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On the Accuracy of Vortex Methods

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The accuracy of the vortex method depends on the choice of the cutoff function and of the cutoff length δ and on the initialization of the vorticity distribution. The practical effect of these choices on the vortex method for inviscid flows in the absence of boundaries is investigated. In our examples the vorticity is radially symmetric and has bounded support. The consistency error and its components, the smoothing error and the discretization error, for high-order cutoff functions and several values of the cutoff length δ are considered. The numerical experiments indicate that for smooth flows, high-order cutoffs improve the accuracy of the approximation. The best value of δ is larger than h, the initial distance between the vortices; it is time dependent in the sense that longer time integration requires a larger δ. In addition the optimal choice of δ is insensitive to the smoothness of the flow. If δ is close to h then the accuracy is lost in a relatively short time. This loss of accuracy is caused by the growth of the discretization error. © 1985 Academic Press, Inc.

INTRODUCTION

The vortex method is a grid-free method that simulates incompressible fluid flow by approximating the vorticity by a finite sum of functions of small support and computing their evolution. These functions, called cutoff or core functions, are parametrized by δ and approximate the delta function as δ tends to zero. A general discussion of vortex methods is given by Chorin [9] and by Leonard [16]. The vortex method as presented by Chorin in [8] has been successfully used to simulate high Reynolds number fluid flow. Applications of the vortex method include the calculation of unstable boundary layers (Chorin [9]), aerodynamic calculations (Cheer [7], Spalart [25], Leonard and Spalart [17]), flow through heart valves (McCracken and Peskin [18]), the simulation of turbulent mixing layers (Ashurst [3]), the modelling of turbulent combustion (Ghoniem, Chorin and Oppenheim [12], Sethian [24]), and flows of variable density (Anderson [1]).

The convergence of the vortex method has been established for two-dimensional inviscid flows in the absence of boundaries. Hald [13] showed that the vortex method can converge with second order accuracy to the solution of Euler's equations as the number of vortices increases. Subsequently Beale and Majda [4, 5] extended Hald's results to obtain higher order methods in two and three dimensions. Recently Beale and Majda's results were simplified by Cottet [11]. A

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simpler version of Beale and Majda's and Cottet's proofs is given by Anderson and Greengard [2]. The convergence proofs are based on consistency and stability estimates.

In this paper we investigate the practical accuracy of the vortex method for inviscid flows in the absence of boundaries. We assume that the vorticity is radially symmetric and has bounded support. Thus the solution of Euler's equations can be given explicitly. We look at the consistency error and its components: the smoothing error and the discretization error.

The accuracy of the vortex method depends on how the delta function is approximated. Hald [13] presented several cutoff functions which give second order accuracy. Beale and Majda [6] suggest a class of infinitely differentiable cutoff functions which in theory provide high order accuracy. We examine the accuracy obtained with these cutoff functions for smooth and nonsmooth flows.

Another factor that affects the accuracy of the approximation is the choice of the cutoff parameter δ. Theoretically δ is chosen so that the smoothing error and the discretization error are of the same order. Chorin [8, 9] chooses $\pi\delta$ equal to the average distance between the vortices created along a boundary. Chorin's choice of δ is much larger than the average distance between the vortices. Hald [13] chooses $\delta = \sqrt{h}$, where h is the initial distance between the vortices. Beale and Majda [5] suggest that for sufficiently smooth flows we can choose δ close to h and obtain an order of accuracy almost as high as the order of the cutoff function. Our numerical experiments indicate that with a suitable choice of δ the vortex method converges. However, if δ is close to h the accuracy in the velocity and vorticity approximations is lost in a relatively short time. The best choice of δ is time dependent, in the sense that longer time integration requires a larger δ. In addition the optimal choice of δ is quite insensitive to the smoothness of the flow. The loss of accuracy is caused by the discretization error, which comes from approximating a convolution integral by the trapezoidal rule. Since the discretization error decreases as δ increases while the smoothing error increases with δ we can choose a larger value of δ to preserve the accuracy over a fixed time interval. A possible explanation for the growth of the discretization error in time is the disorganization of the computational points. However, we have not found an explanation to the observed decrease in the order of accuracy.

The initial vorticity distribution can be approximated in two different ways. Hald [13] assigns to each computational point the vorticity contained in the blob surrounding it, while Beale and Majda [5] assign the value of the vorticity at the point times the area of the blob. Our numerical experiments indicate that Hald's choice leads to second order accuracy for any cutoff function, while Beale and Majda's approximation can provide high order accuracy.

Earlier numerical experiments with radially symmetric vorticity distributions were presented by Hald and Del Prete [14]. They used cutoff functions of the type introduced by Chorin [8] and observed second order accuracy. Nakamura, Leonard and Spalart [20] tested the accuracy of the vortex method for inviscid shear layers. Numerical experiments with high-order cutoff functions were presented by Perlman [22]. Additional numerical experiments are given by Beale and Majda [6].

This paper is divided into four parts. In Section 1 we present the derivation of the vortex method and a summary of the existent convergence proofs. Section 2 contains our test problems and how we measure the errors. In Section 3 we present our numerical experiments. We study the behavior of the consistency error as a function of h, δ, and of the time t and look at its components: the smoothing error and discretization error. In Section 4 we compare the two different approximations of the initial vorticity distribution.

1. The Vortex Method in Two Dimensions

Consider the vorticity-stream function formulation of Euler's equations in the (x, y) plane:

$$\omega_t + (u \cdot \nabla)\,\omega = 0, \tag{1.1}$$

$$\Delta \Psi = -\omega, \tag{1.2}$$

$$u_1 = \Psi_y, \qquad u_2 = -\Psi_x, \tag{1.3}$$

where $u = (u_1, u_2)$ is the velocity vector, $z = (x, y)$ is the position vector, ω is the vorticity, and Ψ is the stream function.

By solving the Poisson equation (1.2) we obtain

$$\Psi(z) = \int G(z - z')\,\omega(z')\,dz'.$$

where $G(z) = -(1/2\pi)\log|z|$, with $|z|^2 = x^2 + y^2$, is the fundamental solution of the Laplace equation (see [15, p. 75]) and $dz' = dx'\,dy'$. The velocity u is obtained by differentiating the stream function with respect to y and x, and is given by the integral

$$u(z, t) = \int K(z - z')\,\omega(z')\,dz', \tag{1.4}$$

where

$$K(z) = -\frac{1}{2\pi}\begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} G(z) = -\frac{1}{2\pi|z|^2}\begin{pmatrix} y \\ -x \end{pmatrix}.$$

In the Lagrangian description of the flow, we follow the motion of material points of the fluid. Thus if $\alpha = (\alpha_1, \alpha_2)$ denote the Lagrangian coordinates, then the path of a particle starting at the point $z = a$ is determined by

$$\frac{dz}{dt}(\alpha, t) = u(z(\alpha, t), t), \qquad z(\alpha, 0) = a. \tag{1.5}$$

It follows from Eq. (1.1) that the vorticity is conserved along particle paths. More precisely, $(d\omega/dt)(\alpha, t) = 0$ or equivalently $\omega(z(\alpha, t), t) = \omega(\alpha, 0)$, see Chorin and Marsden [10]. By using this fact and the fact that the flow is incompressible we can write the right-hand side of Eq. (1.5) in the following way:

$$\begin{aligned} u(z(\alpha, t), t) &= \int K(z - z')\, \omega(z', t)\, dz' \\ &= \int K(z - z(\alpha, t))\, \omega(z(\alpha, t), t)\, d\alpha \\ &= \int K(z - z(\alpha, t))\, \omega(\alpha, 0)\, d\alpha. \end{aligned} \tag{1.6}$$

