Symbolic Methods

Symbolic state-space traversal for finite-state systems

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What you'll learn

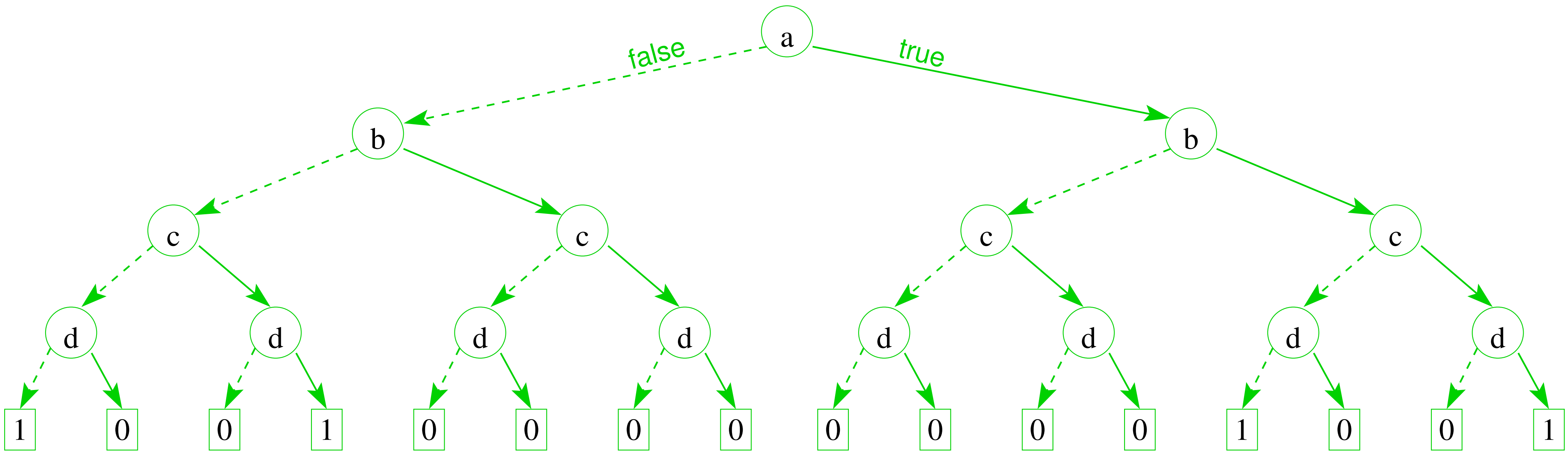
- reduced ordered binary decision diagrams
- symbolic methods for state reachability
  - SAT-based procedures for bounded state reachability
  - full reachability via BDDs
- symbolic CTL model checking

Reduced ordered binary decision diagrams

(RO)BDDs

Binary decision diagrams

An ordered decision tree for $(a \Leftrightarrow b) \wedge (c \Leftrightarrow d)$:



Size exponential in number of variables!

ROBDDs

Obs.: A lot of the tests in the decision diagram are redundant.

