int_0^T s(w,z) - \epsilon \|z\|^2 \, dt \\ W(\xi(T)) &\leq W(\xi(0)) + \int_0^T -s(w,z) \, dt \end{aligned}$$

Since we know that V has a local minimum (possibly non-strict) at $x = 0$, it is easy to see that for $x = 0$ we must have $s(w,z) - \epsilon\|z\|^2 \geq 0$. Repeating the argument for W we get that $s(w,z) \leq 0$ whenever $\xi = 0$. Combining the two we get that whenever $(x,\xi) = (0,0)$ we have $(w,z) = (0,0)$. This implies (using linearity) that the interconnection is well posed, and hence the solutions exist on $[0,\infty)$.

Adding the two dissipation inequalities and using the non-negativity of the storage functions give

$$\int_0^T \|z\|^2 \, dt \leq \frac{1}{\epsilon}\left(V(x(0)) + W(\xi(0))\right)$$

Since this holds for all T, the configuration is robustly stable. ■

Remark 7: It can be argued that the lemma is of limited interest since it is very restrictive to assume that perturbations are linear time invariant[1]. Notice, however, that these assumptions on Δ are only used to guarantee existence and uniqueness of the state trajectories; the $\mathcal{L}_2$-bound on z follows directly from the storage functions. Luckily, there exist other ways of guaranteeing existence and uniqueness when Δ is non-LTI: Local well-posedness may be established through linearizations; assuming that

[1] A similar point was also emphasized in [8]

Δ is strictly causal removes algebraic loops; including w in z and assuming properness of W guards against finite escape times; finally one may assume incremental properties on Δ and use the input-output setting of e.g. [10, 11].

To avoid these details we will in the remainder of the paper always assume that any feedback connection which we analyse is well-posed in the sense that there exist unique signals which solve the describing equations. We remark that if one is willing to make the assumption that Δ is LTI, then well-posedness is guaranteed by the stability conditions we derive. $\square$

In this remainder of the paper we consider analysis of systems Σ connected in feedback as in figure 1 (a) with a perturbation Δ, which is dissipative with respect to the p supply rates

$$-s_i = -(w' \;\; z')Q_i \begin{pmatrix} w \\ z \end{pmatrix}, \qquad i \in \{1, \dots, p\} \tag{3}$$

Without loss of generality we assume that Q_i are symmetric. We use the symbol $\boldsymbol{\Delta}$ to denote this particular class of *multi-dissipative* perturbations, i.e.

$$\boldsymbol{\Delta} = \{\Delta : \text{ dissipative w.r.t. } -s_i;\; i \in \{1, \dots, p\}\} \tag{4}$$

To each $\Delta \in \boldsymbol{\Delta}$ and each supply rate $-s_i$ corresponds a storage function W_i defined on the state space of Δ.

4 Robust stability

It was noted already in [12] that the storage functions for a dynamical system with respect to a single supply rate form a convex set. For multi-dissipative systems this fact extends immediately to the following:

Lemma 8: Let Σ be a dynamic system with state space $\mathbb{X}$, input space $\mathbb{W}$ and output space $\mathbb{Z}$. Let $\mathcal{V}$ be a linear space of *candidate storage* functions $\mathbb{X} \to \mathbb{R}$ and let $\mathcal{S}$ be a linear space of supply rates $\mathbb{W} \times \mathbb{Z} \to \mathbb{R}$. Then the set

$$\{(V, s) \in \mathcal{V} \times \mathcal{S} |\; V \geq 0, \text{ and } (V, s) \text{ satisfy (2) } \}$$

is a convex cone. $\square$

Proof: Follows immediately by inserting into the Dissipation Inequality (2) $\blacksquare$

In particular, the set of supply rates, with respect to which the system is dissipative, is a convex cone. This enables us to apply convex search routines among the supply rates.

We see that if the perturbation Δ is dissipative w.r.t. $-s_i$ for $i \in \{1, \ldots, p\}$ then Δ is also dissipative w.r.t. any supply rate $-s$ in the convex hull of $\{-s_i | \; i \in \{1, \ldots, p\}\}$. If furthermore Σ is strongly dissipative w.r.t. just *one* of these supply rates, we have robust stability. This leads to our first result:

Theorem 9: Given the system Σ defined in (1) and the class of perturbations $\boldsymbol{\Delta}$ of (4). The configuration $(\Sigma, \; \boldsymbol{\Delta})$ is robustly stable if the following linear matrix inequality

$$\begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} - \sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} + \epsilon \begin{bmatrix} C' \\ D' \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \leq 0 \tag{5}$$

is satisfied for some $P = P' \geq 0, \; \epsilon > 0, d_i \geq 0$ □

We emphasize that the theorem gives a less restrictive condition than e.g. small gain criterion or positive real criterion because of the extra freedom associated with d_i.