We will now describe the discretization of the system of ordinary differential equations (1.5). Assume that at time $t = 0$ the support of the vorticity is contained in the region Ω. We introduce a square grid in the α plane. The squares B_j are centered at the grid points $jh = (j_1, j_2)\, h$ and have length and width h. We denote by $z_j(t) = z(jh, t)$ the position a time t of a fluid particle starting at the point jh at time $t = 0$. Let $u_j(t) = u(z_j(t), t)$ be the velocity at the point z_j. By using the grid points z_j that are contained in the support Ω of the initial vorticity distribution, we approximate the right-hand side of (1.6) by

$$u^h(z, t) = \sum_j K(z - z_j(t))\, c_j, \tag{1.7}$$

where the c_j's have one of the following two forms:

$$c_j = \int_{B_j} \omega(z)\, dz, \tag{1.7a}$$

$$c_j = \omega(jh)\, h^2. \tag{1.7b}$$

One possible numerical method consists of replacing Eq. (1.5) by the system of ordinary differential equations

$$\frac{d\tilde{z}_i(t)}{dt} = \tilde{u}_i^h(t), \qquad \tilde{z}_i(0) = ih, \tag{1.8}$$

where

$$\tilde{u}_i^h(t) = \sum_{j \neq i} K(\tilde{z}_i(t) - \tilde{z}_j(t)) c_j. \tag{1.9}$$

Thus we expect that the $\tilde{z}_j$'s will approximate the particle positions. The algorithm (1.8), (1.9) is called the point vortex method. It was introduced by Rosenhead [23] to study the behavior of vortex sheets. Since $u^h(z, t) = K * \sum \delta(z - z_j(t))\, c_j$ we see that u^h is the velocity corresponding to a collection of point vortices with strength c_j.

Since the kernel K is singular at the origin the velocity tends to infinity as the distance between two particles tends to zero. To overcome this difficulty, Chorin [8] replaced the kernel K by a kernel K_δ, which is bounded at the origin. The kernel K_δ can be obtained by convolving K with a smooth cutoff function ψ_δ

$$K_\delta(z) = K * \psi_\delta(z) = \int K(z - z')\, \psi_\delta(z')\, dz', \tag{1.10}$$

where ψ_δ is a radially symmetric function and satisfies $\psi_\delta(z) = \delta^{-2}\psi(z/\delta)$ and $\int \psi(z)\, dz = 1$. Thus ψ_δ approximates the Dirac delta function as $\delta \to 0$. The velocity for the point vortex method is then replaced by

$$u^h(z, t) = \sum_j K_\delta(z - z_j(t))\, c_j. \tag{1.11}$$

We can then compute the particle trajectories by solving the system of ordinary differential equations

$$\frac{d\tilde{z}_i}{dt} = \tilde{u}_i^h, \qquad \tilde{z}_i(0) = ih, \tag{1.12}$$

where

$$\tilde{u}_i^h = \sum_{j \neq i} K_\delta(\tilde{z}_i(t) - \tilde{z}_j(t))\, c_j. \tag{1.13}$$

The algorithm (1.12), (1.13) is called the vortex blob method. Since $u^h(z) = K * \sum \psi_\delta(z - z_j(t))\, c_j$, we see that u^h is the velocity field corresponding to the vorticity distribution $\omega^h(z, t) = \sum \psi_\delta(z - z_j(t))\, c_j$. Thus we arrive at Chorin's interpretation of the vortex method [8], namely that the vorticity is approximated by a sum of vortex blobs of common shape ψ_δ centered at $z_j(t)$ and with strength c_j.

The accuracy of the vortex method depends on the smoothness of the flow, on the initial approximation of the vorticity, and on the choice of cutoff function ψ.

Numerical experiments by Hald and Del Prete [14] indicate that the rate of convergence for the vortex method with Chorin's cutoff functions is essentially second order. Hald [13] showed that the vortex method can converge with second order accuracy in the L^2 norm, for arbitrarily long time intervals. Hald's cutoff functions ψ are twice continuously differentiable, have support in the disk $|z| \leqslant 1$ and are constructed so that the first three moments of $K - K_\delta$ vanish. In addition Hald [13] uses (1.7a) to define c_j, i.e., he lets c_j be the vorticity contained in the square B_j. Our numerical experiments, presented in Section 4, and Cottet's results [11, Theorem 4.1] show that by using Hald's vorticity approximation and cutoff functions the rate of convergence for the vortex method can never be larger than quadratic.

Beale and Majda [5] have improved Hald's results by showing that the vortex method can converge with arbitrarily high-order accuracy, provided the initial vorticity ω is sufficiently smooth and that the velocity and vorticity are approximated using the c_j's defined in (1.7b) and finally that the cutoff function ψ satisfies

$$\text{(i)} \quad \psi \varepsilon C^2(R^2). \tag{1.14a}$$

$$\text{(ii)} \quad \int \psi(z)\, dz = 1, \tag{1.14b}$$

$$\int z^\gamma \psi(z)\, dz = 0, \qquad \gamma = (\gamma_1, \gamma_2), \quad 1 \leqslant |\gamma| \leqslant p - 1.$$

(iii) For some $L > 0$, and for any multi-index β the Fourier transform $\hat{\psi}(\xi)$ satisfies

$$\sup_{\xi \in R^2} |D_\xi^\beta \hat{\psi}(\xi)| \leqslant C_\beta (1 + |\xi|)^{-L - |\beta|}. \tag{1.14c}$$

The second condition is called the moment condition. Beale and Majda's results are summarized in

THEOREM (Beale and Majda [5]). *Assume that the cutoff function ψ satisfies (1.14a)–(1.14c) for some $2 \leqslant L \leqslant \infty$ and for some $p \geqslant 2$. Choose $\delta = h^q$, with $0 < q < 1$ if $L = \infty$ and $q < (L-1)/(L+p)$ if L if finite. If the velocity field $u(z, t)$ is sufficiently smooth for $z \in R^2$ and $0 \leqslant t \leqslant T$ and the initial vorticity has bounded support, then for any $1 < \mu < \infty$ and $T > 0$ there exists an $h_0 > 0$ such that for all $h < h_0$*

$$\max_{0 \leqslant t \leqslant T} \|z_j(t) - \tilde{z}_j(t)\|_{L_h^\mu} \leqslant C h^{pq},$$

$$\max_{0 \leqslant t \leqslant T} \|u_j(t) - \tilde{u}_j^h(t)\|_{L_h^\mu} \leqslant C h^{pq}.$$

The convergence proofs for the vortex method by Hald [13], Beale and Majda [5], Cottet [11], and Anderson and Greengard [2] are based on consistency and stability estimates. The convergence is proved by estimating the distance between the exact velocity u defined in (1.4) and the computed velocity $\tilde{u}^h$ defined in (1.11). By using the triangle inequality we can estimate the distance by

$$\|u(t) - \tilde{u}^h(t)\| \leqslant \|u(t) - u^h(t)\| + \|u^h(t) - \tilde{u}^h(t)\|.$$

Here u^h is evaluated by using the exact particle positions z_j in Eq. (1.7). The first term $\|u - u^h\|$ is called the consistency error. It is the distance between the exact velocity u and the discrete velocity u^h obtained by replacing the continuous vorticity distribution by a collection of vortex blobs ψ_δ centered at $z_j(t)$ and with strength $\omega_j h^2$. The second error term $\|u^h - \tilde{u}^h\|$ is called the stability error. It measures how the computed particle paths differ from the exact ones.

In their proof, Beale and Majda further estimate the consistency error by the sum of two terms:

$$\begin{aligned}\|u(t)-u^h(t)\| &= \left\| \int K(z-z')\,\omega(z',t)\,dz' - \sum_j K_\delta(z-z_j(t))\,\omega_j h^2 \right\| \\ &\leqslant \left\| \int K(z-z')\,\omega(z')\,dz' - \int K_\delta(z-z')\,\omega(z')\,dz' \right\| \\ &\quad + \left\| \int K_\delta(z-z')\,\omega(z')\,dz' - \sum_j K_\delta(z-z_j(t))\,\omega_j h^2 \right\| \\ &= \|u-u^\delta\| + \|u^\delta - u^h\|.\end{aligned}$$

The first error term $\|u-u^\delta\|$ is called the smoothing error. It arises because the kernel K is replaced by the kernel $K_\delta = K * \psi_\delta$. The smoothing error depends on the cutoff parameter δ and on the time t, but does not depend on the grid size h. The second term $\|u^\delta - u^h\|$ is called the discretization error. It represents the error in the numerical integration of the function $K_\delta(z-z')\,\omega(z')$ by the trapezoidal rule. The discretization error depends on the mesh length h, on the cutoff parameter δ and on the time t.