Idea: Combine equivalent sub-cases,
i.e. reduce size of the diagram by

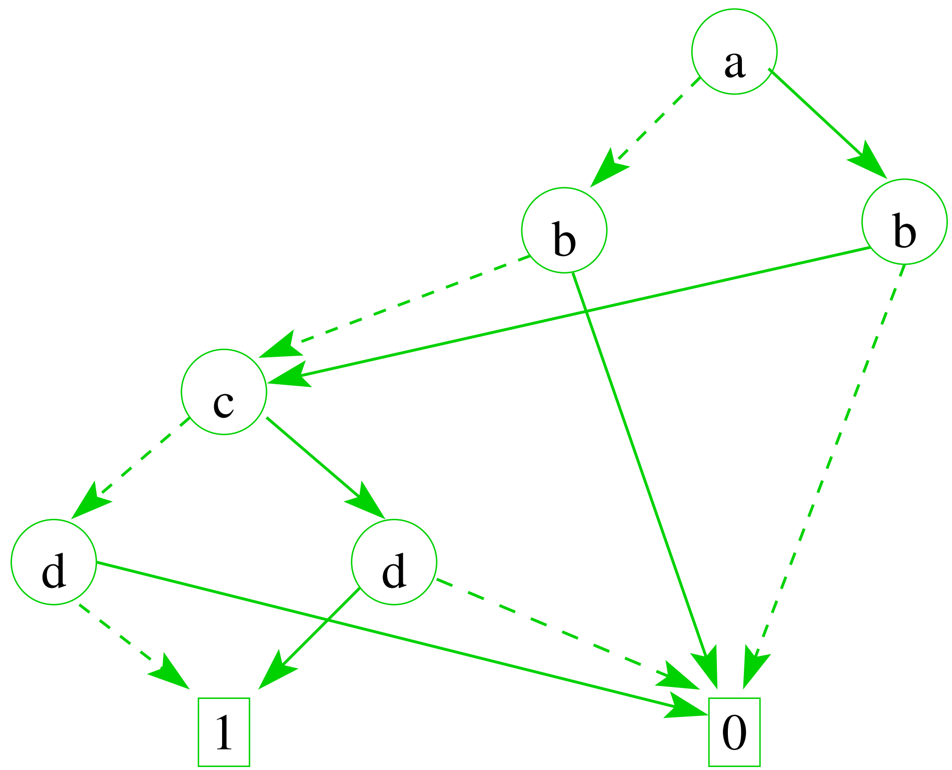
1. omitting nodes that have equivalent left and right sons,
2. sharing common sub-trees:
   - remove duplicate terminal nodes; share instead
   - remove duplicate internal nodes; share instead

Def.: The decision diagrams obtained by above rules are called reduced ordered binary decision diagrams (ROBDDs).

May expect good performance if many substructures are equivalent!

ROBDDs

An ROBDD for $(a \Leftrightarrow b) \wedge (c \Leftrightarrow d)$, using node order $a < b < c < d$:



Note how variable order affects size: Using $a < c < b < d$ would yield a layer with 4 nodes.
For n-bit comparison, we obtain a layer with 2^n nodes if poor order is chosen, yet maximum layer width 2 with appropriate order.

ROBDDS: Some properties

- Given a variable ordering, ROBDDs provide a canonical representation for Boolean functions
  - simple equivalence check, once the ROBDDs have been built:
    - linear in size of BDDs
    - $O(1)$ if sharing across BDDs is used
- Applying a connective to two ROBDDs can be done by simultaneous recursive descent through the two ROBDDs (+acceleration by dynamic programming)
  - $(\text{if } x \text{ then } \phi_t \text{ else } \phi_e) \wedge (\text{if } x \text{ then } \psi_t \text{ else } \psi_e) \equiv$ $(\text{if } x \text{ then } \phi_t \wedge \psi_t \text{ else } \phi_e \wedge \psi_e)$

  $\rightarrow$ efficient

  $\rightarrow$ can construct ROBDDs for non-trivial circuits
- Variable order strongly affects size.
  - need reordering heuristics,
  - even then, some circuits don't permit any good order: e.g., multipliers yield exponentially sized BDDs

ROBDD operations

Negation:

Operation: Constructs from an ROBDD B an ROBDD $not(B)$ with $f_{not(B)} = \neg f_B$, where f_B is the truth function encoded by B.

Algorithm: Swap the terminal nodes:

- node 0 is replaced with 1
- node 1 is replaced with 0.

Complexity: $O(1)$.

ROBDD operations

Boolean junctors:

Operation: Constructs from two ROBDDs B_1, B_2 and a Boolean junctor $\oplus$ an ROBDD $\mathrm{apply}(\oplus, B_1, B_2)$ with $f_{\mathrm{apply}(\oplus, B_1, B_2)} = f_{B_1} \oplus f_{B_2}$.

Algorithm: Recursively proceed as follows:

- If both B_1 and B_2 are terminal nodes then yield terminal node $f_{B_1} \oplus f_{B_2}$.
- If the top nodes of B_1 and B_2 agree on their variable v then
  1. compute $L = \mathrm{apply}(\oplus, \mathrm{left}(B_1), \mathrm{left}(B_2))$,
  2. compute $R = \mathrm{apply}(\oplus, \mathrm{right}(B_1), \mathrm{right}(B_2))$,
  3. build the OBDD (v, L, R),
  4. reduce it.
- If the top nodes of B_1 and B_2 have different variables v_1, v_2 with $v_1 < v_2$ in the variable order then
  1. compute $L = \mathrm{apply}(\oplus, \mathrm{left}(B_1), B_2)$,
  2. compute $R = \mathrm{apply}(\oplus, \mathrm{right}(B_1), B_2)$,
  3. build the OBDD (v, L, R),
  4. reduce it.
- ...

