CounterExample-Guided Abstraction Refinement (CEGAR)

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(with many slides © S. Ratschan[b])

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- but finding the right abstraction is hard
  - may be too coarse $\rightsquigarrow$ verification fails
  - may be too fine $\rightsquigarrow$ state-space exploration impossible
  - may even be too fine in some places and too coarse in others

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Can we do the same within abstraction-based model checking?

- Upon a failing proof, let the model-checker analyze the reasons and
- refine the abstraction as necessary.

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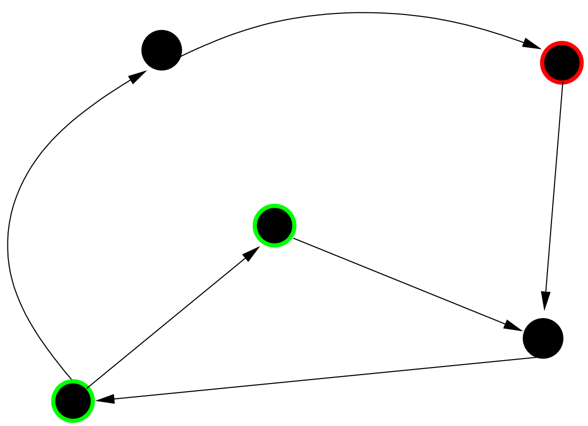

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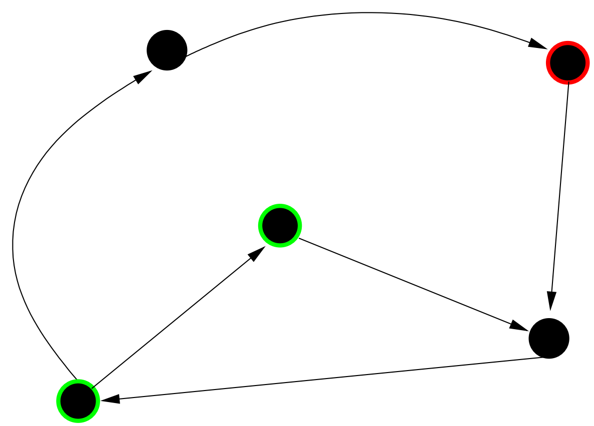

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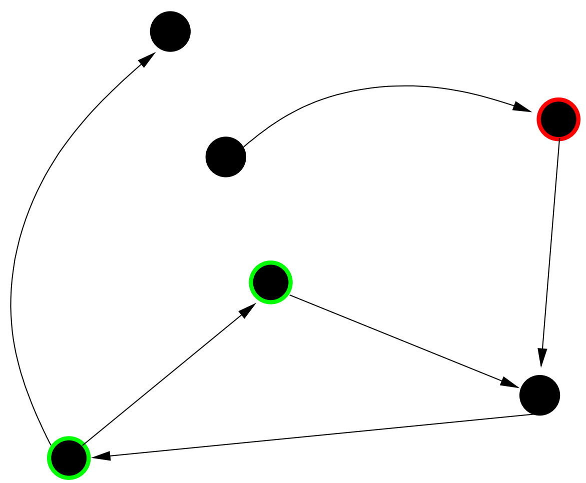

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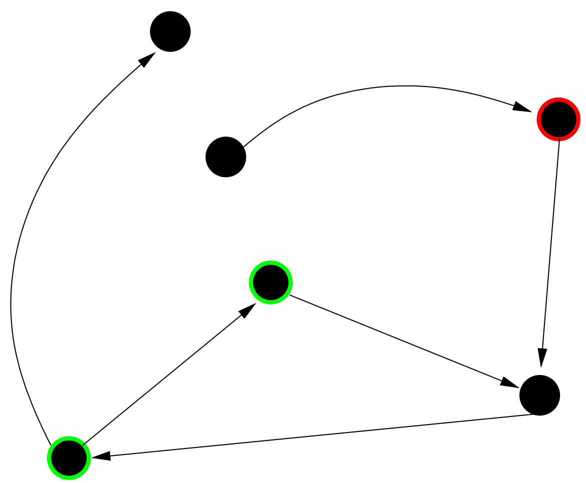

- if abstraction safe, done
- while abstraction not safe, refine it
- counter-example based: refine to remove a given spurious counter-example (Clarke et al. 03, Alur et al. 03)

Basic CEGAR

Spurious counterexample

Def: Let $A \succ C$ be an homomorphic abstraction wrt. abstraction function h. Let ϕ be an $\forall$CTL formula and $\pi = (c_1, c_2 \dots)$ be an anchored path of C witnessing violation of ϕ on C.
Then π is called a counterexample for ϕ on C.