Beale and Majda have shown that if the flow is smooth then the smoothing error is of order δ^p, where p measures the number of moments of the cutoff function that vanish. The discretization error is of order $\delta^{-L}h^{-L-1-\varepsilon}$, where $\varepsilon > 0$ and L depends on the rate of decay of the Fourier transform of ψ. Thus, the consistency error can be bounded by $C_1 \delta p + C_2 (h/\delta)^L \delta^{-1-\varepsilon}$, where C_1 and C_2 are independent of δ and h. For a fixed mesh length h we would like to choose δ so that the consistency error is as small as possible. Beale and Majda choose $\delta = h^q$ with $q = (L-1-\varepsilon)/(L+p)$. With this choice the smoothing error and the discretization error are of order h^{pq}. For smooth cutoff functions L may be arbitrarily large. Thus we can choose δ close to h and obtain in principle a p^{th}-order method.

The last choice is valid only for smooth flows. If the flow is not infinitely differentiable, then the exponent L in the estimate of the discretization error cannot be larger than the number of derivatives of the vorticity, see Lemma 2.5 by Cottet [11] or the Discretization Lemma by Anderson and Greengard [2]. The estimate of the smoothing error also depends upon the smoothness of the flow. Thus a higher order cutoff does not always lead to more accurate results.

2. Choice of Test Problems

In this section we describe the various test problems we have used in the numerical experiments to check the accuracy of the vortex method. We consider the radially symmetric initial vorticity distribution

$$\omega^{(1)}(z) = \begin{cases} (1-|z|^2)^7 & |z| \leqslant 1 \\ 0 & |z| > 1. \end{cases}$$

The corresponding velocity field is given by

$$u^{(1)}(z,t) = f(|z|)\begin{pmatrix} y \\ -x \end{pmatrix}$$

where

$$f(|z|) = \begin{cases} -\dfrac{1}{16|z|^2}(1-(1-|z|^2)^8) & |z| \leqslant 1 \\ -\dfrac{1}{16|z|^2} & |z| > 1. \end{cases}$$

The velocity field $u(z, t)$ is in $C^7(R^2)$ and is C^∞ for $|z| \neq 1$. The flow is radially symmetric and rotates about the origin. Fluid particles at different radii move at different speeds. The particles near the origin complete one rotation at time $t = 4\pi$, while the particles on $|z| = 1$ complete one rotation a $t = 32\pi$.

At time $t = 0$ we place the particles at the points $jh = (j_1 h, j_2 h)$ on a square grid on the (x, y) plane. Since $\omega \equiv 0$ outside the unit circle all our computational points lie inside the unit circle.

Our second test problem is a C^∞ radially symmetric vorticity distribution

$$\omega^{(2)}(z) = e^{-12|z|^2}.$$

The corresponding velocity field is given by

$$u^{(2)}(z,t) = -\frac{1}{24|z|^2}(1-e^{-12|z|^2})\begin{pmatrix} y \\ -x \end{pmatrix}.$$

The flow is radially symmetric and rotates about the origin. The vorticity distribution does not have compact support, but decays rapidly at infinity. To prove convergence of the vortex method Cottet [11] assumes that the vorticity and its derivatives decay rapidly at infinity. Thus our choice of ω is within the range of validity of his theory. As in the previous test case we place the particles at the points $jh = (j_1 h, j_2 h)$ on a square grid on the $(x, y,)$ plane. We neglect those particles z_j for which $\omega^{(2)}(z_j) < 10^{-6}$. Our numerical experiments indicate that this does not affect the qualitative behavior of the error.

In the third and last test case all the fluid particles inside the unit circle rotate at constant speed. The vorticity distribution is given by

$$\omega^{(3)}(z) = \begin{cases} 1 & |z| \leqslant 1 \\ 0 & |z| > 1 \end{cases}$$

and is discontinuous at $|z|=1$. The corresponding velocity field is given by

$$u^{(3)}(z,t)=g(|z|)\begin{pmatrix} y \\ -x \end{pmatrix},$$

where

$$g(|z|)=\begin{cases} \dfrac{1}{2} & |z|\leqslant 1 \\ \dfrac{1}{2|z|^2} & |z|>1. \end{cases}$$

To test the accuracy of the vortex method we have used Gaussian cutoff functions of different orders:

$$\text{(i)}\quad p=2, \qquad \psi_\delta=\frac{1}{2\pi\delta^2}e^{-r^2/2\delta^2}, \tag{2.1}$$

$$\text{(ii)}\quad p=4, \qquad \psi_\delta=\frac{1}{\pi\delta^2}\left(2e^{-r^2/\delta^2}-\frac{1}{2}e^{-r^2/2\delta^2}\right), \tag{2.2}$$

$$\text{(iii)}\quad p=6, \qquad \psi_\delta=\frac{1}{\pi\delta^2}\left(\frac{8}{3}e^{-r^2/\delta^2}-e^{-r^2/2\delta^2}+\frac{1}{12}e^{-r^2/4\delta^2}\right), \tag{2.3}$$

$$\text{(iv)}\quad p=8, \qquad \psi_\delta=\frac{1}{\pi\delta^2}\left(\frac{64}{21}e^{-r^2/\delta^2}-\frac{4}{3}e^{-r^2/2\delta^2}+\frac{1}{6}e^{-r^2/4\delta^2}-\frac{1}{168}e^{-r^2/8\delta^2}\right), \tag{2.4}$$

where $r^2=x^2+y^2$. These cutoff functions have $L=\infty$ and are suggested by Beale and Majda in [6].

The numerical experiments by Hald and Del Prete [14], Anderson [1], and Nakamura, *et al.* [20] have shown that the vortex method is stable. In this paper we will therefore investigate the consistency error in detail. As suggested by the numerical results presented in Section 4 and by Cottet's observation [11, Lemma 4.1] we assign to each particle z_j the vorticity value $c_j=\omega_j h^2$. Here $\omega_j=\omega(z_j)$ and h^2 is the area of the square B_j centered at z_j. Thus we approximate the vorticity and the velocity by

$$u^h(z,t)=\sum_j K_\delta(z-z_j(t))\,\omega_j h^2,$$
$$\omega^h(z,t)=\sum_j \psi_\delta(z-z_j(t))\,\omega_j h^2.$$

The behavior of the consistency error for the velocity and for the vorticity as a function of h, δ, and t will suggest a choice of the cutoff parameter δ for a fixed time interval $[0, T]$.

We measure the consistency error for the velocity and the vorticity in the discrete L^2 norm:

$$E_u = \left(\sum_j |u(z_j, t) - u^h(z_j, t)|^2 h^2 \right)^{1/2} \tag{2.5}$$

$$E_\omega = \left(\sum_j |\omega(z_j, t) - \omega^h(z_j, t)|^2 h^2 \right)^{1/2}. \tag{2.6}$$

We also compute the relative errors $E_u/\|u\|$ and $E_\omega/\|\omega\|$, where $\|u\|$ and $\|\omega\|$ are the discrete L^2 norms of the velocity and the vorticity. Similarly we measure the smoothing error and the discretization error for the vorticity in the discrete L^2 norm:

$$E^S_\omega = \left(\sum_j |\omega(z_j) - \omega^\delta(z_j, t)|^2 h^2 \right)^{1/2}, \tag{2.7}$$

$$E^D_\omega = \left(\sum_j |\omega^\delta(z_j, t) - \omega^h(z_j, t)|^2 h^2 \right)^{1/2}, \tag{2.8}$$

where $\omega^\delta = \psi_\delta * \omega$.