Complexity: $O(|B_1| \cdot |B_2|)$ if memoization is used to save recomputations which may arise due to sharing of subgraphs.

ROBDD operations

Quantification:

Operation: Constructs from an ROBDD B and a variable v an ROBDD $exists(v, B)$ with $f_{exist(v,B)} = \exists v.f_B$.

Algorithm:

1. Replace each sub-BDD of B which has a root node n labeled with v by the ROBDD $apply(\vee, left(n), right(n))$.
2. Reduce the resulting BDD.

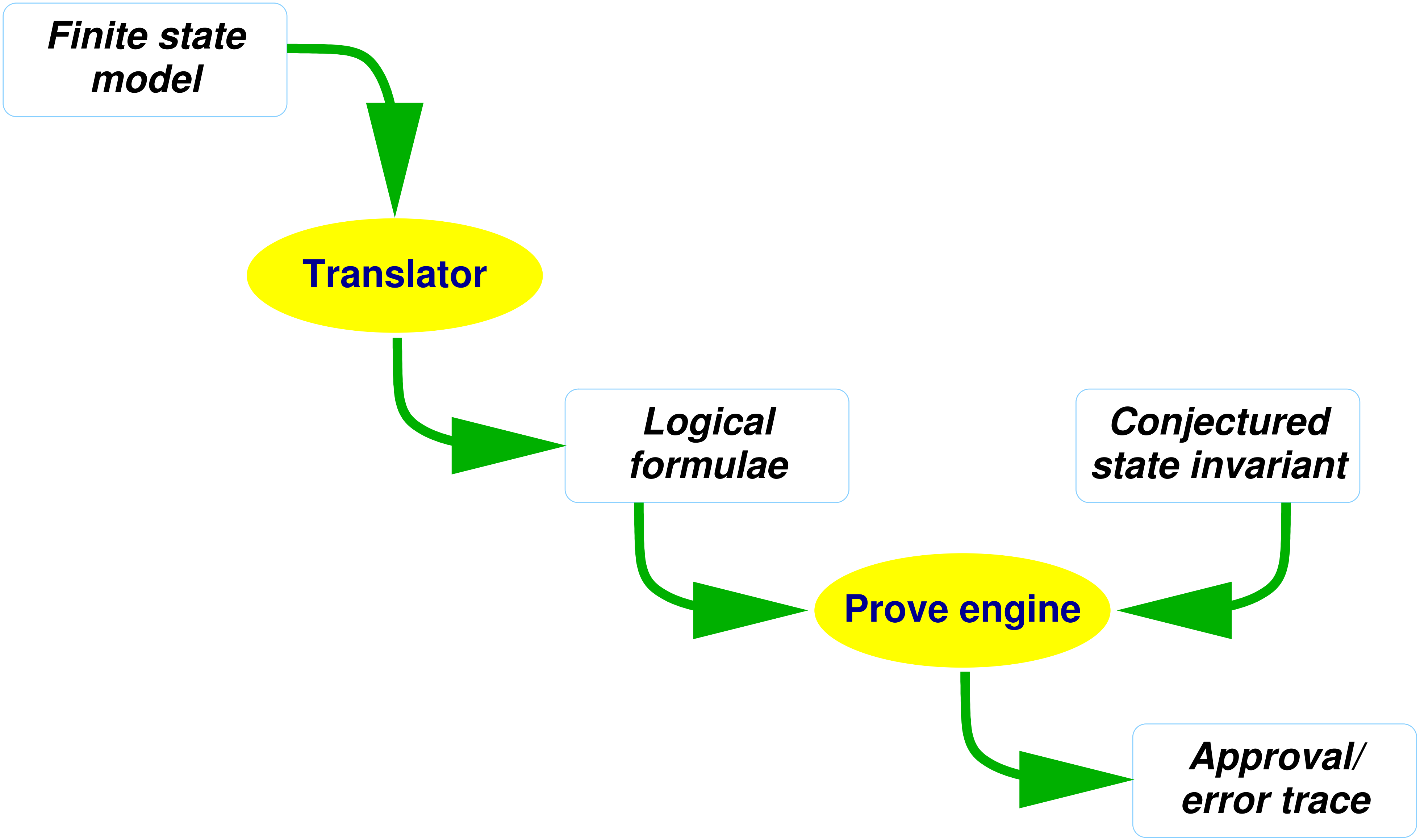
Complexity: $O(|B|^2)$.