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Furthermore, $h(\pi) = (h(c_1), h(c_2), \ldots)$ then is an anchored path of A which violates ϕ, i.e. a counterexample on A. We do then call $h(\pi)$ the abstract counterexample corresponding to π and we call π the concrete counterexample corresponding to $h(\pi)$.

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Def: If π_A is a counterexample on the abstraction $A \succ C$ which has no corresponding concrete counterexample on C then we call π_A a spurious counterexample.

Abstraction Refinement

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Idea: Whenever there is a spurious counterexample in A, identify an abstraction refinement A' that lacks that particular spurious counterexample.

CEGAR algorithm (simple version: invariants)

To verify $C \models AGp$ do

1. build finite Kripke structure $A \succ C$,
2. model-check $A \models AGp$,
3. if this holds then report $C \models AGp$ and stop,
4. otherwise validate the counterexample on C, i.e., find a corresponding concrete counterexample,
5. if a corresponding concrete counterexample exists then report $C \not\models AGp$ and stop,
6. otherwise use the spurious counterexample to refine A and restart from 2.

The crucial ingredients of CEGAR

- Model checking,
- validation/concretization of counterexample,
- guided refinement of abstraction.

Validation of counterexample

Given: $A \succ C$ and an abstract counterexample $\phi = (a_1, a_2, \ldots, a_n)$ on A.

Alg: Provided we can effectively manipulate pre-images of the abstraction morphism h, proceed as follows:

1. Compute $S_1 := h^{-1}(a_1) \cap I_C$, where I_C is the set of initial states of C,
2. For $i = 2$ to n, compute $S_i := h^{-1}(a_i) \cap \mathrm{Post}(S_{i-1})$.
   Abort as soon as some S_i becomes $\emptyset$.
   In this case, the counterexample has been shown to be spurious.
3. In case of proper termination of the loop, the counterexample is real.

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N.B. Assumes that $h^{-1}(a_i)$, $\mathrm{Post}(S_i)$, and their intersections are computable (in the sense of an effective emptiness test)!

State splitting

Idea: For a set $C_i = h^{-1}(a_i)$ of concrete states represented by an abstract state a_i occurring in the spurious counterexample, split it into $C_i \cap \mathrm{Post}(h^{-1}(a_{i-1}))$ and $C_i \setminus \mathrm{Post}(h^{-1}(a_{i-1}))$, provided both non-empty (or into $C_1 \cap I_C$ and $C_1 \setminus I_C$ in case $i = 1$).

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Approach: Replace a_i by two states a_i^+ and a_i^- representing $C_i \cap \mathrm{Post}(h^{-1}(a_{i-1}))$ and $C_i \setminus \mathrm{Post}(h^{-1}(a_{i-1}))$, resp.

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Technique: Replace the Kripke structure $A = (V, E, L, I)$ by $A' = (V', E', L', I')$ with

- $V' = V \setminus \{a_i\} \cup \{a_i^+, a_i^-\}$, where the latter are $\notin V$,
- $E' = E \cap (V' \times V') \cup \{(a_i^+, a_i^-), (a_i^-, a_i^+)\} \cup \{(a, a_i^+) \mid (a, a_i) \in E\} \cup$
  $\{(a, a_i^-) \mid (a, a_i) \in E, a \neq a_{i-1}\} \cup \{(a_i^+, a), (a_i^-, a) \mid (a_i, a) \in E\}$
- $L'(v) = \begin{cases} L(v) & \text{if } v \in V, \\ L(a_i) & \text{if } v \in \{a_i^+, a_i^-\}, \end{cases}$
- $I' = \begin{cases} I & \text{if } C_i \cap I_C = \emptyset, \\ I \setminus \{a_i\} \cup \{a_i^+\} & \text{otherwise.} \end{cases}$

Resulting morphism

$$h'(c) = \begin{cases} a_i^+ & \text{if } c \in C_i \cap \mathrm{Post}(h^{-1}(a_{i-1})), \\ a_i^- & \text{if } c \in C_i \setminus \mathrm{Post}(h^{-1}(a_{i-1})), \\ h(c) & \text{otherwise.} \end{cases}$$

Refining E': transition pruning

Observation: Pre- and post-images of $h'^{-1}(a_i^+)$ or $h'^{-1}(a_i^-)$ may well have empty intersections with sets that the pre- or post-set of $h'^{-1}(a_i)$ did intersect with.
In such cases, E' contains spurious edges.

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Solution: Remove such edges by pruning E' to

$$E'' = \{(s,t) \in E' \mid \mathrm{Post}(h'^{-1}(s)) \cap h'^{-1}(t) \neq \emptyset\}$$

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7. *perform transition pruning* on the resulting refinement A',
8. goto 2.

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Concrete version is just an example, variants of split/prune rules abound.

