Revenue Management Under a General Discrete Choice Model of Consumer Behavior

Kalyan Talluri
Department of Economics and Business, Universitat Pompeu Fabra, Ramon Trias Fargas 25-27, 08005 Barcelona, Spain, kalyan.talluri@upf.es

Garrett van Ryzin
Graduate School of Business, Columbia University, New York, New York 10027, gjv1@columbia.edu

Customer choice behavior, such as buy-up and buy-down, is an important phenomenon in a wide range of revenue management contexts. Yet most revenue management methodologies ignore this phenomenon—or at best approximate it in a heuristic way. In this paper, we provide an exact and quite general analysis of this problem. Specifically, we analyze a single-leg reserve management problem in which the buyers’ choice behavior is modeled explicitly. The choice model is very general, simply specifying the probability of purchase for each fare product as a function of the set of fare products offered. The control problem is to decide which subset of fare products to offer at each point in time. We show that the optimal policy for this problem has a quite simple form. Namely, it consists of identifying an ordered family of “efficient” subsets \( S_1, \ldots, S_m \), and at each point in time opening one of these sets \( S_k \), where the optimal index \( k \) is increasing in the remaining capacity \( x \) and decreasing in the remaining time. That is, the more capacity (or less time) available, the further the optimal set is along this sequence. We also show that the optimal policy is a nested allocation policy if and only if the sequence of efficient sets is nested, that is \( S_1 \subseteq S_2 \subseteq \cdots \subseteq S_m \). Moreover, we give a characterization of when nesting by fare order is optimal. We also develop an estimation procedure for this setting based on the expectation-maximization (EM) method that jointly estimates arrival rates and choice model parameters when no-purchase outcomes are unobservable. Numerical results are given to illustrate both the model and estimation procedure.

Key words: yield management; revenue management; discrete choice theory; dynamic programming; maximum likelihood estimation; EM method

History: Accepted by Wallace J. Hopp; received September 13, 2000. This paper was with the authors 24 months for 3 revisions.

1. Introduction and Overview

Yield (or revenue) management is a practice that dates back to the deregulation of the U.S. airline industry in the late 1970s. It was developed as an outgrowth of the need to manage capacity sold at discounted fares, which were targeted to leisure travelers, while simultaneously minimizing the dilution of revenue from business travelers willing and able to pay full fares. Using statistical forecasting techniques and mathematical optimization methods, airlines developed automated systems to dynamically control the availability of the many discounted fares that emerged in the postderegulation era. The practice has since spread beyond airlines to the hospitality, rental car, cruise line, railway, energy, and broadcasting industries. Significant revenue benefits have been documented from such techniques—often an improvement of 2%–8% in revenue over no revenue management or ad hoc, manual controls (Smith et al. 1992).

Concurrent with the evolution of industry practice, a considerable amount of management science literature on yield management has been published over the last 20 years. The earliest work on capacity control was Littlewood’s (1972) analysis of a simple, two-fare-class model of capacity allocations on a single flight leg. The problem with more than two fare products (we define a fare product as a fare—rate or price—along with an associated set of restrictions to qualify for this fare) is considerably more complex, but Belobaba (1987a, b; 1989) developed two simple and effective heuristics for the single-leg problem based on the concept of expected marginal seat revenue (EMSR-a and EMSR-b) that are still in widespread use today.

On a theoretical level, single-leg models, in which demand for each fare product is assumed to occur in nonoverlapping periods, have been developed and analyzed by Brumelle and McGill (1993), Curry (1989), Robinson (1991), and Wollmer (1992). A key
result of this work is that the optimal policy can be implemented using a set of so-called nested allocations. (See Brumelle and McGill 1993, for a precise definition of nested allocations.) Lee and Hersh (1993) introduced and analyzed a discrete-time Markov model that allows for an arbitrary order of arrivals. For further work on single-leg allocation problems, see Brumelle et al. (1990), Kleywegt and Papastavrou (1998), Lautenbacher and Stidham (1999), Liang (1999), Stone and Diamond (1992), Subramanian et al. (1999), and Zhao (1999). For analysis of multiple-leg (network) allocation problems, see Cooper (2000), Curry (1989), Dror et al. (1988), Glover et al. (1982), Simpson (1989), Talluri (1996), Talluri and van Ryzin (1998, 1999), and Williamson (1988, 1992). A recent survey of yield management research is provided by McGill and van Ryzin (1999); Barnhart and Talluri (1996) provide an overview of yield management and other airline operation research areas.