By using the cutoff functions (2.1)–(2.4) we compute the velocity u^h and the vorticity ω^h with $0.05 \leqslant h \leqslant 0.2$ and $\delta = h^q$, $0.5 < q < 1$, and in the time interval $[0, 20]$. This corresponds to 60 to 950 vortices. We compute $\omega^\delta = \psi_\delta * \omega$ by numerical integration. Specifically we use the routine D01DAF of the NAG library [19] with an error tolerance of 10^{-7}. The method in this routine is described by Patterson in [21]. Finally we estimate the rate of convergence of the vortex method by using two successive values of h:

$$\text{rate of convergence} = \frac{\log(E_{h_1}/E_{h_2})}{\log(h_1/h_2)}. \tag{2.9}$$

3. Numerical Results

For a fixed time interval $[0, T]$ we will consider the approximations to the velocity and the vorticity to be accurate if the rate of convergence is constant over the time interval. The accuracy of the vortex method depends on

(i) the approximation of the initial vorticity distribution,

(ii) the choice of cutoff function ψ for some L and p,

(iii) the cutoff parameter $\delta = h^q$, for some $0 < q < 1$.

We present now the results of the numerical experiments for the first test problem. Our numerical experiments show that the consistency errors E_u and E_ω are qualitatively similar for the three cutoff functions (2.2)–(2.4) and differ from

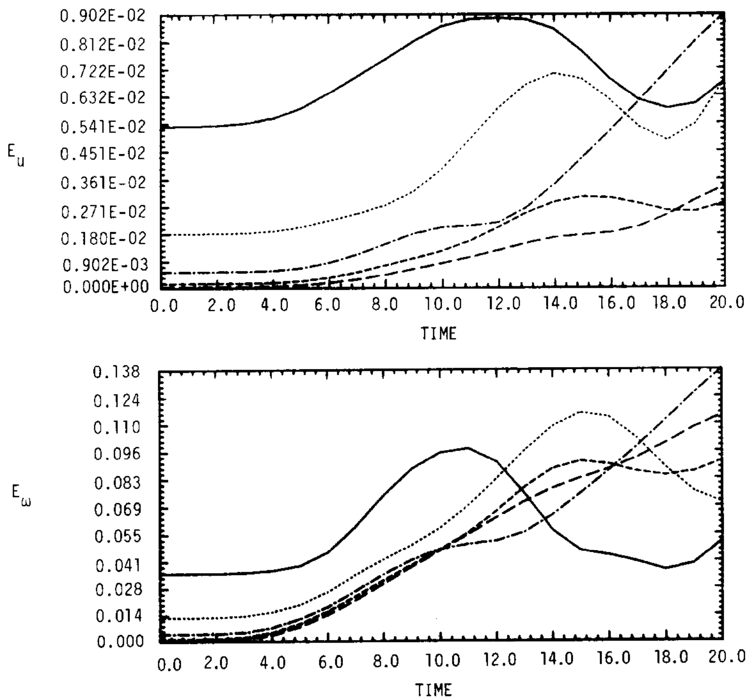

FIG. 3.1. Velocity and vorticity consitency errors in the time interval $[0, 20]$, with $\omega = \omega^{(1)}$, $p = 4$ and $\delta = h^{0.95}$. h: (—) 0.2, (···) 0.14, (-·-) 0.10, (---) 0.07, (– – –) 0,05.

those of (2.1). Hence, we contrast the results obtained with these higher order cutoff functions with those obtained for the second order cutoff (2.1).

Consider one of the higher order cutoff functions. We find that for a fixed $\delta = h^{1-\varepsilon}$ with ε small and $0.05 \leqslant h \leqslant 0.2$ both E_u and E_ω increase sharply in time. However, E_u and E_ω do not increase without bound; they reach a local maximum at time T_* and oscillate around it later on. The time T_* increases as h decreases. We can observe this behavior of E_u and E_ω for the cutoff function (2.2) with $\delta = h^{0.95}$ in Fig. 3.1.

In addition to the sharp increase of the error as a function of t, we find that as a function of h and with $\delta = h^{1-\varepsilon}$ with ε small, neither E_u nor E_ω decrease uniformly as h decreases. The rate of convergence is kept constant for a short time interval and then decreases sharply. This can be seen in Fig. 3.2 for the cutoff function (2.2) and $\delta = h^{0.95}$. This time interval becomes shorter as h decreases and as p, the order of the cutoff function, increases (see Fig. 3.3). We also find that for this choice of δ and some $T > 0$, the errors do not decrease with h. We observe in Fig. 3.1 that this effect is more pronounced in the consistency error of the vorticity than in the consistency error of the velocity.

We consider now the error as a function of δ, with h fixed and set $\delta = h^q$, with

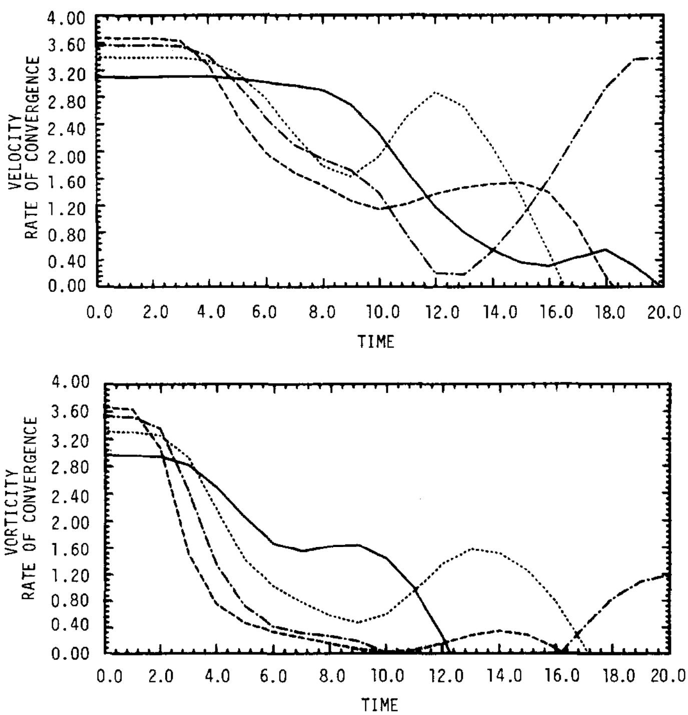

FIG. 3.2. Order of convergence of the velocity and vorticity approximations in the time interval [0, 20], with $\omega = \omega^{(1)}$, $p = 4$, and $\delta = h^{0.95}$. h: (—) 0.2, (···) 0.14, (-·-) 0.10, (----) 0.07.

$0.5 < q < 1$. The theoretical estimates by Beale and Majda [5] and Cottet [11] show that if $\delta = h^q$ with $q < 1$, then the consistency error is of order h^{pq}, where p is the order of the cutoff function. Hence the errors should increase as q decreases. We find that this holds for a short time interval $[0, T^*]$. This time interval becomes shorter as h descreases and as p increases. Table 3.1 shows the consistency errors E_u and E_ω for $p = 4$, $h = 0.07$ (465 vortices), and $\delta = h^q$ with $0.5 < q < 1$. We observe that at time $t = 0$ the errors increase as q decreases. This agrees with the theoretical estimates. However, at time $t = 10$, the velocity error is the smallest for $q = 0.85$, while the vorticity error is the smallest for $q = 0.75$. At time $t = 20$ the smallest velocity error is obtained when $\delta = h^{0.8}$, while the smallest vorticity error is obtained when $\delta = h^{0.7}$.

As δ increases the sharp increase of the error in time is gradually attenuated and we observe a more uniform behavior of the error as $h \to 0$ over the time interval [0, 20] (compare Fig. 3.1 with Fig. 3.4).

One could think that the behavior of the error for $\delta = h^{1-\varepsilon}$ with ε small, is due to the fact that the flow in our test case is not infinitely differentiable. Therefore to choose δ close to h may not be consistent with the theoretical estimates, and a larger δ has to be chosen.

