The Set Partitioning Problem
– using the structure

Jesper Larsen
jla@imm.dtu.dk

Informatics and Mathematical Modelling
Technical University of Denmark
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The Vehicle Routing Problem

The Vehicle Routing Problem is central in distribution management. It is defined as:

- Let $G = (V, A)$ be a graph where $V = \{0, 1, \ldots, n\}$ is the vertex set, and $A = \{(i, j) : i, j \in V, i \neq j\}$ is the set of arcs.
- 0 represents the depot. At the depot we have $m$ identical vehicles. Each vehicle has a capacity of $Q$.
- Associated with each customer (node) $i$ is demand $q_i$ and service time $s_i$.
- Each edge $(i, j)$ has a non-negative cost $c_{ij}$.
The VRP – The objective

The objective of the VRP is to:

- determine $m$ routes, starting and ending at the depot, such that,
- each customer is visited exactly once and
- total demand of any vehicle route does not exceed $Q$, and
- total duration does not exceed an upper limit $D$, and
- the total costs of all routes are minimized.
The VRP – Some remarks

- The VRP is a hard optimization problem which reduces to the Traveling Salesman Problem (TSP) when \( m = 1 \) and both \( Q \) and \( D \) sufficiently large.

- Generally, instances of up to size 100 are currently solvable to optimality.

- Since the first definitions of the problem in the 50’s several exact methods and a huge number of heuristics have been defined for the problem.
Developing the Set Partition Formulation

- In the SPP formulation
  - each of the columns represent a (feasible) route, and
  - each of the rows represent a customer.
- So $a_{ij} = 1$ if customer $i$ is included into route $j$.
- For each subset of customers $S_j$ the cost $c_j$ is determined by solving a TSP over the customers.
SPP – Challenges

Two big issues must be addressed to get success with the SPP formulation:

- Total number of routes can be extremely large. Example: Consider an instance with 40 customers. Assume every route consisting of 7 customers is feasible. This will result in the order of 46 million routes.

- The SPP is NP-hard.
SPP – One approach

In routing applications the solution of the LP relaxation seldom produces a naturally integer solution. Therefore an approach could be:

- First solve the LP relaxation
- Resolve fractionality (Branch-and-Bound)

This approach will never work without the addition of valid inequalities.
SPP – A clever alternative

An alternative is to carefully select a subset of customers from amongst all possible legal duties, in a way so that:

- selected should contain an optimal solution,

AND

- improve the natural integer property.
OUR MISSION

We attempt to construct a small SPP which contains the optimal solution of the full SPP, and which also has a relaxed LP with a significant (better all) variables integer.
Strength of Integrality Conditions

Matrices with an integer optimal vertex

- Perfect matrices

Balanced matrices

- Total unimodular matrices
Unimodular zero-one matrices

- Determinant of all square submatrices of \( A \) is either 0, 1 or \(-1\).
- A square 0-1 matrix is Eulerian if all rows and columns sums are even.
- A zero-one matrix is totally unimodular if it does not contain any Eulerian submatrices with \( \sum \sum a_{ij} \) not divisible by 4.
- Difficult to check for the forbidden submatrices
- Difficult to associate with practical problems
Balanced zero-one matrices

- A zero-one matrix is balanced if it does not contain any odd order 2-cycle submatrices, that is,
- it does not contain an odd order submatrix with row and column sums equal to 2.
The concept of a petal

- A route with petal structure visits all customers in a particular sector with center in the depot.
- Routes of a petal structure radiate from the depot like petals of a flower, and do not overlap.
- The enumeration of all feasible routes with petal structure can be implemented efficiently.
- As the cost we need to compute the length of the corresponding TSP
A spanning petal

- A spanning petal is a collection of petals that contains every customer exactly ones. It can be described by a sub-sequence of deliveries taken from the cyclic order.

- The set of petals 
  \{(9, 10), (11, 12), (6, 7, 8), (4, 5), (13, 1, 2, 3)\} can be described by \{4, 6, 9, 11, 13\}.

- So an optimal set of routes is the same as finding a minimum cost spanning petal.
From petals to generalized petals

- One key observation is that the integral nature of our petals does not depend upon an radial ordering.

- We can reorder our customers in a non-radial cyclic order before applying the petal generation scheme.

- **Example**: use the ordering $3, 7, 5, 8, 10, 6, 4, 2, 9, 1, 12, 13, 11$ and we get.
Optimal Petal Selection

- Solve the LP-relaxation of the SPP
- Use a shortest path reformulation of the problem
OPS by Shortest Path

Petals are represented by a weighted cyclic digraph $H = (N, E)$. $H$ is called the petal graph.

- The nodes $N$ correspond to customers.
- The arcs $E$ corresponds to generalized petals.
- Each generalized petal is represented by an arc from node $i$ to node $j$ where $i$ is the first customer in the route and $j$ the first subsequent customer in the cyclic order.
- The cost of an arc equals the cost of the petal.
OPS and Compact cycles

- So a solution, that is, an OPS, is a cycle in the petal graph.
- So the optimal solution is to find the shortest cycle in the petal graph.
Extending the notion of petals

- The columns we have represent one route – let us call them *1-petals*.

- We can now extend the notion of petals to *r*-petals, where an *r*-petal is a set of *r* routes which collectively service all customers within a given sector centred at the depot.