Despite the success of this body of work, most of the above-mentioned models make a common, simplifying—and potentially problematic—assumption: that consumer demand for each of the fare products is completely independent of the controls being applied by the seller. That is, the problem is modeled as one of determining which exogenously arriving requests to accept or reject, and it is assumed that the likelihood of receiving a request for any given fare product does not depend on which other fares are available at the time of the request. However, casual observation—and a brief reflection on one’s own buying behavior—suggests that this is not the case in reality. The likelihood of selling a full-fare ticket may very well depend on whether a discount fare is available at that time; the likelihood that a customer buys at all may depend on the lowest available fare, etc. Clearly, such behavior could have important revenue management consequences and should be considered when making control decisions.

We lay no claim to uncovering this deficiency. Indeed, many researchers have tried to address buy-up (buying a higher fare when lower fares are closed) and buy-down (substituting a lower fare for a high fare when discounts are open) effects in the context of traditional models. Phillips (1994) proposed a “state-contingent” approach to yield management that adjusts controls based on forecasts that depend on the controls in effect (the system “state”) at any point in time. Belobaba (1987a) proposed a correction to the EMSR heuristics that introduces a probability of buying a higher fare when a low fare is closed. While conceptually appealing for a two-fare-class model, such pairwise buy-up probabilities are problematic in a multiple-fare-class setting. The probability of buying a given high fare should depend on which other high fares are also available. Also, one cannot directly observe buy-up, so how does one separate “original” sales from buy-up sales? How are the probabilities (forecasts) adjusted when there are price changes etc.? Despite the methodological difficulties, several airlines have experimented with consumer choice models for revenue management. The most significant research is the work of Andersson (1998) and Algers and Besser (2001), who report a research and development effort at Scandinavian Airline Systems (SAS) to apply logit choice models to estimate buy-up and recapture factors at one of SAS’s hubs.

Another stream of work on understanding choice behavior is the passenger origin and destination simulator (PODS) studies of Belobaba and Hopperstad (1999). PODS is a detailed simulation model of passenger purchase behavior developed by Hopperstad and colleagues at Boeing. It includes factors for airline preference, time preference, path preference, and price sensitivity. While using a very detailed simulation model, the focus of the PODS studies is to test the performance of traditional forecasting and optimization methods under conditions of complex passenger choice behavior, rather than developing new estimation and optimization methods. Nevertheless, the PODS studies have provided many useful insights and clearly demonstrate the significant impact that choice behavior has on the performance of yield management systems.

The only theoretical models and methods that partially address choice behavior issues are dynamic pricing models, such as those studied by Bitran et al. (1998), Feng and Gallego (2000), and Gallego and van Ryzin (1994, 1997). While these models allow demand to depend on the current price (the control in this case), they assume only one product is sold at one price at any point in time. Thus, customers face a binary choice: to buy or not to buy. In reality, firms offer many fares simultaneously and customers choose among them based on price together with their preferences for nonprice factors, such as refundability and whether they can meet various restrictions (e.g., Saturday night stay, minimum stay, and maximum stay). The above dynamic pricing models do not capture this complexity.

In summary, while many attempts have been made to understand the impact of choice behavior on traditional yield management methods and to develop heuristics that partially capture buy-up and buy-down behavior, to date there is no methodology that directly and completely addresses the problem. In this paper, we develop a methodology that we believe substantially fills this void. We analyze a single-leg yield management problem in which we explicitly model consumer choice behavior using a general
choice model, which specifies the probability of purchasing each fare product as a function of the set of available fare products. This general formulation includes most choice models of practical interest.

Given this general model of consumer choice behavior, we then formulate the single-leg, multiple-fare-class yield management problem as one of selecting a subset of fare products to offer at each point in time. We derive optimality conditions for the resulting dynamic program. While the policy might appear to be potentially complex under this model, we show that it has a simple form. First, we show that the optimal subsets can be reduced to an ordered family, \( S_1, \ldots, S_m \), of efficient subsets, which are those sets that provide the most favorable trade-off between total probability of purchase and expected revenue. (see Definition 1 in §3.1). Typically, this family of subsets is much smaller than the number of total possible subsets. The optimal policy then consists of opening one of the efficient sets \( S_k \) in the sequence, where the optimal index \( k \) is increasing in the remaining capacity and decreasing in the remaining time. That is, the more capacity (or less time) we have available, the further the optimal set is along the sequence. Moreover, we show that the optimal policy can be implemented using nested protection levels (a so-called nested allocation policy) if and only if the family of efficient subsets is increasing—that is, \( S_1 \subseteq S_2 \subseteq \ldots \subseteq S_m \). This provides a complete and general characterization of the cases in which nested allocation policies are optimal. We also provide conditions that guarantee the nesting is by fare-class order. We use these conditions to show that for the traditional, independent-demand model, the optimal policy is nested by fare-class order. The same conditions show that for the classical multinomial logic (MNL) choice model, the optimal policy is nested by fare-class order as well.