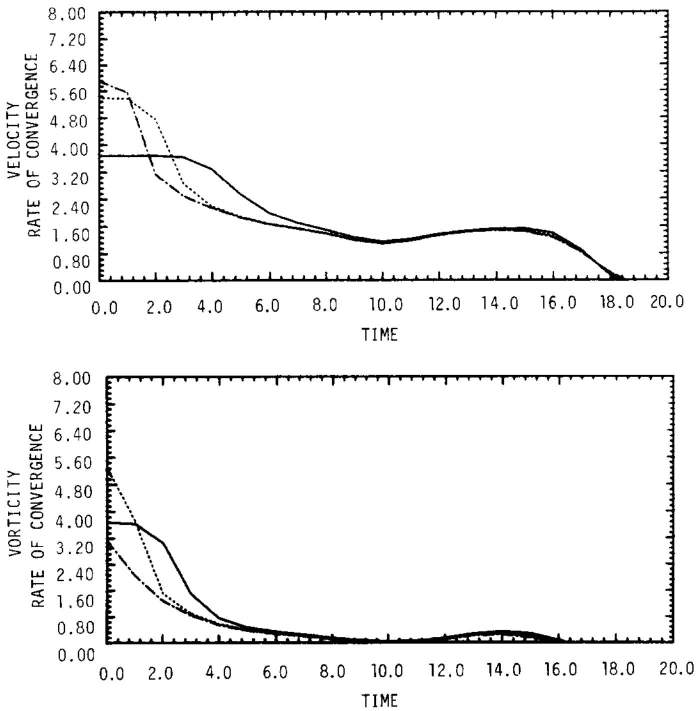

FIG. 3.3. Order of convergence of the velocity and vorticity approximations in the time interval [0,20] for $\omega=\omega^{(1)}$, $p=4$, (—), 6 (···), 8 (-·-); $h=0.07$; and $\delta=h^{0.95}$.

We consider now the second test problem to check how the smoothness of the flow will affect the choice of δ and as a consequence the behavior of the error. Since the flow is infinitely differentiable, in the estimates of the consistency error we can take L to be arbitrarily large. This allows us the choice of $\delta=h^{1-\varepsilon}$ with ε small, in accordance with the theory. Since we are interested only in the qualitative behavior of the error we computed the consistency errors E_u and E_ω using only the 4^{th} order cutoff function (2.2). It follows from Fig. 3.5 that the behavior of the error as a function of h and t is similar to the behavior of the error observed in the previous test case.

In contrast to the first two test problems, we observe that for the third test problem the consistency errors E_u and E_ω are constant in time. This is not surprising since the particles rotate as a rigid body and therefore the distance between the computational points is constant in time. We observe that the errors decrease with h and for a fixed h the smallest errors are obtained for δ close to h. It follows from Tables 3.2a and 3.2b that the errors are reduced by a factor slightly higher than two when we increase the order of the cutoff function from $p=2$ to $p=4$, however, for $p\geqslant 4$ the accuracy is not improved by increasing the order of the cutoff function.

TABLE 3.1

Velocity and Vorticity Consistency Errors at $t=0$, 10, 20 with $\omega=\omega^{(1)}$, $p=4$, and $h=0.07$

	E_u			E_ω		
$\delta=h^q$	$t=0$	$t=10$	$t=20$	$t=0$	$t=10$	$t=20$
$q=0.95$	1.628-4	1.252-3	2.809-3	1.173-3	4.729-2	9.122-2
$q=0.90$	2.694-4	8.873-4	2.158-3	1.946-3	2.933-2	6.103-2
$q=0.85$	4.427-4	7.009-4	1.647-3	3.179-3	1.732-2	3.969-2
$q=0.80$	7.211-4	7.932-4	1.348-3	5.136-3	1.076-2	2.558-2
$q=0.75$	1.161-3	1.178-3	1.411-3	8.189-3	9.709-3	1.787-2
$q=0.70$	1.844-3	1.848-3	1.927-3	1.284-2	1.318-2	1.657-2
$q=0.65$	2.880-3	2.881-3	2.907-3	1.976-2	1.983-2	2.106-2
$q=0.60$	4.410-3	4.410-3	4.418-3	2.971-2	2.972-2	3.018-2
$q=0.55$	6.598-3	6.598-3	6.601-3	4.348-2	4.349-2	4.366-2

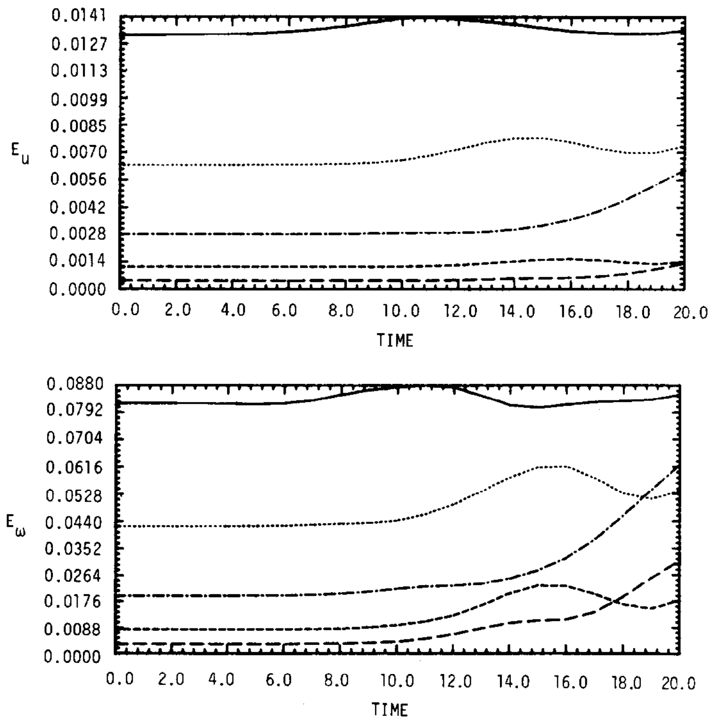

FIG. 3.4. Velocity and vorticity consistency errors in the time interval [0, 20] with $\omega=\omega^{(1)}$, $p=4$, and $\delta=h^{0.75}$. h: (—) 0.2, (···) 0.14, (- · -) 0.10, (- - -) 0.07, (— — —) 0.05.

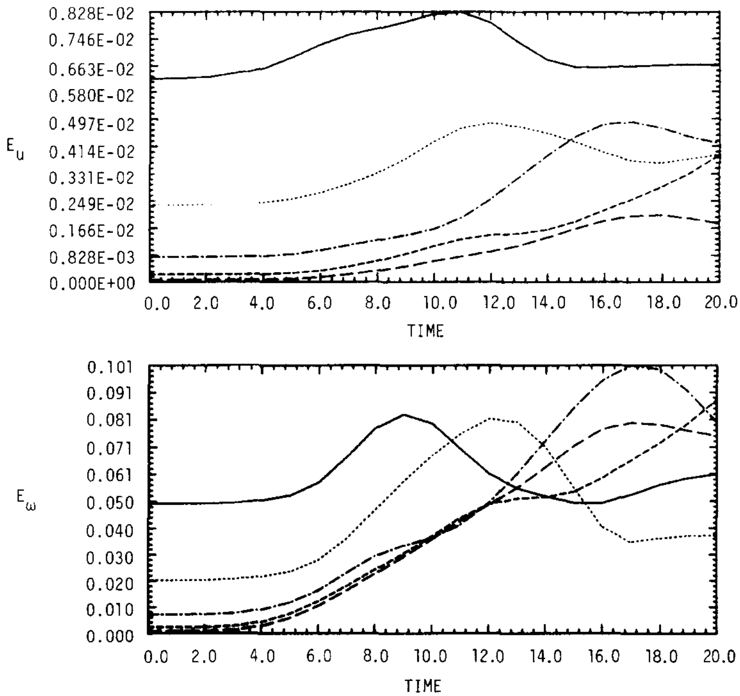

FIG. 3.5. Velocity and vorticity consistency errors in the time interval [0, 20] with $\omega=\omega^{(2)}$, $p=4$, and $\delta=h^{0.95}$. h: (—) 0.2, (···) 0.14, (-·-) 0.10, (---) 0.07, (– – –) 0.05.