Remark 10: One will often examine the following LMIP in $\bar{P}$ and $\bar{d}$ in stead:

$$\begin{bmatrix} \bar{P}A + A'\bar{P} & \bar{P}B \\ B'\bar{P} & 0 \end{bmatrix} - \sum_{i=1}^{p} \bar{d}_i \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} < 0$$

$$\bar{P} > 0, \; \bar{d}_i > 0 \tag{6}$$

Feasibility of this LMIP is a sufficient condition for robust stability of $(\Sigma, \; \boldsymbol{\Delta})$ and - under weak assumptions on the data - equivalent to feasibility of (5). □

Proof: Given a solution pair P, d_i, ϵ, the function $x'Px$ acts as a strong storage function for Σ with respect to the supply rate $\sum_i d_i s_i$ and since any $\Delta \in \; \boldsymbol{\Delta}$ is dissipative w.r.t. $-\sum_i d_i s_i$ (lemma 8) we have shown robust stability (lemma 5).

It is also easy to see that feasibility of (6) implies feasibility of (5): In fact, given solutions $\bar{P}, \bar{d}_i$ to (6), one may find sufficiently small $\epsilon > 0$ such that $\bar{P}, \bar{d}_i, \epsilon$ solves (5). ■

5 An illustrative example

In this section we will consider a simple single-input single-output example in order to illustrate theorem 9.

Consider the spring-mass system in figure 2, which is a simple model of a one-dimensional position regulator system. The force u is the output of an LTI controller. Our goal is to establish non-conservative conditions for the stability of the interconnected system, i.e. we consider the small mass as an unmodelled parasitic, and the parameters associated with it to be very uncertain.

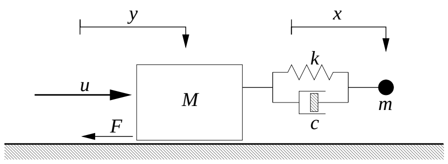

Figure 2: A Position Regulator with Parasitic Dynamics

The overall interconnection of the small mass and the remaining system may be written in the form of figure 1 (a). The error signal z is then the velocity $\dot{y}$ of the large mass while the disturbance w is the force acting from small mass on the large mass. With this formulation, Δ is given by

$$\Delta(s) = \frac{(k+cs)ms}{ms^2+cs+k}$$

The transfer function Δ is positive real, i.e. dissipative w.r.t. the supply rate $s_0(z,w) = zw$. It is also small gain, i.e. dissipative w.r.t. the supply rate $s_1(z,w) = \gamma^2 z^2 - w^2$ for $\gamma \geq \|\Delta\|_\infty \approx km/c\sqrt{1+c^2/km}$.

One can easily imagine situations where physical considerations provide a reasonable bound on γ but where we nevertheless do not wish to estimate m, c and k. Indeed, in many situations we do not wish to specify the order of Δ.

The suitable robustness analysis question is then: When is Σ robustly stable with respect to perturbations Δ which are (LTI and) dissipative with respect to s_i for $i \in \{0,1\}$? We have already in the previous established *sufficient* conditions (theorem 9). In order to construct necessary and sufficient conditions through frequency domain considerations, notice that the requirement that Δ is LTI, passive and small gain may be stated as[2]

$$\forall s \in \bar{\mathbb{C}}^{+} : \ \Delta(s) \in A = \{s \in \mathbb{C} | \mathbf{Re}\, s \geq 0 \wedge |s| \leq 1\} \tag{7}$$

A is sketched in figure 3. The interconnection is unstable if and only if

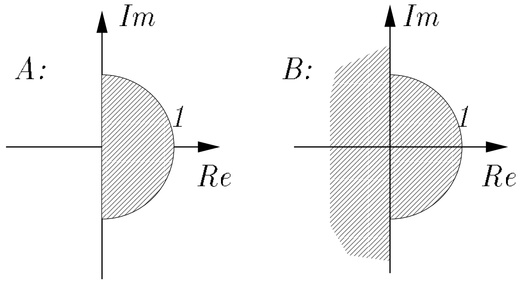

Figure 3: The subsets A and B of the complex plane

there exists an $s \in \bar{\mathbb{C}}^{+}$ such that

$$\Sigma(s)\Delta(s) = 1$$

So the interconnection is stable *for all* Δ such that (7) holds if and only if

$$\forall s \in \bar{\mathbb{C}}^{+} : \ \Sigma(s) \in B = \{s \in \mathbb{C} | \ \frac{1}{s} \notin A\} \tag{8}$$

$$= \{s \in \mathbb{C} | \ \mathbf{Re}\, s < 0 \vee |s| < 1\}$$

(B is sketched in figure 3). Notice that the set B may be described by

$$B = \{s \in \mathbb{C} | \ \exists \alpha > 0 : \ |s + \alpha| < |i + \alpha|\}$$

So the condition (8) holds if and only if there exists a map $\alpha : \bar{\mathbb{C}}^{+} \rightarrow \mathbb{R}^{+}$ such that

$$\forall s \in \bar{\mathbb{C}}^{+} : \ |\Sigma(s) + \alpha(s)| < |i + \alpha(s)|$$

The inequality in this expression can be restated as

$$(\bar{\Sigma}(s) \ 1) \begin{bmatrix} 1 & \alpha(s) \\ \alpha(s) & -1 \end{bmatrix} \begin{pmatrix} \Sigma(s) \\ 1 \end{pmatrix} < 0 \tag{9}$$

[2]Here we have rescaled Δ such that $\gamma = 1$.