Note that BDDs obtained by quantifying *multiple* variables may thus grow exponentially in the number of quantified variables.

Symbolic techniques II:

State reachability in finite-state reactive systems

The general framework



Mapping models to formulae (essence of)

- Each control location s is assigned a proposition p_s;
  each symbolic variable v is assigned $\lceil \log_2 |\mathrm{dom}\, v| \rceil$ propositional variables;
- for describing transitions, propositional variables are duplicated:
  - undecorated version encodes pre-state,
  - primed version encodes post-state,

$$\boxed{s} \xrightarrow{g \,/\, v := e} \boxed{t} \quad \mapsto \quad \phi_{tr} \;\equiv\; p_s \wedge \underbrace{[g]}_{} \wedge \underbrace{[v' = e]}_{\text{proposit. encodings}} \wedge p'_t$$

$$\mathrm{trans}(x, x') \;\equiv\; \bigwedge_{s \text{ state}} \Big(p_s \Longrightarrow \bigvee_{tr \text{ transition from } s} \phi_{tr} \Big)$$

- similar for describing initial state set, yielding predicate $\mathrm{init}(x)$.

> - Translation can be done componentwise, using conjunction for encoding parallel composition.
>
> $\Rightarrow$ This saves computing the automaton product!

Verification/Falsification

Given: Transition pred. $trans(x, x')$, initial state pred. $init(x)$, conj. invar. $\phi(x)$.

QBF-based algorithm:

1. Start with $R_0(x) = init(x)$.
2. Test for satisfiability of $R_i(x) \wedge \neg\phi(x)$. If test succeeds then report violation of goal.
3. Else build $R_{i+1}(x) = R_i(x) \vee \exists\tilde{x}. (R_i(\tilde{x}) \wedge trans(\tilde{x}, x))$.
4. Test whether $R_{i+1}(x) \implies R_i(x)$. If so then report satisfaction of goal. Otherwise continue from step 2, with $i + 1$ instead of i.

BF-based algorithm:

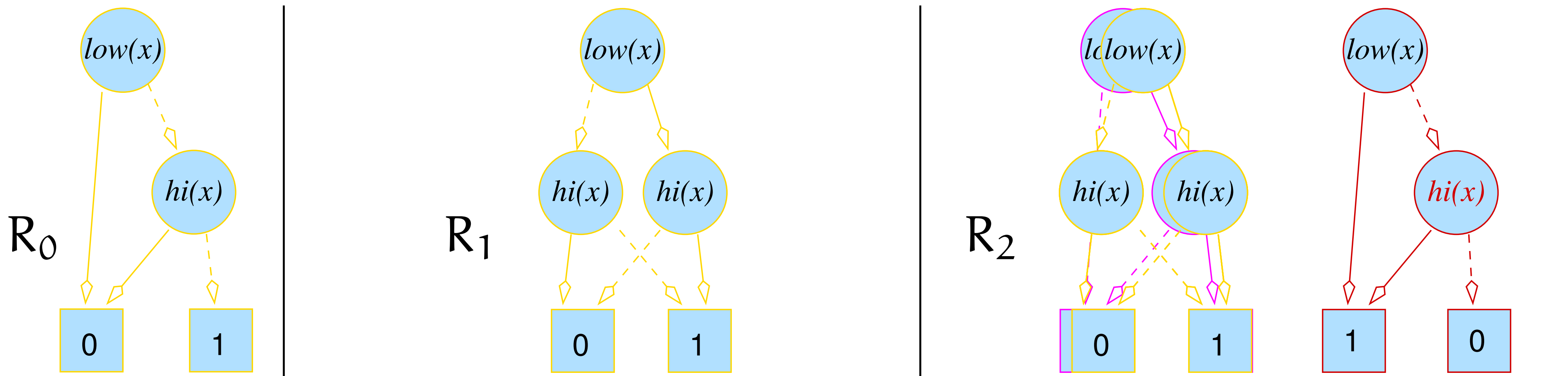
1. For given $i \in \mathbb{N}$ check for satisfiability of
$$\neg\left(\begin{array}{ll} & init(x_0) \wedge trans(x_0, x_1) \wedge \ldots \wedge trans(x_{i-1}, x_i) \\ \Rightarrow & \phi(x_0) \wedge \ldots \wedge \phi(x_i) \end{array}\right).$$
If test succeeds then report violation of goal.
2. Otherwise repeat with larger i.