TABLE 3.2a

Relative Velocity errors for $\omega=\omega^{(3)}$ and $\delta=h^{0.95}$

h	$p=2$	$p=4$	$p=6$	$p=8$
		(a) Relative velocity errors		
0.20	2.621-2	1.267-2	1.180-2	1.151-2
0.14	1.561-2	6.984-3	6.401-3	6.209-3
0.10	9.271-3	4.344-3	4.051-3	3.957-3
0.07	5.520-3	2.331-3	2.126-3	2.059-3
0.05	3.291-3	1.511-3	1.415-3	1.384-3
0.03	2.018-3	8.549-4	7.834-4	7.605-4
		(b) Relative vorticity errors		
0.2	1.682-1	1.354-1	1.319-1	1.304-1
0.14	1.382-1	1.113-1	1.082-1	1.068-1
0.10	1.113-1	8.771-2	8.485-2	8.362-2
0.07	9.303-2	7.306-2	7.044-2	6.929-2
0.05	7.553-2	5.806-2	5.579-2	5.481-2
0.03	6.477-2	5.030-2	4.834-2	4.748-2

TABLE 3.3

Relative Velocity and Vorticity Errors for $\omega = \omega^{(1)}$ with the Optimal Values of δ at Time $t = 10$ and $t = 20$

	$h = 0.07$		$h = 0.05$	
	$E_u/\|u\|$	$E_\omega/\|\omega\|$	$E_u/\|u\|$	$E_\omega/\|\omega\|$
		(a) $t = 10$		
$p = 2,\ \delta = h^{.90}$	5.009-2	8.691-2	2.764-2	4.865-2
$p = 4,\ \delta = h^{.80}$	5.407-3	2.351-2	2.030-3	1.351-2
$p = 6,\ \delta = h^{.75}$	2.563-3	1.398-2	7.626-4	6.270-3
$p = 8,\ \delta = h^{.70}$	2.181-3	8.795-3	5.056-4	2.540-3
		(b) $t = 20$		
$p = 2,\ \delta = h^{.90}$	5.044-2	8.964-2	2.869-2	6.976-2
$p = 4,\ \delta = h^{.65}$	1.981-2	4.603-2	9.866-3	3.123-2
$p = 6,\ \delta = h^{.60}$	1.385-2	3.527-2	6.001-3	2.042-2
$p = 8,\ \delta = h^{.60}$	2.870-3	2.546-2	3.935-3	1.823-2

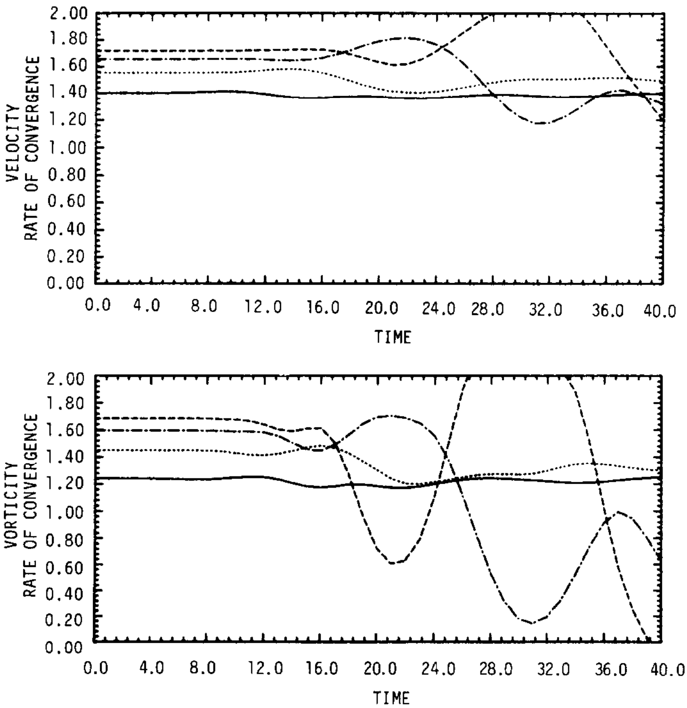

FIG. 3.6. Order of convergence of the velocity and vorticity approximations in the time interval [0, 20] with $\omega = \omega^{(1)}$, $p = 2$, and $\delta = h^{0.90}$. h: (—) 0.2, (···) 0.14, (-·-) 0.10, (---) 0.07.

We conclude that the qualitative behavior of the consistency error is quite insensitive to the smoothness of the flow and that the optimal choice of δ, a δ that will preserve a uniform accuracy over a finite time interval $[0, T]$, depends on the final time. For the first test problem the optimal choice of δ in the time interval $[0, 10]$ is $\delta = h^{0.8}$ for $p = 4$, $\delta = h^{0.75}$ for $p = 6$ and $\delta = h^{0.7}$ for $p = 8$ while in the time interval $[0, 20]$, the optimal choice of δ is $\delta = h^{0.65}$ for $p = 4$ and $\delta = h^{0.6}$ for $p = 6$ and $p = 8$. Tables 3.3a and 3.3b show the relative errors $E_u/\|u\|$ and $E_\omega/\|\omega\|$ for $h = 0.07$ and $h = 0.05$ with the optimal choice of δ for $p = 4, 6, 8$ at times $t = 10$ and $t = 20$. We find that the errors are substantially reduced as we increase the order of the cutoff function. For $p = 4$ and $h = 0.05$ the velocity error at $t = 10$ is 0.2% while for $p = 8$ it is reduced to 0.05%. At $t = 20$ the error for $p = 4$ and $h = 0.05$ is 0.99% and for $p = 8$ the velocity error is 0.39%.

In contrast to the higher order cutoff functions, we find that if we use the second order cutoff function (2.1) and $\delta = h^{0.9}$, we do not observe a loss of accuracy in the time interval $[0, 20]$ and the errors do not have the sharp increase in time that is observed when we used higher order cutoffs. We therefore obtain essentially second order accuracy with a relative velocity error of 2.9% at time $t = 20$ and with $h = 0.05$ (see Tables 3.3a and 3.3b). Although we are able to choose δ close to h in the time interval $[0, 20]$ and obtain second order accuracy, we observe from Fig. 3.6 that to preserve the accuracy over a longer time interval δ will have to be larger, as in the case of the higher order cutoff functions.

To understand the behavior of the consistency error, and the time dependency of the cutoff parameter δ, and following the spirit of the proof in [5], we look at the components of the consistency error, the smoothing and discretization error, defined in (2.7) and (2.8).

The smoothing error E_ω^S is the difference between the exact vorticity ω and $\omega^\delta = \psi_\delta * \omega$. Since ω and ω^δ do not change in time, E_ω^S remains constant for all time. It is therefore enough to look a E_ω^S at time $t = 0$. Table 3.4 contains the smoothing error for the first test problem. We observe that E_ω^S is asymptotically of order δ^p, for a p^{th} order cutoff function and a smooth enough vorticity.