To recap, the feedback system is stable if and only if there exists an $\alpha : \bar{\mathbb{C}}^+ \to \mathbb{R}^+$ such that this holds for all $s \in \bar{\mathbb{C}}^+$. This is exactly the type of stability conditions that appear in [4] (see also the references therein).

If we pose the restriction that $\alpha(s)$ is independent of s, then we get the condition that (a state-space realization of) Σ is strongly dissipative with respect to the supply rate

$$s_\alpha(w,z) = (w\ z)\begin{bmatrix} 1 & -\alpha \\ -\alpha & -1 \end{bmatrix}\begin{pmatrix} w \\ z \end{pmatrix} \tag{10}$$

This is exactly the sufficient condition of remark 10, and we see that it has the interpretation that Σ maps the right half of the complex plane into the inner of some circle centered in $-\alpha < 0$ and containing i - see figure 4. Using the maximum modulus theorem, this will be the case if and only if Σ is stable and its Nyquist plot is contained in such a circle. It is evident that this condition is more conservative than (9), however it may be desirable to accept this conservatism in order to avoid the frequency dependence of α. Furthermore. if Δ is non-LTI then Σ being dissipative w.r.t. s_α still guarantees robust stability (assuming well-posedness) whereas the frequency-dependent condition (9) does not.

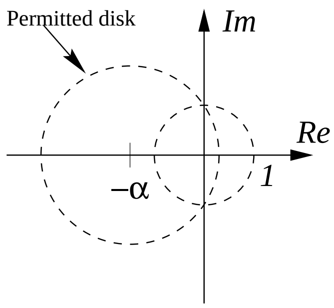

Figure 4: Permitted area for $\Sigma(\bar{\mathbb{C}}^+)$ with $\alpha(s)$ independent of s

Notice, for $\alpha(s) \to 0$, we retrieve the condition that Σ has $\mathcal{L}_2$-gain less than 1, while for $\alpha(s) \to \infty$ the permittable circle approaches the entire left half plane, and thus we get the condition that Σ is Strictly Negative Real. For high order plants, the latter condition is often difficult or even impossible to obtain, while the former may impose too severe constraints on bandwidth. Also taking $\alpha \in (0, \infty)$ into account obviously increases the possibility of reaching a good design.

6 Parameter uncertainty

A popular model of parameter uncertainty is that the system matrices A, B, C and D (are possibly time varying and) belong to a given polytope:

$$\begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \in \mathbf{Co}\left(\left\{\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} \mid j = 1, \ldots, m\right\}\right) \tag{11}$$

This situation covers not only parameter uncertainties but also non-linear systems with sector-bounded non-linearities; in [1] a model like this is called a *Polytopic Linear Differential Inclusion* and many properties of such models are reduced to feasibility of LMIs. It is therefore not surprising that also robustness in the presence of multi-dissipative perturbations can be guaranteed with LMIs. In order to discuss robust stability of the configuration in figure 1 (a), we modify the definition 3 in the following obvious way: the signals w and z must be well defined for all initial times and initial conditions, and z must be bounded in $\mathcal{L}_2$ uniformly over the initial time. For this situation, a sufficient condition for robust stability in the presence of multi-dissipative perturbations is given by the following:

Theorem 11: Let Σ satisfy (11). Assume that each of the supply rates s_i is concave in z and that the following LMIP in P, d_i and ϵ is feasible:

$$\forall j \in \{1, \ldots, m\}:$$

$$\begin{bmatrix} PA_j + A_j'P & PB_j \\ B_j'P & 0 \end{bmatrix} - \sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C_j' \\ I & D_j' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C_j & D_j \end{bmatrix}$$

$$+\epsilon \begin{bmatrix} C_j' \\ D_j' \end{bmatrix} \begin{bmatrix} C_j' & D_j' \end{bmatrix} \leq 0$$

$$P \geq 0,\ d_i \geq 0,\ \epsilon \geq 0$$

Then the feed-back configuration $(\Sigma,\ \boldsymbol{\Delta})$ is robustly stable. □

Remark 12: The assumption that s_i is concave in z may be written

$$Q_i^{zz} \leq 0 \text{ where } Q_i = \begin{bmatrix} Q_i^{ww} & Q_i wz \\ Q_i^{zw} & Q_i^{zz} \end{bmatrix}$$

and essentially says, that when the input w is zero, the flow s_i is zero or directed out of the system Σ. □

Proof: Due to the condition (11) we know that there exists parameters $\lambda_j(t) \geq 0$, $j = 1, \dots, m$, such that

$$[A(t), B(t), C(t), D(t)] = \sum_{j=1}^{m} \lambda_j(t)[A_j, B_j, C_j, D_j], \qquad \sum_{j=1}^{m} \lambda_j(t) = 1$$