Application to hybrid systems

- Above procedure is effective if $h^{-1}(a_i)$, $\mathrm{Post}(S_i)$, and their intersections are computable (in the sense of an effective emptiness test).
- This is in general not true for hybrid systems.

$\Rightarrow$ Need to embed an appropriate form of approximation of the above sets into CEGAR.

CEGAR on hybrid states

Conservative approximation of state sets

Application to hybrid systems

- "Naive" CEGAR procedure is effective if $h^{-1}(a_i)$, $\mathrm{Post}(S_i)$, and their intersections are computable (in the sense of an effective emptiness test).
- In general not true for hybrid systems, thus embed an appropriate form of approximation of the above sets into CEGAR.
- Main difficulty is computation of successor states: explicit (jumps) and implicit transitions (flows, defined by ODE)
  - Multiple shapes of overapproximation can be used
    - various effective representations of subsets of $\mathbb{R}^n$: rectangular boxes, zonotopes, polyhedra, ellipsoids, . . .,
    - multiple techniques for conservatively approximating hybrid transitions (jumps & flows)
  - can be combined to obtain an adaptive CEGAR algorithm
    - e.g., proceeds from coarse to fine, investing computational effort to increase precision when necessary.

Computing successors

- CEGAR algorithm applies different approximations of successor computation in sequence,
- proceeds from coarse to fine, investing more computational effort to increase precision only when necessary,
- hope is that crucial deductions (absence of counterexamples, non-concretizability of a certain counter-example) can often be obtained on coarse abstractions,
- CEGAR needs to compute different relative successors $\mathit{Succ}(X, Y) = \mathit{Post}(X) \cap Y$, where $X, Y \in \mathcal{P}(\mathbb{R}^n)$.
- Can approximate these by any operation $\mathit{SUCC} : \mathcal{P}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ with
  1. Overapproximation: $\mathit{SUCC}(X, Y) \supseteq \mathit{Post}(X) \cap Y$,
  2. Reasonability: $\mathit{SUCC}(X, Y) \subseteq Y$.

Validation of counterexample

Given: $A \succ C$ and an abstract counterexample $\phi = (a_1, a_2, \ldots, a_n)$ on A.

Alg: For a sequence of successively tighter overapproximations $(SUCC_i)_{i=1,\ldots,k}$, proceed as follows:

1. Start with $i = 1$, i.e., the coarsest approximation.
2. Compute $S_1^i := overapprox_i(h^{-1}(a_1) \cap I_C)$, where I_C is the set of initial states of C,
3. For $j = 2$ to n, compute $S_j^i := SUCC_i(S_{j-1}^i, h^{-1}(a_i))$
   Abort as soon as some S_j^i becomes $\emptyset$.
   In this case, the counterexample is spurious.
4. In case of proper termination of the inner loop, restart at 1. with $i := i + 1$, i.e., the next finer approximation, if $i < k$.
5. If the inner loop terminates regularly for $i = k$, then the abstract counterexample can't be refuted by any of the overapproximations. (Probably is real.)

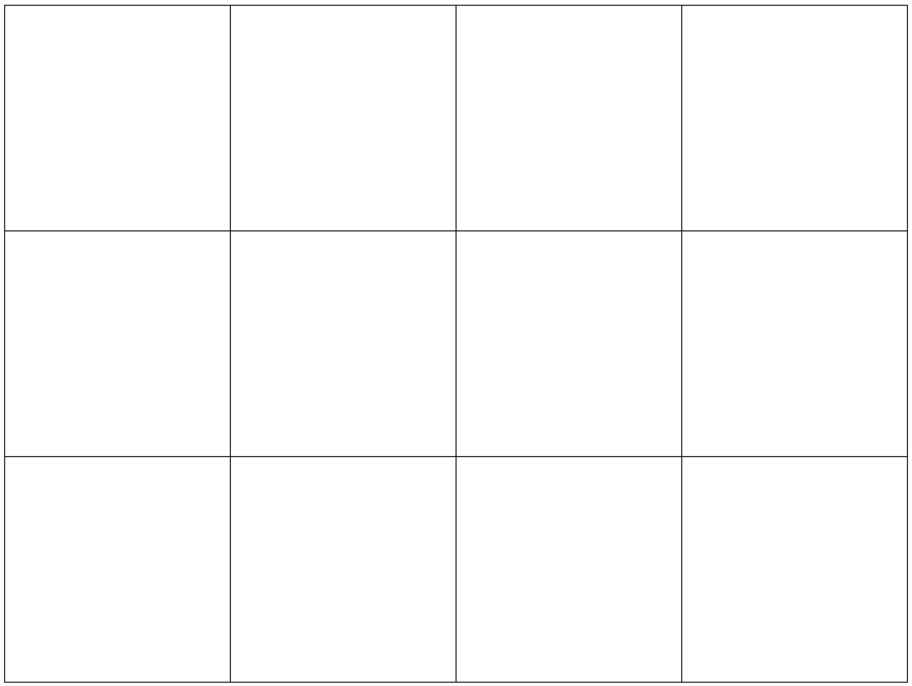
HSolver

Overapproximation via Constraint-based Reasoning

Stefan Ratschan, Czech Academy of Sciences
Shikun She, MPII, Saarbrücken

Starting Point: Interval Grid Method

Stursberg/Kowalewski et. al., one-mode case:



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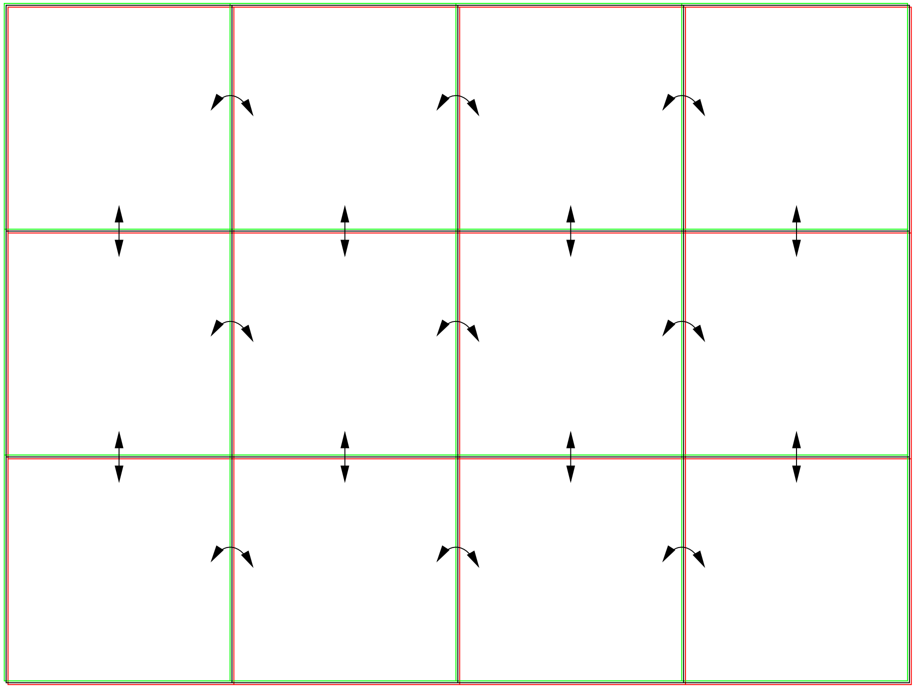

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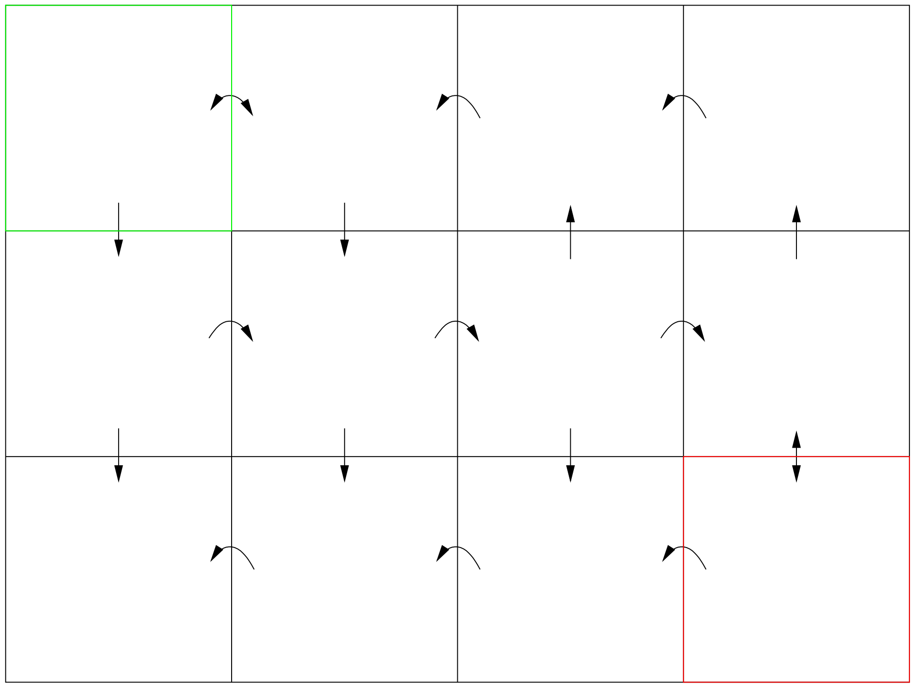

$$\dot{x} \in [-5, -1]$$

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- remove impossible transitions/marks (interval arithmetic check on boundaries/boxes)