We also develop a practical estimation procedure for our model. One major difficulty in estimating choice models in the yield management setting is that one typically cannot observe no-purchase decisions. In many industries, sales are conducted remotely and anonymously and the only available data are purchase transactions. Thus, it is often impossible to distinguish between periods with no arrival and periods in which there was an arrival and the arriving customer decided not to purchase. (An exception is when sales are direct, for example, from the firm’s own website, in which case considerable information on no-purchases can potentially be gathered).

We overcome this incomplete data problem by applying the expectation-maximization (EM) method of Dempster et al. (1977) to the traditional maximum-likelihood discrete-choice parameter estimation. The method allows us to simultaneously estimate both the parameters of the choice model and the arrival rates using only transaction data on sales. Together, our estimation procedure and optimization model provide a theoretically sound and quite complete approach to the single-leg problem with choice behavior.

The remainder of the paper is organized as follows: In §2 we define the choice-based model of the problem. In §3 we formulate a dynamic program and analyze the resulting optimal policy, and in §4 we analyze the optimality of nested allocation policies. Section 5 provides a theoretically sound and quite complete characterization of consumer choice behavior under a choice model of consumer behavior. Section 6 describes our EM-based estimation procedure. Finally, some brief numerical examples are given in §6, and our conclusions are given in §7.

2. Model

Time is discrete and indexed by \( t \), and the indices run backwards in time (e.g., smaller values of \( t \) represent later points in time). Time \( t = 0 \) represents the deadline for the sale of the inventory. In each period there is at most one arrival. (This assumption is the same as in Lee and Hershe 1993; see also Gerchak et al. 1985.) The probability of arrival is denoted by \( \lambda \), which we assume is the same for all time periods \( t \). While extending our results to time-varying arrival probabilities is very straightforward, it is cumbersome notationally, so we omit the details to simplify the exposition. There are \( n \) fare products and \( N = \{1, \ldots, n\} \) denotes the entire set of fare products. Each fare Product \( j \in N \) has an associated revenue (fare) \( r_j \), and without loss of generality we index fare products so that \( r_1 \geq r_2 \geq \cdots \geq r_n \geq 0 \).

In each period \( t \), the firm must choose a subset \( S \subseteq N \) of fare products to offer. When the fares \( S \) are offered, the probability that a customer chooses Product \( j \in S \) is denoted \( P_j(S) \) and we assume \( P_j(S) = 0 \) if \( j \not\in S \). (For theoretical reasons, we also allow the choice of the null set, \( S = \emptyset \), which corresponds to not offering any products for sale.) We let \( j = 0 \) denote the no-purchase choice, that is, the event that the customer does not purchase any of the fares offered in \( S \). \( P_0(S) \) denotes the no-purchase probability. It is possible to allow the choice probabilities to be a function of time \( t \) as well, but to keep the notation simple we assume that the probabilities do not depend on time. The probability that a sale of Product \( j \) is made in period \( t \) is therefore \( \Lambda P_j(S) \), and the probability that no sale is made is \( \Lambda P_0(S) + (1 - \Lambda) \). Note this last expression reflects the fact that having no sales in a period could be due either to no arrival at all or an arrival that does not purchase; as mentioned, this leads to an incomplete data problem when estimating the model.

The only conditions we impose on the choice probabilities \( P_j(S) \) is that they define a proper probability function. That is, for every set \( S \subseteq N \), the probabilities satisfy \( P_j(S) \geq 0 \) for all \( j \in S \) and \( \sum_{j \in S} P_j(S) + P_0(S) = 1 \).
1. This includes most choice models of interest. For example, some psychologists have shown that customers can be overwhelmed by more choices, and they may become more reluctant to purchase as more options are offered (see Iyengar and Lepper 2000). Such cases would be covered by a suitable choice of $P_i(S)$ that is decreasing in $S$. It also includes most discrete choice models used in practice, such as those described in Ben-Akiva and Lerman (1985). The only real limitation is that we assume the choices are only a function of the set $S$ of open fares at the time of purchase. In particular, we do not model potential strategic behavior (e.g., when a buyer’s choice depends on the seller’s policy or the strategies of other buyers) or history-dependent choice behavior (e.g., when a buyer’s choice depends on his or her past choices or past events in the system).