We omit the time argument after signals and use the notation

$$z_j = C_j x + D_j w$$

Our candidate storage function for Σ is $x'Px$. We then get

$$\begin{aligned}
\frac{d}{dt} x'Px &= (x' \;\; w') \begin{bmatrix} PA(t) + A'(t)P & PB(t) \\ B'(t)P & 0 \end{bmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \\
&= (x' \;\; w') \left(\sum_{j=1}^{m} \lambda_j \begin{bmatrix} PA_j + A_j'P & PB_j \\ B_j'P & 0 \end{bmatrix} \right) \begin{pmatrix} x \\ w \end{pmatrix} \\
\leq \sum_{j=1}^{m} \lambda_j (x' \;\; w') & \left(\sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C_j' \\ I & D_j' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C_j & D_j \end{bmatrix} - \epsilon \begin{bmatrix} C_j' \\ D_j' \end{bmatrix} [C_j \;\; D_j] \right) \begin{pmatrix} x \\ w \end{pmatrix} \\
&= \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{p} d_i (w' \;\; z_j') Q_i \begin{pmatrix} w \\ z_j \end{pmatrix} - \epsilon z_j' z_j \right) \\
&= \sum_{i=1}^{p} d_i \left(\sum_{j=1}^{m} \lambda_j (w' \;\; z_j') Q_i \begin{pmatrix} w \\ z_j \end{pmatrix} \right) - \epsilon \sum_{j=1}^{m} \lambda_j z_j' z_j \\
&\leq \sum_{i=1}^{p} d_i \left((w' \;\; z') Q_i \begin{pmatrix} w \\ z \end{pmatrix} \right) - \epsilon z'z
\end{aligned}$$

We have thus show that the time-invariant function $x'Px$ is a strong storage function for the time-varying system Σ w.r.t. the supply rate $\sum_i d_i s_i$ and hence we may conclude robust stability of the interconnection (Σ, Δ). ∎

7 Guaranteed $\mathcal{H}_2$ Performance

We now expand the system with a noise input $v(t)$ and a performance output $y(t)$:

$$\begin{aligned} \dot{x} &= Ax + Bw + Gv \\ z &= Cx + Dw \\ y &= Hx + Jw \end{aligned} \tag{12}$$

As before, we have $w = \Delta z$ where $\Delta \in \mathbf{\Delta}$. As a measure of performance for the unknown closed-loop system (Σ, Δ) we use its $\mathcal{H}_2$-norm.

When Δ is non-LTI one needs to specify what is meant by the $\mathcal{H}_2$-norm of the interconnection, since it cannot be represented by a transfer function. Two possibilities exist: One can use the $\mathcal{L}_2$-norm of the impulse response which we will call the deterministic $\mathcal{H}_2$-norm of the interconnection, or one can assume that v is white noise (i.e, rewrite (12) as a Itô diffusion [5]) and consider the steady-state variance of y. However, for general non-LTI perturbations we cannot even guarantee that a steady-state exist. In the following we circumvent this difficulty.

The mathematical model of the system Σ in interconnection in figure 1 (b) is the Itô diffusion

$$dx = (Ax + Bw)dt + GdB$$

where B is brownian motion with respect to an underlying filtration $\mathcal{F}_t$. We notice that since Δ is a deterministic causal system w will be $\mathcal{F}_t$-adapted. The 'white noise' signal v in figure 1 (b) can loosely be identified with the derivative dB/dt. With this model we may define the $\mathcal{H}_2$-norm (here, $\mathcal{E}_t$ denotes expectaion conditioned on $\mathcal{F}_t$):

Definition 13: The stochastic $\mathcal{H}_2$-norm of the interconnection in figure 1 (b) is

$$||F_l(\Sigma, \Delta)||^2_{\mathcal{H}_2} = \limsup_{T \to \infty} \frac{1}{T} \mathcal{E}_0 \{ \int_0^T |y(t)|^2 \ dt \}$$

□

A couple of remarks are appropriate. First, notice that in case the process y converges to a stationary ergodic process then the $\mathcal{H}_2$-norm of the system equals the root mean square of the limit distribution. Secondly, in case Δ is non-LTI one should think that the choice of initial state (in deterministic case, also the initial time and amplitude of the impulses) becomes important. For a single specified Δ there may indeed be ambiguity, but under the worst-case considerations associated with $\Delta \in \mathbf{\Delta}$ this ambiguity disappears. This will be clear in the proof of the following bound on the $\mathcal{H}_2$-norm of the interconnection, which holds whether we use a stochastic or a deterministic interpretation:

Theorem 14: The $\mathcal{H}_2$ norm of the closed loop system from v to y is bounded above by

$$\|F_l(\Sigma, \Delta)\|^2_{\mathcal{H}_2} \leq \inf_{P, d_i, \epsilon} \text{trace } G'PG$$

where the infimization is subject to

$$P = P' \geq 0, \; d_i \geq 0, \; \epsilon > 0$$

$$\begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} - \sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q_i \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}$$
$$+ \begin{bmatrix} H' \\ J' \end{bmatrix} \begin{bmatrix} H & J \end{bmatrix} + \epsilon \begin{bmatrix} C \\ D \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \leq 0$$

□

Remark 15: Computation of the upper bound on $\mathcal{H}_2$-performance is an LMIP in P, d_i and ϵ.