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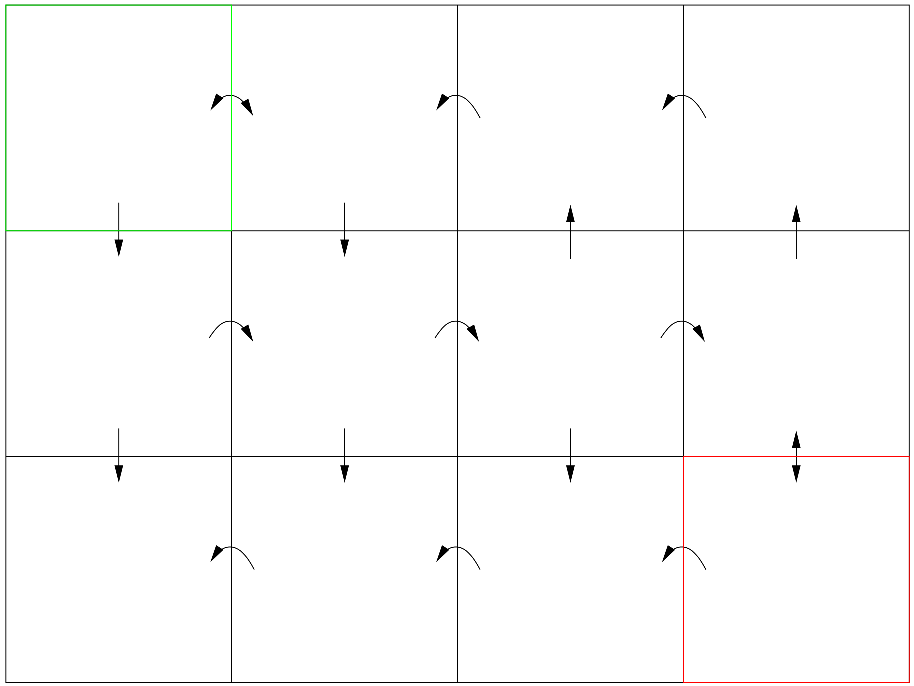

$\dot{x} \in [-5, 1]$

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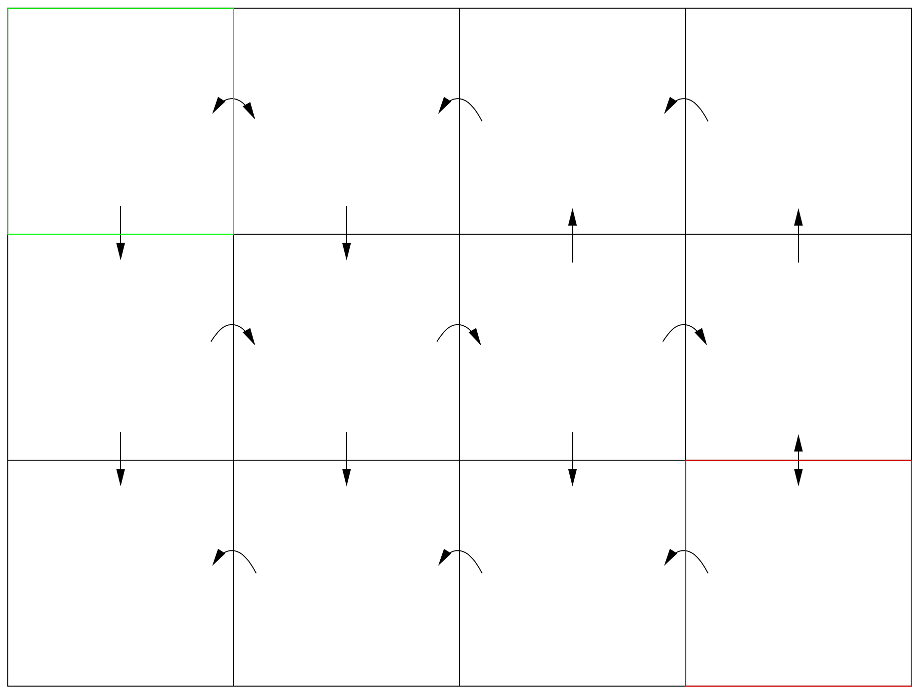