We will use the following running example to illustrate the model and analysis:

**Example 1.** An airline offers three fare products, $Y$, $M$, and $K$. These products differ in terms of revenues and conditions, as shown in Table 1. The airline has five segments of customers: two business segments, and three leisure segments. The segments differ in the restrictions that they qualify for and the fares they are willing to pay. The data describing each segment are given in Table 2. The second column of Table 2 gives the probability that an arriving customer is from each given segment.

Given these data for Example 1, Table 3 describes the choice probabilities that would result. To see how the probabilities in Table 3 are derived, consider the set $S = \{Y, K\}$. If $S = \{Y, K\}$ is offered, segments Business 1 and Business 2 buy the $Y$ fare, because they cannot qualify for both the SA stay and 21-day advance restrictions on $K$, so $P_Y = 0.1 + 0.2 = 0.3$. Similarly, Leisure 1 cannot qualify for the SA stay restriction of $K$ and is not willing to purchase $Y$, so these customers do not purchase at all. Leisures 2 and 3, however, qualify for both restrictions on $K$ and purchase $K$. Hence, $P_K = 0.2 + 0.3 = 0.5$. Product $M$ is not offered, so $P_M = 0$. The other rows of Table 3 are filled out similarly.

Again, this particular method of generating choice probabilities is only for illustration. Other choice models could be used, and in general any proper set of probabilities could be used to populate Table 3.

### 3. Optimization

We next formulate a single-leg problem based on this general choice model. Let $C$ denote the aircraft capacity, $T$ the number of time periods, $t$ the number of remaining periods (recall time is indexed backwards), and $x$ the number of remaining inventory units. Define the value function $V_t(x)$ as the maximum expected revenue obtainable from periods $t, t-1, \ldots, 1$, given that there are $x$ inventory units remaining at time $t$. Then, the Bellman equation for $V_t(x)$ is

$$V_t(x) = \max_{S \subseteq N} \left\{ \sum_{j \in S} AP_j(S)(r_j + V_{t-1}(x-1)) \right\} + \lambda(V_{t-1}(x) - \Delta V_{t-1}(x)\right\} + V_{t-1}(x), \quad (1)$$

where $\Delta V_{t-1}(x) = V_{t-1}(x) - V_{t-1}(x-1)$ denotes the marginal cost of capacity, and we have used the fact that for all $S$,

$$\sum_{j \in S} P_j(S) + P_0(S) = 1.$$

The boundary conditions are

$$V_t(0) = 0, \quad t = 1, \ldots, T \quad \text{and} \quad V_0(x) = 0, \quad x = 1, \ldots, C. \quad (2)$$

We can write (1) in more compact form as

$$V_t(x) = \max_{S \subseteq N} \left\{ \lambda(R(S) - Q(S)\Delta V_{t-1}(x))\right\} + V_{t-1}(x), \quad (3)$$

---

**Table 1**

<table>
<thead>
<tr>
<th>Fare product (class)</th>
<th>SA stay</th>
<th>21-day adv.</th>
<th>Revenue ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$</td>
<td>No</td>
<td>No</td>
<td>800</td>
</tr>
<tr>
<td>$M$</td>
<td>No</td>
<td>Yes</td>
<td>500</td>
</tr>
<tr>
<td>$K$</td>
<td>Yes</td>
<td>Yes</td>
<td>450</td>
</tr>
</tbody>
</table>

**Table 2**

<table>
<thead>
<tr>
<th>Segment</th>
<th>Prob.</th>
<th>Qualifies for restrictions?</th>
<th>Willing to buy?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$SA$ stay</td>
<td>21-day adv.</td>
</tr>
<tr>
<td>Bus. 1</td>
<td>0.1</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Bus. 2</td>
<td>0.2</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Leis. 1</td>
<td>0.2</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Leis. 2</td>
<td>0.2</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Leis. 3</td>
<td>0.3</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

**Table 3**

<table>
<thead>
<tr>
<th>Segment</th>
<th>Qualifies for restrictions?</th>
<th>Willing to buy?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>$M$</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$K$</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$S$</th>
<th>$P_i(S)$</th>
<th>$P_m(S)$</th>
<th>$P_k(S)$</th>
<th>$Q(S)$</th>
<th>$R(S)$</th>
<th>Efficient?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>Yes</td>
</tr>
<tr>
<td>${Y}$</td>
<td>0.3</td>
<td>0</td>
<td>0</td>
<td>0.7</td>
<td>0.3</td>
<td>240</td>
</tr>
<tr>
<td>${M}$</td>
<td>0</td>
<td>0.4</td>
<td>0</td>
<td>0.6</td>
<td>0.4</td>
<td>200</td>
</tr>
<tr>
<td>${K}$</td>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>225</td>
</tr>
<tr>
<td>${Y, M}$</td>
<td>0.1</td>
<td>0.6</td>
<td>0</td>
<td>0.3</td>
<td>0.7</td>
<td>380</td>
</tr>
<tr>
<td>${Y, K}$</td>
<td>0.3</td>
<td>0</td>
<td>0.5</td>
<td>0.2</td>
<td>0.8</td>
<td>465</td>
</tr>
<tr>
<td>${M, K}$</td>
<td>0</td>
<td>0.4</td>
<td>0.5</td>
<td>0.1</td>
<td>0.9</td>
<td>425</td>
</tr>
<tr>
<td>${Y, M, K}$</td>
<td>0.1</td>
<td>0</td>
<td>0.4</td>
<td>0.5</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
where