Notice that if we remove Δ, the LMI in P reduces to $PA + A'P + H'H \leq 0$, i.e. we retrieve the standard way of computing the $\mathcal{H}_2$-norms of known system using the Gramians [13]. □

Proof: Given P, d_i and ϵ, we know that Σ *with* $v \equiv 0!$ is strongly dissipative w.r.t. $\sum_i d_i s_i - \|y\|^2$. This implies strong dissipativity w.r.t. $\sum_i d_i s_i$ and hence it is reasonable to assume well-posednes of the interconnection of Σ and Δ, cf. remark 7.

Deterministic interpretation: Assume that the interconnection (Σ, Δ) is at rest for $t < 0$ and that we at $t = 0$ excite the interconnection with an impulse at v, i.e. $v(t) = v_0 \delta(t)$ where $\delta(\cdot)$ is the Dirac delta. We then have $x(0^+) = Gv_0$. Assume that P, d_i and ϵ solve the LMIP in the theorem, then the integral dissipation inequality for the interconnection reads

$$\int_0^T \|y\|^2 \; dt + \int_0^T \epsilon \|z\|^2 \; dt \leq x'(0^+) P x(0^+)$$

(and holds because $v(t) = 0$ for $t > 0$). Hence,

$$\int_0^\infty \|y\|^2 \; dt \leq v_0' G' P G v_0$$

Now let v_j be the jth unit vector in the input space $\mathbb{V} = \mathbb{R}^{n_v}$ ($v(t) \in \mathbb{V}$) and let $y_j(t)$ be the impulse response of the interconnection (Σ, Δ) to the input $v(t) = v_j \delta(t)$. We then have

$$\|F_l(\Sigma, \Delta)\|_{\mathcal{H}_2}^2 = \sum_{i=1}^{n_v} \int_0^\infty \|y_j(t)\|^2 \; dt \leq \sum_{i=1}^{n_v} v_i' G' P G v_i = \text{trace } G' P G$$

Since this holds for any P, d_i and ϵ that solve the LMIP the conclusion in the theorem follows.

Stochastic interpretation: The Itô model is

$$dx = (Ax + Bw)dt + GdB$$

where B is standard Brownian motion [5] w.r.t. the underlying filtration $\mathcal{F}_t$. Notice that w is adapted to $\mathcal{F}_t$ since Δ is causal. Now consider the function $V(x) = x'Px$. Itô's lemma yields:

$$dV = 2x'P(Ax + Bw) \; dt + \text{tr}(GPG') \; dt + 2x'PG \; dB$$

which gives

$$\mathcal{E}_0\{\Delta V(T)\} = \mathcal{E}_0\{\int_0^T 2x'P(Ax + Bw) + \text{tr}(GPG') \; dt\}$$

where $\Delta V(T)$ is shorthand for $V(x(T)) - V(x(0))$. Now let $\gamma \geq \text{tr}(GPG')$ and assume that P, d, ϵ solve the LMIP in the theorem. Now notice that the dissipation properties of Δ impliy

$$\mathcal{E}_0\{\Delta W_i(T)\} \leq -\mathcal{E}_0\{\int_0^T s_i \; dt\}$$

since Δ is deterministic (the dissipation inequality holds for all realizations and hence in particular also in expectation). Combine this with the dissipation property of Σ which is implied by the LMI and use the positive definiteness of the storage functions to obtain:

$$\mathcal{E}_0\{\int_0^T \epsilon|z|^2 + |y|^2 \; dt\} \leq V(x(0)) + \sum d_i W_i(\xi(0)) + \gamma T$$

In particular

$$\limsup_{T\to\infty} \mathcal{E}_0\{\frac{1}{T}\int_0^T |y|^2\ dt\} \le \gamma$$

This holds for all feasible P, d, ϵ from which the conclusion follows. ■

8 Finite Signal-to-Noise Ratios

In this section we examine the system in figure 1 (c) where $\Delta \in \mathbf{\Delta}$ is a multi-dissipative perturbation as in (4) and v_1 is an exogenous white noise signal with unit intensity as in section 7. The new component Δ_F is a *noise generator* with a signal-to-noise ratio of 1, i.e. v_2 is a white noise signal with intensity equal to the power of y_2. Such *FSN* models were introduced by Skelton in [6] to describe uncertain sensors and actuators etc, see [7].

The FSN models in these references focus on steady-state behavior. Therefore we specify

Definition 16: A linear FSN model is a model of the form

$$\begin{aligned} dx &= (Ax+Bw)dt + G_0 dB_0 + \sum_{j=1}^{m} G_j \nu_j dB_j \\ y_j &= Hx + Jw\ , \qquad\qquad j = 0,\ldots,m \\ z &= Cx + Dw \end{aligned}$$

where ν_j, $j = 1,\ldots,m$ is the scalar output of a causal deterministic system Δ_j with input y_j and $\mathcal{L}_2$-gain less than or equal to a specified *noise-to-signal ratio* $\sigma_j > 0$ and where B_j, $j = 0,\ldots,m$ are standard Brownian motion w.r.t. an underlying fitration $\mathcal{F}_t$. Finally w is adapted to $\mathcal{F}_t$. □

Notice that y_0 and z are not essential to the FSN model but they are needed in the present application. In figure 1 (c) the white noise signal v_1 can loosely be identified with $\nu_1 dB_1/dt$. Without loss of generality we assume $\sigma_j = 1$, $j = 1,\ldots,m$.