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Result: finite abstraction

Interval arithmetic

Is a method for calculating an interval *covering* the possible values of a real operator if its arguments range over intervals:

$$
\begin{aligned}
[a, A] \overset{\circ}{+} [b, B] &= [a + b, A + B] \\
[a, A] \overset{\circ}{\cdot} [b, B] &= [\min\{ab, aB, Ab, AB\}, \max\{ab, aB, Ab, AB\}] \\
\overset{\circ}{\min} ([a, A], [b, B]) &= [\min\{a, b\}, \min\{A, B\}] \\
\overset{\circ}{\sin} ([a, A]) &= \begin{bmatrix} \min\{\sin x \mid x \in [a, A]\}, \\ \max\{\sin x \mid x \in [a, A]\} \end{bmatrix} \\
\overset{\circ}{f} ([a, A], [b, B], \ldots) &= \begin{bmatrix} \min\{f(\vec{x}) \mid \vec{x} \in [a, A] \times [b, B] \times \ldots\}, \\ \max\{f(\vec{x}) \mid \vec{x} \in [a, A] \times [b, B] \times \ldots\} \end{bmatrix}
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$$

Theorem: For each term t with free variables $\vec{v}$:

$$\{t(\vec{v} \mapsto \vec{x}) \mid \vec{x} \in [a, A] \times [b, B] \times \ldots\} \subseteq \overset{\circ}{t}(v_1 \mapsto [a, A], v_2 \mapsto [b, B], \ldots)$$

Is the approximation tight?

1. In the limit: **yes!**

$$t(\vec{v} \mapsto \vec{x}) = \overset{\circ}{t}\,(v_1 \mapsto [x_1, x_1], v_2 \mapsto [x_2, x_2], \ldots)$$

$$t(\vec{v} \mapsto \vec{x}) = \lim_{\varepsilon \to 0} \overset{\circ}{t}\,(v_1 \mapsto [x_1 - \varepsilon, x_1 + \varepsilon], v_2 \mapsto [x_2 - \varepsilon, x_2 + \varepsilon], \ldots)$$

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2. In general: **No!** If $a < A$ then

$$x - x(x \mapsto [a, A]) \;=\; [a, A] \mathbin{\mathring{-}} [a, A] \;=\; [a - A, A - a] \;\neq\; [0, 0]$$

Dependency problem of interval arithmetic:

☹ Tight bounds only if each variable occurs at most once!

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Jumps: also check using interval arithmetic

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Let's remove them!

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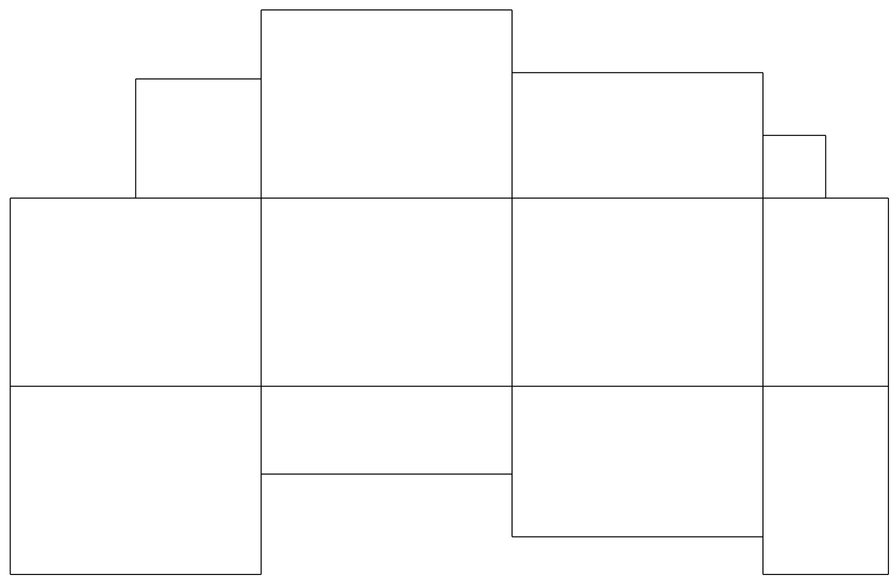

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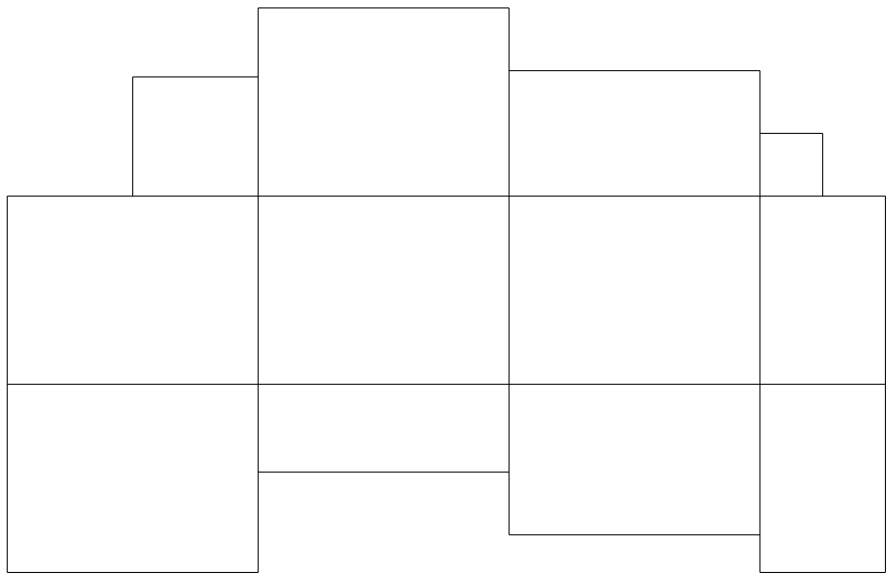