The discretization error E_ω^D is the difference between $\omega^\delta = \psi_\delta * \omega$ and its

TABLE 3.4

Smoothing Error for $\omega = \omega^{(1)}$ and $p = 2, 4, 6, 8$

δ	$p = 2$	$p = 4$	$p = 6$	$p = 8$
0.2	1.415-1	2.810-2	1.415-2	9.430-3
0.15	9.186-2	1.118-2	3.975-3	2.055-3
0.1	4.485-2	2.646-3	5.109-4	1.641-4
0.075	2.695-2	8.954-4	1.060-4	2.307-5
0.05	1.237-2	1.858-4	1.068-5	1.268-6
0.04	7.994-3	7.725-4	2.850-6	2.460-7

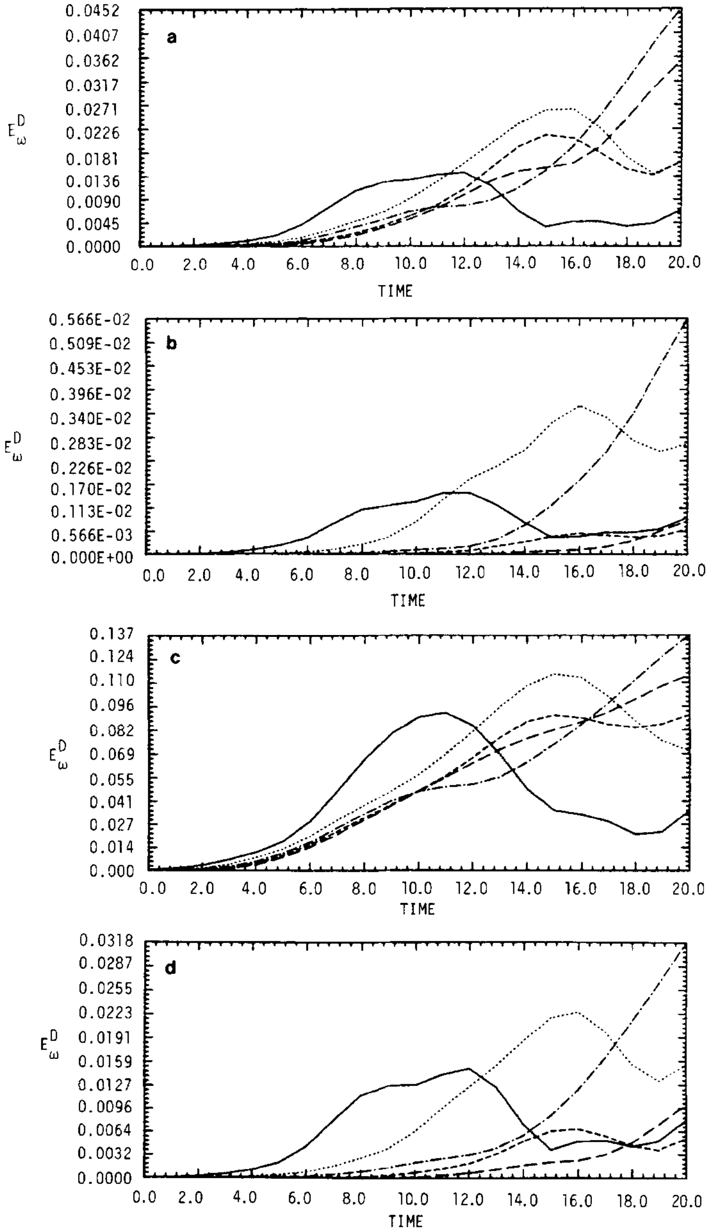

FIG. 3.7. (a) Discretization error E^D_ω in the time interval [0,20] with $\omega = \omega^{(1)}$, $p = 2$, and $\delta = h^{0.95}$. (b) Discretization error E^D_ω in the time interval [0, 20] with $\omega = \omega^{(1)}$, $p = 2$, and $\delta = h^{0.65}$. (c) Discretization error E^D_ω in the time interval [0, 20] with $\omega = \omega^{(1)}$, $p = 4$, and $\delta = h^{0.95}$. (d) Discretization error E^D_ω in the time interval [0, 20] with $\omega = \omega^{(1)}$, $p = 4$, and $\delta = h^{0.65}$. h: (—) 0.2, (···) 0.14, (–·–) 0.10, (---) 0.07, (–––) 0.05.

trapezoidal rule approximations ω_h. E_ω^D is of order $(h/\delta)^L$, (see Lemma 2.5 by Cottet [11] or the Discretization Lemma by Anderson and Greengard [2]), where L depends on the smoothness of the flow and of the cutoff function ψ. Thus if we choose $\delta = h^q$, with $q < 1$, the error should decrease for any $q < 1$.

Our numerical experiments indicate that the discretization error E_ω^D has the same qualitative behavior for all functions, including the second order cutoff function. Hence, we will now describe the discretization error for these cutoff functions as a function of h, δ, and t. The numerical experiments relate to the first test problem. Partial runs for the second test problem, not presented here, indicate a similar behavior of the discretization error.

We find that E_ω^D behaves in an unexpected fashion both as a function of h and of the time t, while E_ω^D has the expected behavior as a function of δ, i.e., the error decreases as δ increases.

Consider a fixed h and $\delta = h^q$ with $0.5 < q < 1$. We present in Figs. 3.7a–d the discretization error for $p = 2$ and $p = 4$ with $\delta = h^{0.95}$ and $\delta = h^{0.65}$. We find that the E_ω^D increases sharply in time, however, it does not increase without bound. The rate at which the error increases, decreases in time. For example, for $p = 4$ and $h = 0.05$, with $\delta = h^{0.95}$ the error at $t = 2$ is 8 times larger than the error at time $t = 1$, while the error at $t = 20$ is only 1.05 times larger than the error at $t = 19$. The major difference in the approximation of ω^δ by ω^h is the position of the computational points. When one observes the flow, one can see that as time increases, there is a rapid decrease in the degree of organization of the flow. At time $t = 0$, when the computational points are equally spaced, the approximation is extremely accurate; but as soon as the points become disorganized, there is a sharp increase in the error. On the other hand further disorganization of the computational points does not affect the approximation drastically. This is seen in the small increase of the error from time $t = 19$ to $t = 20$, and in longer time computations. It would seem that the accuracy of the approximation depends on the organization of the computational points.

As a function of h and for any $\delta = h^q$, with $0.5 < q < 1$, we observe a loss of accuracy over time in the approximation of ω^δ by ω^h. This loss of accuracy is more pronounced for δ close to h, $\delta = h^q$ with $0.75 < q < 1$. For $0.05 \leqslant h \leqslant 0.2$ we observe that for any δ and for some $T > 0$ the error does not decrease as we increase the number of vortices, (see Figs. 3.7a–d). For $h \leqslant 0.05$, which corresponds to more than 900 vortices, we find that while the errors do not decrease for δ close to h and some $T > 0$, they do decrease for larger δ over the time interval $[0, 20]$.

If h is fixed then E_ω^D decreases as δ increases (see Fig. 3.8). This agrees with the negative powers of δ which occur in the theoretical estimates of E_ω^D, see Beale and Majda [5], Cottet [11], Anderson and Greengard [2]. The decrease of E_ω^D as δ increases allows us to create a balance between the smoothing and discretization errors, to obtain accurate results over a fixed time interval.

If we compare the discretization error for different cutoff functions, with a fixed h and δ, we find that the discretization error for $p = 2$ is substantially smaller than the discretization error for higher order cutoff functions. The latter are of comparable

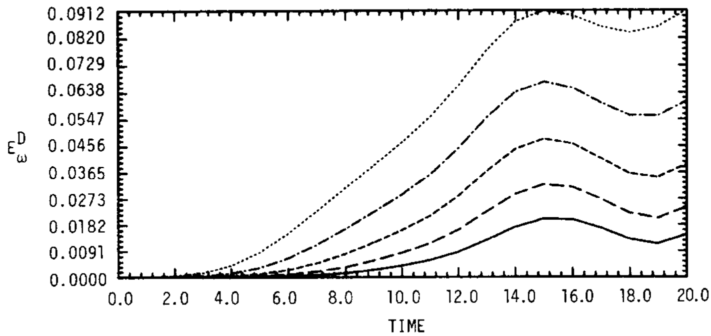

FIG. 3.8. Discretization error E^D_ω, with $\omega = \omega^{(1)}$, $p = 4$, $h = 0.07$, and $\delta = h^q$, $0.75 \leqslant q \leqslant 0.95$. g: (—) 0.95, (-·-) 0.90, (---) 0.85, (–––) 0.80, (—) 0.75.

size. On the other hand the smoothing error decreases as we increase the order p of the cutoff function (see Table 3.4). Because of these two facts we need to choose a larger δ as p increases. This is not consistent with the theory for $L = \infty$, i.e., for infinitely differentiable flows, but is qualitatively consistent with the theory if L is finite.

Having observed the behavior of the smoothing and discretization errors, we can understand how the consistency error develops as a function of h, δ, and t. Consider the second order cutoff function. As we mentioned above, the discretization error E^D_ω increases in time and for some $T > 0$ does not decrease as h decreases, however, it is small relative to the size of the smoothing error, which is of order δ^2. Thus the behavior of E^D_ω is not felt in the consistency error and we obtain an accuracy of $2q$, with $q \leqslant 0.9$.