Examples of suitable systems Δ_j are

$$\begin{aligned} \Delta_j :&\quad \nu_j(t) = \ y_j(t) \qquad \text{(for } y_j \text{ scalar)} \\ \Delta_j :&\quad \nu_j(t) = \ |y_j(t)| \\ \Delta_j :&\quad \nu_j(t) = \ \sqrt{\int_0^\infty \omega \exp(-\omega\tau)|y_j(t-\tau)|^2\ d\tau} \end{aligned}$$

The first case that of *multiplicative noise*, c.f. e.g. [2] and the references therein. We emphasize that we are interested in results which holds for *any* choice of Δ_j. In general ν_j^2 may be thought of some estimator of the variance of y_j.

Notice that if a steady-state exists then asymptotically the intensity of v_j will be (less than or equal to) the variance of y_j.

We have the following result bounding the variance of y_0:

Theorem 17: Let Σ be a linear FSN system as in definition 16 and let $w = \Delta z$ where $\Delta \in \boldsymbol{\Delta}$. Then

$$\limsup_{T\to\infty} \mathcal{E}_0\{\frac{1}{T}\int_0^T |y_0(t)|^2\ dt\} \leq \mathrm{tr}(G_0' P G_0)$$

provided $P = P' \geq 0$, $d \geq 0$ and $\epsilon > 0$ satisfy

$$\begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} - \sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} + \epsilon \begin{bmatrix} C' \\ D' \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}$$

$$+ \sum_{j=1}^{m} \mathrm{tr}(G_i' P G_i) \begin{bmatrix} H_j' \\ J_{w,j}' \end{bmatrix} \begin{bmatrix} H_j & J_{w,j} \end{bmatrix} + \begin{bmatrix} H_0' \\ J_{w,0}' \end{bmatrix} \begin{bmatrix} H_0 & J_{w,0} \end{bmatrix} \leq 0$$

□

Proof: Let the storage functions for Δ w.r.t. $-s_i$ be W_i and let the storage functions for Δ_j w.r.t. $|y_j|^2 - |\nu_j|^2$ be U_j. Let $V(x) = x'Px$ where P, d, ϵ solve the LMIP in the theorem. Let $\gamma \geq \mathrm{tr}(G_0' P G_0)$. Then some elementary manipulations of the dissipation inequalities and the linear matrix inequality in P, d, ϵ yields:

$$\mathcal{E}_0\{\Delta V + \sum_i d_i \Delta W_i + \sum_j \mathrm{tr}(G_j P G_j) \Delta U_j\} \leq \mathcal{E}_0\{\int_0^T -\epsilon|z|^2 + \gamma - |y_0|^2\ dt\}$$

Combining with the positive semidefiniteness of the storage functions yields the desired result as in theorem 14. ■

9 State feedback design

In this section we discuss design of state feedback controllers for FSN systems with multi-dissipative perturbations. The reason for focusing on this particular problem is that it contains control for robust stability as well as control for guaranteed $\mathcal{H}_2$ performance as special cases.

To this end we consider plants with the FSN description

$$\begin{array}{rcl} dx & = & Ax + B_u u + B_w w + G_0 dB_0 + \sum_{j=1}^m G_j \nu_j dB_j \\ y_j & = & Hx + J_{u,j} u + J_{w,j} w \qquad j = 1, \ldots, m \\ z & = & Cx + D_u u + D_w w \end{array} \tag{13}$$

where u is a control signal given by some static state-feedback law $u(t) = Fx(t)$. We wish to find F such as to minimize the bound on the steady-state variance of y_0 which can be guaranteed by theorem 17. I.e. we are interested in the following minimization problem

$$\inf_{P,d,F,\epsilon} \mathrm{tr} G_0 P G_0'$$

subject to the constraints

$$P = P' \geq 0, \quad d_i \geq 0, \quad \epsilon > 0$$

$$\Phi \leq 0$$

where Φ is shorthand for

$$\begin{aligned} \Phi = & \begin{bmatrix} P(A+B_u F)+(A+B_u F)'P & PB_w \\ B_w' P & 0 \end{bmatrix} \\ & - \sum_{i=1}^p d_i \begin{bmatrix} 0 & (C+D_u F)' \\ I & D_w' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C+D_u F & D_w \end{bmatrix} \\ & + \epsilon \begin{bmatrix} (C+D_u F)' \\ D_w' \end{bmatrix} \begin{bmatrix} C+D_u F & D_w \end{bmatrix} \\ & + \sum_{j=1}^m \mathrm{tr}(G_j' P G_j) \begin{bmatrix} (H_j+J_{u,j} F)' \\ J_{w,j}' \end{bmatrix} \begin{bmatrix} H_j+J_{u,j} F & J_{w,j} \end{bmatrix} \\ & + \begin{bmatrix} (H_0+J_{u,0} F)' \\ J_{w,0}' \end{bmatrix} \begin{bmatrix} H_0+J_{u,0} F & J_{w,0} \end{bmatrix} \end{aligned}$$