Method: form constraints that hold on reachable parts of state space, remove non-solutions by constraint solver

Reach Set Pruning

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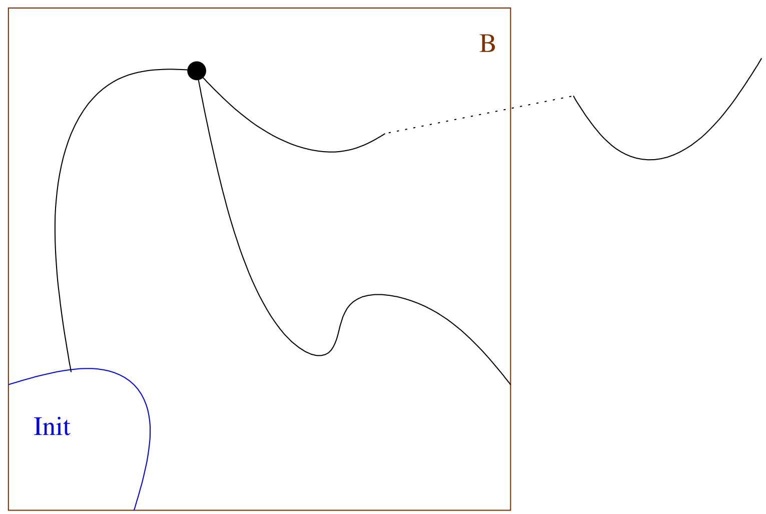
A point in a box B can be reachable

- from the initial set via a flow in B
- from a jump via a flow in B
- from a neighboring box via a flow in B



formulate corresponding constraints, remove all points from box that do not fulfill one of these constraints

Constraints in Specification

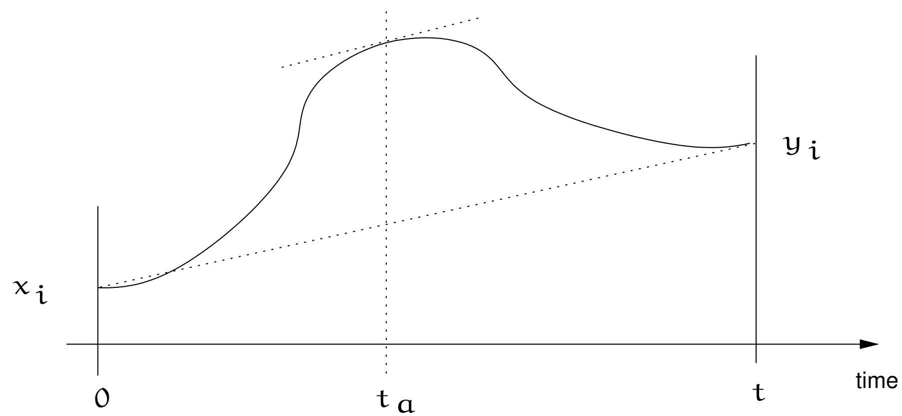
We specify system using constraints:

- $\mathrm{Flow}(s, \vec{x}, \dot{\vec{x}})$ (e.g., $s = \mathtt{off} \rightarrow \dot{x} = x \sin(x) + 1 \quad \ldots)$
  - ˙purely syntactic!
  - even implicit and algebraic!
- $\mathrm{Jump}(s, \vec{x}, s', \vec{x}')$ (e.g.,
  $(s = \mathtt{off} \wedge x \geq 10) \; \rightarrow \; (s' = \mathtt{on} \wedge x' = 0))$
- $\mathrm{Init}(s, \vec{x})$

Reachability Constraints

Lemma (n-dimensional mean value theorem): For a box B, mode s, if a point $(y_1, \ldots, y_n) \in B$ is reachable from a point $(x_1, \ldots, x_n) \in B$ via a flow in B then

$$\exists t \in \mathbb{R}_{\geq 0} \bigwedge_{1 \leq i \leq n} \exists a_1, \ldots, a_k, \dot{a}_1, \ldots, \dot{a}_k[(a_1, \ldots, a_k) \in B \wedge \mathrm{Flow}(s, (a_1, \ldots, a_k), (\dot{a}_1, \ldots, \dot{a}_k)) \wedge y_i = x_i + \dot{a}_i \cdot t]$$



Denote this constraint by $flow_B(s, \vec{x}, \vec{y})$.