For higher order cutoff functions and $\delta = h^q$, with $0.75 < q < 1$, the sharp increase of the consistency error in time and its behavior as h decreases is caused by the discretization component. We observe that for some $T > 0$ the consistency error is almost equal to its discretization component. This indicates that except for a short initial time, the dominant term in the consistency error is the discretization error. This neutralizes the advantages provided by higher order cutoff functions. Because E^S_ω increases with δ, while E^D_ω decreases as δ increases, by choosing δ larger we are able to eliminate the sharp increase of the consistency error as a function of t and we obtain a more uniform decrease of the error as a function of h. In doing so, we lose some of the increased accuracy provided by the higher oder cutoff functions.

We conclude this section with a summary of the results of our numerical experiments:

If the flow is smooth enough, the accuracy of the vortex method is improved by increasing the order of the cutoff function. This is not the case for nonsmooth flows, as we showed in the numerical experiments with the third test problem.

Our numerical experiments indicate that for the values of h tested, the choice of δ is quite insensitive to the smoothness of the flow. We find that if δ is close to h, as suggested by Beale and Majda [5], then the accuracy is lost in a relatively short time, even for infinitely differentiable flows. By increasing δ we are able to preserve the accuracy of the method over longer time intervals. Thus the predicted p^{th} order accuracy is reduced to pq, with q closer to $\frac{1}{2}$ than to 1. In addition we observe that the discretization error does not grow without bound. Therefore by choosing δ large enough so that the smoothing error is larger than the discretization error we can obtain accurate results over long time intervals.

4. The Approximation of the Initial Vorticity Distribution

The initial vorticity distribution can be approximated by one of the two forms,

$$\omega_1^h = \sum_j \psi_\delta(z - z_j) \int_{B_j} \omega(z)\, dz \tag{4.1}$$

or

$$\omega_2^h = \sum_j \psi_\delta(z - z_j)\, \omega_j h^2, \tag{4.2}$$

where the z_j's are the grid points and h^2 is the area of the square B_j centered at z_j. The corresponding velocity approximations are given by $u_l^h = K * \omega_l^h$, for $l = 1,2$. The approximation (4.2) is the approximation of the convolution integral $\psi_\delta * \omega$ by the trapezoidal rule. The error is of order $\delta^p + (h/\delta)^L$, where p is the order of the cutoff function and L depends on the number of derivatives of the vorticity (see Anderson [1]). Cottet has shown that if the vorticity is approximated by (4.1) there is an additional error of order h^2. Thus if the cutoff function is of order $p \geqslant 4$ and $\sqrt{h} \leqslant \delta \leqslant h$, the approximation (4.1) is only second order accurate. On the other hand, for sufficiently smooth flows and δ close to h the vorticity approximation (4.2) is, at least at time $t = 0$, p^{th} order accurate for a p^{th} order cutoff function.

The numerical experiments presented in Section 3 indicate that to preserve the accuracy of the velocity and vorticity approximation over a fixed time interval $[0, T]$ the smoothing error should be larger than the discretization error. This is always the case at $t = 0$. As our initial vorticity is radially symmetric the smoothing error is independent of time. Thus we compare the velocity and vorticity approximations at time $t = 0$. We use the first test problem of Section 2 and compute the discrete velocity and vorticity approximations u_l^h and ω_l^h $(l = 1,2)$ at time $t = 0$ for the cutoff functions (2.1)–(2.4), $0.03 \leqslant h \leqslant 0.2$ and $\delta = h^q$ with $0.5 < q < 1$. We measure the consistency errors $t = 0$ in the discrete L^2 norm

$$E_u^l = \left(\sum_j |u(z_j) - u_l^h(z_j)|^2\, h^2 \right)^{1/2}.$$

TABLE 4.1

Velocity Errors Obtained with Approximations (4.1) and (4.2) and $\delta = h^{0.95}$

	$p=2$		$p=4$		$p=6$		$p=8$	
h	E_u^1	E_u^2	E_u^1	E_u^2	E_u^1	E_u^2	E_u^1	E_u^2
0.2	3.394-2	3.219-2	7.885-3	5.333-3	5.156-3	2.607-3	4.301-3	1.731-3
0.14	2.006-2	1.894-2	3.209-3	1.821-3	1.999-3	6.002-4	1.735-3	2.989-4
0.1	1.123-2	1.058-2	1.286-3	5.621-4	8.565-4	1.132-4	7.985-4	3.811-5
0.07	6.065-3	5.711-3	5.343-4	1.628-4	4.022-4	1.877-5	3.919-4	3.984-6
0.05	3.210-3	3.025-3	2.348-4	4.547-5	1.970-4	2.903-6	1.954-4	5.113-7
0.03	1.680-3	1.585-3	1.083-4	1.245-5	9.793-5	4.685-7	9.768-5	1.979-7

and

$$E_\omega^l = \left(\sum_j |\omega(z_j) - \omega_l^h(z_j)|^2 \, h^2 \right)^{1/2},$$

where $l = 1,2$. We estimate the rate of convergence by using two successive values of h in Eq. (2.9).

With both approximations the velocity and vorticity errors decrease as $h \to 0$ for any $\delta = h^q$, with $0.5 < q < 1$, and for a fixed h the errors increase as δ increases. For a fixed h and q both E^1 and E^2 decrease as the order of the cutoff function increases.

We find that for any $0.03 \leqslant h \leqslant 0.2$ and for any $\delta = h^q$, with $0.5 < q < 1$, the errors E_u^1 and E_ω^1 are larger than the errors E_u^2 and E_ω^2. Table 4.1 compares the velocity errors obtained with both approximations. For the second order cutoff function (2.1), E^1 and E^2 are of the comparable order. Both approximations are $2q$ order accurate for $\delta = h^q$. For higher order cutoff functions and $\delta = h^q$, with $0.75 \leqslant q < 1$, there is a significant difference between the two approximations. For

TABLE 4.2

Velocity Errors Obtained with Approximations (4.1) and (4.2) and $\delta = h^{0.65}$

	$p=2$		$p=4$		$p=6$		$p=8$	
h	E_u^1	E_u^2	E_u^1	E_u^2	E_u^1	E_u^2	E_u^1	E_u^2
0.2	6.224-2	6.114-2	1.814-2	1.941-2	1.567-2	1.368-2	1.341-2	1.146-2
0.14	4.703-2	4.634-2	9.687-3	1.110-2	7.895-3	6.785-3	6.287-3	5.221-3
0.1	3.241-2	3.378-2	6.490-3	5.865-3	3.538-3	2.954-3	2.555-3	1.999-3
0.07	2.403-2	2.377-2	3.209-3	2.880-3	1.429-3	1.129-3	9.277-4	6.400-4
0.05	1.640-2	1.625-2	1.504-3	1.336-3	5.405-4	3.876-4	3.270-4	1.745-4
0.03	1.095-2	1.087-2	6.785-4	5.935-4	2.007-4	1.222-4	1.247-4	4.164-5

example, with $h=0.05$, $\delta=h^{.95}$ and $p=6$, E_u^1 is approximately 10^{-4} while E_u^2 is approximately 10^{-6}. In addition we find that for these values of δ and $p=6$ and 8 u_1^h and ω_1^h are quite insensitive to the order of cutoff function and to the choice of δ. With these values of δ the rate of convergence of the approximation (4.1) decreases to 2, while the rate of convergence obtained with (4.2) increases as $h \to 0$.

Our numerical examples show that for $\delta=h^q$, with $0.5<q\leqslant 0.75$, the difference between E^1 and E^2 is not so drastic, although $E^1>E^2$ still holds (see Table 4.2). We find that for a fixed q in the range specified above, the rate of convergence of (4.1) increases up to approximately 3 and decreases again for smaller values of h, however, it does not decrease to 2. This can be explained by the fact that these values of δ are not in the region where the asymptotic estimates are valid.

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