Notice that for fixed P, d, ϵ the problem is quadratic in F in the sense that Φ is quadratic in F and that for fixed F the problem is convex in P, d, ϵ. Notice also that the optimum is found in the limit $\epsilon \to 0$. These observations suggest the following conceptual algorithm:

Algorithm 1:

1. Set $\epsilon = 0$ and fix an initial guess F (e.g. $F = 0$).
2. For fixed F, solve the minimization problem

$$\min_{P,d,\gamma} \gamma \text{ s.t. } \Phi \leq \gamma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \; P \geq 0, \; d \geq 0$$

   using convex programming, e.g. [3].
3. With the optimal P, d, solve the optimization problem

$$\min_{F,\gamma} \gamma \text{ s.t. } \quad \Phi \leq \gamma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

   using completion of the squares (see appendix A).
4. If the optimal $\gamma > 0$, then go to step 2, else continue.
5. For fixed F, solve the minimization problem

$$\min_{P,d} \mathrm{tr} G_0' P G_0 \text{ s.t. } \Phi \leq 0, \; P \geq 0, \; d \geq 0$$

   using convex programming.
6. For fixed P, d solve the minimization problem

$$\min_{F,\gamma} \gamma \text{ s.t. } \quad \Phi \leq \gamma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

   using completion of the squares.
7. If convergence or satisfactory performance, then stop, else go to step 5.

It is evident that γ is non-increasing through the iterations of steps 2 and 3 and hence γ will either converge to some value $\gamma^* \geq 0$ or, after a finite number of iterations of steps 2 and 3 we will continue to step 5. If this happens it is evident that the performance bound $\mathrm{tr} G_0' P G_0$ is non-increasing through the iterations of steps 5 and 6 and hence convergent. No convergence properties except these are claimed.

We demonstrate the method on a small numerical example. The system is given by

$$\left[\begin{array}{c|c|c} A & B_u & B_w \\ \hline C & D_u & D_w \\ \hline H & J_u & J_w \end{array}\right] = \left[\begin{array}{rrr|r|r} -0.3554 & -2.5164 & 0.6644 & 0.9904 & -0.0029 \\ -0.0848 & 0.3994 & -1.9891 & -1.1318 & 0.0032 \\ -0.5176 & -0.0744 & -1.3373 & 0.2110 & -0.0001 \\ \hline -0.2280 & -0.4355 & -0.9947 & 0.0040 & 0.0045 \\ \hline -0.6135 & -1.4450 & -1.1157 & 1.1226 & 1.0967 \\ 1.7707 & 1.1957 & -0.8505 & 0.0263 & -1.5387 \\ -0.8594 & 1.4081 & 1.7341 & -0.6585 & -0.5794 \end{array}\right]$$

where $m = 3$ and where we have lumped H_1, H_2 and H_3 into H and similarly for J_u and J_w. Furthermore we have

$$G = [G_1 \ G_2 \ G_3] = \begin{bmatrix} -0.0189 & 0.0093 & 0.0130 \\ -0.0016 & 0.0053 & 0.0160 \\ 0.0013 & 0.0093 & 0.0066 \end{bmatrix}$$

and Δ is passive and has $\mathcal{L}_2$-gain less than or equal to 1.

For this choice of system parameters, and an initial guess of $F = 0$, γ goes negative already in the second iteration of step 2. The performance index $\mathrm{tr}(G_0' P G_0)$ after step 5 is plotted in figure 5 for the first 10 iterations of steps 5 and 6. Notice that the cirterion seems to converge fairly rapidly.

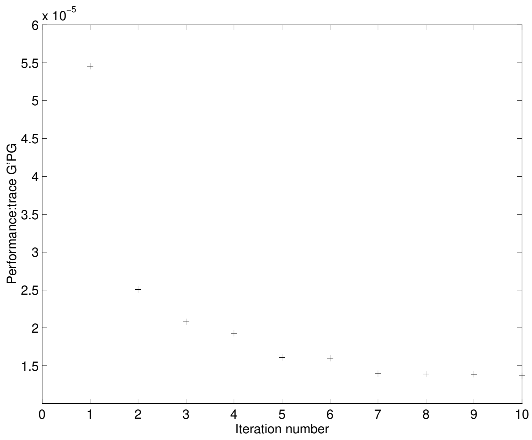

Figure 5: Performance index vs. iteration number

The state feedback gain F is seen in table 1. Notice that we have made no claims that F converges; however in this case it seems so.

Step	Iter. no.	F_1	F_2	F_3
3	1	0.4621	1.1461	0.3919
6	1	0.2297	1.3976	-0.1011
6	2	0.3061	1.2929	0.1544
6	3	0.3586	1.2788	0.3366
6	4	0.4057	1.2698	0.5000
6	5	0.4310	1.2594	0.5803
6	6	0.4587	1.2684	0.6782
6	7	0.4891	1.2818	0.7924
6	8	0.4920	1.2751	0.7983
6	9	0.4865	1.2537	0.7691
6	10	0.4913	1.2451	0.7799

Table 1: The state feedback gain F for the first 10 iterations

The algorithm was progammed naively, i.e. with no regards paid to numerical issues, and tested with a large number of systems with randomly chosen coefficients. The example presented here seems to be representative for systems of low order, i.e. not larger than 6 or so. For systems of higher order the numerical sensitivity becomes critical and further work on the implementation is necessary.