Algorithms by example

Model: $\mathtt{VAR}\ x : \{0 \ldots 3\};\ \mathtt{INIT}\ x = 0;\ \mathtt{NEXT}\ x := 3 - x$

Conjectured Invar.: $\mathtt{ALWAYS}\ x = 0$

QBF: BDD-based MC



$\overline{l_0} \wedge \overline{h_0} \quad \wedge$ $(l_0 \vee h_0)$	$\overline{l_0} \wedge \overline{h_0} \ \wedge\ l_1 = \overline{l_0} \wedge h_1 = \overline{h_0} \ \wedge$ $(l_0 \vee h_0 \ \vee \quad l_1 \vee h_1)$	$\ldots \ \wedge\ \ldots \ \wedge\ l_2 = \overline{l_1} \wedge h_2 = \overline{h_1} \ \wedge$ $(\ldots \vee \ldots \vee \quad l_2 \vee h_2)$

BF: SAT-based BMC

Comparison

BDD-based model-checking:

- Normalization within each step of graph coloring.

$$\Downarrow$$

1. Keeps size of intermediate representations compact.
2. Detects saturation of graph coloring.

- Tackles ≈ 500 *state bits*

SAT-based model-checking:

- Purely syntactic expansion, followed by satisfiability check.

$$\Downarrow$$

- Size of syntactic expansion grows rapidly. E.g. wrt. number of propositional variables used for characterizing n step reachability:

$$\begin{array}{rrcl} & \text{statebits} & \times & (n+1) \\ + & \underbrace{\text{auxbits}}_{>90\%} & \times & n \end{array}$$

- Tackles $\approx 1.000.000$ *propositions*, most of which are auxiliary.

[Use cases: verification of high-level models w. limited arithmetic.]

Symbolic methods III:

Beyond reachability

The *pre* operator

Observation: Given

- a predicative encoding S of a state set (with free variables $\vec{x}$),
- a predicative encoding T of the transition relation (with free variables $\vec{x}, \vec{x}'$),

the set $pre(S)$ of states that have a successor in (i.e., satisfying) S can be expressed symbolicly using QBF operators:

$$pre(S) = \exists \vec{x}'.T \land S[\vec{x}'/\vec{x}]$$

This can be used for determining all sequential predecessors of a whole set of states in one sweep, thus implementing predecessor colouring “in parallel”.

Symbolic CTL model checking

Using the *pre* operator, CTL model checking can be performed by any QBF engine, e.g. by BDDs:

Formula	Algorithm	Result
propos. P	return $[P]$	Formula f_P denoting P-states
EX ϕ	return *pre*(f_ϕ)	Formula $f_{\mathsf{EX}\phi}$ denoting all states satisfying EX ϕ
EG ϕ	Incrementally build $S_0 = f_\phi$, $S_{i+1} = f_\phi \wedge pre(S_i)$ until $(S_n \iff S_{n+1})$ holds	Formula $f_{\mathsf{EG}\phi} = S_n$ denoting all states satisfying EG ϕ
ϕ EU ψ	Incrementally build $S_0 = f_\psi$, $S_{i+1} = f_\psi \vee (f_\phi \wedge pre(S_i))$ until $(S_n \iff S_{n+1})$ holds	Formula $f_{\phi \mathsf{EU} \psi} = S_n$ denoting all states satisfying ϕ EU ψ

If I characterizes initial states then $I \implies f_\phi$ is to be checked finally.