Reachability Constraints

Lemma: For a box $B \subseteq \mathbb{R}^k$, mode s, if $\vec{y} \in B$ is reachable from the initial set via a flow in B then

$$\exists \vec{x} \in B \left[\mathit{Init}(s, \vec{x}) \wedge \mathit{flow}_B(s, \vec{x}, \vec{y}) \right]$$

Reachability Constraints

Lemma: For a box $B \subseteq \mathbb{R}^k$, mode s, if $\vec{y} \in B$ is reachable from the initial set via a flow in B then

$$\exists \vec{x} \in B\,[\mathrm{Init}(s, \vec{x}) \wedge \mathrm{flow}_B(s, \vec{x}, \vec{y})]$$

Lemma: For a box $B \subseteq \mathbb{R}^k$, mode s, $\vec{y} \in B$, $(s, \vec{y})$ is reachable from a jump from a box B^* and mode s^* via a flow in B then

$$\exists \vec{x}^* \in B^* \exists \vec{x} \in B\,[\mathrm{Jump}(s^*, \vec{x}^*, s, \vec{x}) \wedge \mathrm{flow}_B(s, \vec{x}, \vec{y})]$$

Reachability Constraints

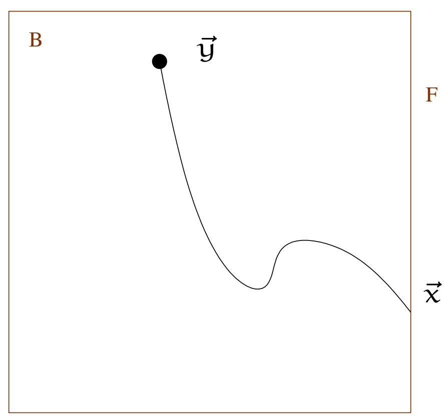
Lemma: For a box $B \subseteq \mathbb{R}^k$, mode s, if $\vec{y} \in B$ is reachable from a neighboring box over a face F of B and a flow in B then

$$\exists \vec{x} \in F \left[\mathrm{incoming}_F(s, \vec{x}) \wedge \mathrm{flow}_B(s, \vec{x}, \vec{y})\right],$$

where $\mathrm{incoming}(s, \vec{x})$ is of the form

$$\exists \dot{x}_1, \ldots, \dot{x}_k [\mathrm{Flow}(s, \vec{x}, (\dot{x}_1, \ldots, \dot{x}_k)) \wedge \dot{x}_j \; r \; 0]$$

where $r \in \{\leq, \geq\}$, $j \in \{1, \ldots, k\}$ depends on the face F



for corners etc. a little bit more involved

Using Constraints

After substituting definitions,

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After substituting definitions, getting rid of quantifiers,

Using Constraints

After substituting definitions, getting rid of quantifiers, interval constraint propagation algorithms can remove parts from boxes not fulfilling such constraints.

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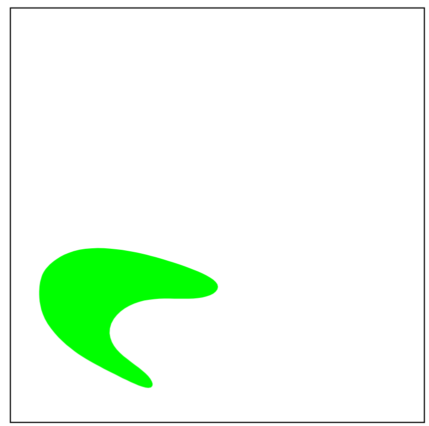
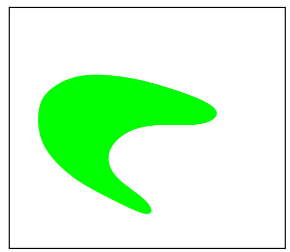


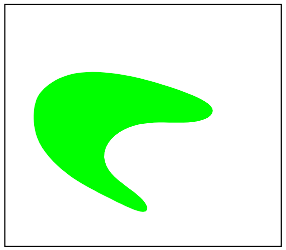
Using Constraints

After substituting definitions, getting rid of quantifiers, interval constraint propagation algorithms can remove parts from boxes not fulfilling such constraints.



Using Constraints

After substituting definitions, getting rid of quantifiers, interval constraint propagation algorithms can remove parts from boxes not fulfilling such constraints.



- correct handling of rounding errors
- almost negligible time
- result not necessarily tight (but tight for $\mathtt{flow}_B(s, \vec{x}, \vec{y})$ in linear case)

`http://rsolver.sourceforge.net`