It should be possible to speed up the convergence by modifying step 6 so that the optimization is done over P^{-1}, γ and FP^{-1}. Such extensions are not explored in the present work, however we note that it it not certain that they will result in less total computational burden, since the optimization in step 6 could then not be done using completion of the squares.

10 Conclusion

We have introduced the idea of expressing system uncertainty in terms of *multi-dissipative dynamic perturbations*, and we emphasize two motivating examples: the block diagonal perturbations similar to those in the μ-framework, and parasitic dynamics which are small gain and passive. The conditions for robust stability and performance that arise are less conservative, than if the supply rates are considered separately, and can be verified with numerically tractable LMIs. The corresponding synthesis matrix inequalities can be approached using coordinate descent type algorithms.

Topics of future research include the study of special supply rates, synthesis of dynamic output feedback controllers, as well as extensions to non-linear and non-stationary systems.

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A Minimization over F

In this section we derive the analytical solution to

$$\min_{F,\gamma} \ \text{s.t.} \ \Phi(F) \leq \gamma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

using completion-of-the-squares arguments. Notice that $\Phi(F)$ may be written

$$\Phi(F) = \begin{bmatrix} \mathcal{A}F + F'\mathcal{A}' + F'\mathcal{B}F + \mathcal{C} & F'\mathcal{D}' + \mathcal{E}' \\ \mathcal{D}F + \mathcal{E} & \mathcal{G} \end{bmatrix}$$

and hence the constraint may be written (assuming $\mathcal{G} < 0$)

$$\mathcal{A}F + F'\mathcal{A}' + F'\mathcal{B}F + \mathcal{C} - (F'\mathcal{D}' + \mathcal{E}')\mathcal{G}^{-1}(\mathcal{D}F + \mathcal{E}) \leq \gamma I \tag{14}$$

It is now clear that the optimal F is found by minimizing the left hand side. Discarding terms which do not include F we rewrite the left hand side:

$$(\mathcal{A} - \mathcal{E}'\mathcal{G}^{-1}\mathcal{D})F + F'(\mathcal{A}' - \mathcal{D}'\mathcal{G}^{-1}\mathcal{E}) + F'(\mathcal{B} - \mathcal{D}'\mathcal{G}^{-1}\mathcal{D})F$$

The optimal F is hence (assuming $\mathcal{B} - \mathcal{D}'\mathcal{G}^{-1}\mathcal{D} > 0$)

$$F^* = -(\mathcal{B} - \mathcal{D}'\mathcal{G}^{-1}\mathcal{D})^{-1}(\mathcal{A}' - \mathcal{D}'\mathcal{G}^{-1}\mathcal{E})$$

What remains is to determine the coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{G}$. Some manipulation of Φ yields

$$
\begin{aligned}
\mathcal{A} &= PB_u - \sum_{i=1}^{p} d_i C' Q_i^{zz} D_u + \epsilon C' D_u + \sum_{j=1}^{m} \mathrm{tr} G_j' P G_j (H_j' J_{u,j}) + H_0' J_{u,0} \\
\mathcal{B} &= -\sum_{i=1}^{p} d_i D_u' Q_i^{zz} D_u + \epsilon D_u' D_u + \sum_{j=1}^{m} \mathrm{tr}(G_j' P G_j) J_{u,j}' J_{u,j} + J_{u,0}' J_{u,0} \\
\mathcal{C} &= PA + A'P - \sum_{i=1}^{p} d_i C' Q_i^{zz} C + \epsilon C' C \\
&\quad + \sum_{j=1}^{m} \mathrm{tr}(G_j' P G_j) H_j' H_j + H_0' H_0 \\
\mathcal{D} &= -\sum_{i=1}^{p} d_i (D_w' Q_i^{zz} + Q_i^{wz}) D_u + \epsilon D_w' D_u \\
&\quad + \sum_{j=1}^{m} \mathrm{tr}(G_j' P G_j) J_{w,j}' J_{u,j} + J_{w,0}' J_{u,0} \\
\mathcal{E} &= B_w' P - \sum_{i=1}^{p} d_i (Q_i^{wz} + D_w' Q_i^{zz}) C + \epsilon D_w' C \\
&\quad + \sum_{j=1}^{m} \mathrm{tr}(G_j' P G_j) J_{w,j}' H_j + J_{w,0}' H_0 \\
\mathcal{G} &= -\sum_{i=1}^{p} d_i (Q_i^{ww} + D_w' Q_i^{zw} + Q_i^{wz} D_w + D_w' Q_i^{zz} D_w) + \epsilon D_w' D_w \\
&\quad + \sum_{j=1}^{m} \mathrm{tr}(G_j' P G_j) J_{w,j}' J_{w,j} + J_{w,0}' J_{w,0}
\end{aligned}
$$

With the optimal F in hand one may of course compute the optimal γ from (14) as the largest eigenvalue of the left hand side.

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