\[ Q(S) = \sum_{j \in S} P_j(S) = 1 - P_S(S) \]

denotes the total probability of purchase and

\[ R(S) = \sum_{j \in S} P_j(S)r_j \]

denotes the total expected revenue from offering set \( S \). Table 3 gives the values \( Q(S) \) and \( R(S) \) for Example 1.

A sequence of sets achieving the maximum in (3) forms an optimal Markovian policy (cf. Bellman 1957 and Bertsekas 1995). For theoretical purposes, we will also consider allowing the seller to randomize over the sets \( S \) that are offered at the beginning of each time period. Because the number of subsets is finite, there is always one set \( S \) that maximizes \( \lambda(R(S) - Q(S)\Delta V_{t-1}(x)) \) (there may be ties or course), so randomizing among the sets provides no additional benefit to the seller. However, allowing this flexibility in policies is theoretically useful.

3.1. Efficient Sets

Potentially, each optimization on the right-hand side of (3) could require an evaluation of all \( 2^n \) subsets. However, we show next that the search can be reduced to an evaluation of only efficient sets. These sets are defined as follows:

**Definition 1.** A set \( T \) is inefficient if there exist probabilities \( \alpha(S), \forall S \subseteq N \) (including the null set \( S = \emptyset \)) with \( \sum_{S \subseteq N} \alpha(S) = 1 \) such that either

\[ Q(T) \geq \sum_{S \subseteq N} \alpha(S)Q(S) \quad \text{and} \quad R(T) < \sum_{S \subseteq N} \alpha(S)R(S). \]

Otherwise, the set \( T \) is efficient.

In words, a set \( T \) is inefficient if we can use a randomization of other sets \( S \) to produce an expected revenue that is strictly greater than \( R(T) \) with no increase in the probability of purchase \( Q(T) \). For Example 1, Table 3 shows which sets are efficient, namely \{Y\}, \{Y, K\}, and \{Y, K, M\}. That these sets are efficient follows from inspection of Figure 1, which shows a scatter plot of the value \( Q(S) \) and \( R(S) \) for all subsets \( S \). Note from this figure and Definition 1 that an efficient set is a point that is on the “efficient frontier” of the set of points \( \{Q(S), R(S)\}, S \subseteq N \), a concept that should be natural to those readers familiar with mean-variance portfolio theory or data envelopment analysis (DEA). Here, “efficiency” is with respect to the trade-off between expected revenue, \( R(S) \), and probability of sale, \( Q(S) \). A very similar notion of efficiency is also developed by Crew and Kleindorfer (1976) in the context of peak load pricing.

The following lemma follows from standard parametric linear programming results (see Bertsimas and Tsitsiklis 1997, Theorem 5.1, p. 213).

**Lemma 1.** The efficient frontier \( \tilde{R}: [0, 1] \rightarrow \mathbb{R} \), defined by

\[ \tilde{R}(q) = \max \left\{ \sum_{S \subseteq N} \alpha(S)R(S) : \sum_{S \subseteq N} \alpha(S)Q(S) \leq q, \sum_{S \subseteq N} \alpha(S) = 1, \alpha(S) \geq 0 \forall S \subseteq N \right\}, \]

is increasing concave in \( q \).

It is also easy to show (see Figure 1) that all efficient sets lie on the efficient frontier \( \tilde{R}(q) \), while all inefficient sets lie strictly below it. Indeed, an alternative characterization of an efficient set, which is useful analytically, is given by the following proposition (see the appendix for a proof, which follows easily from Lemma 1):

**Proposition 1.** A set \( T \) is efficient if and only if, for some value \( v \geq 0 \), \( T \) is an optimal solution to

\[ \max_{S \subseteq N} \left\{ R(S) - vQ(S) \right\}. \]

The significance of inefficient sets is that they can be eliminated from consideration in our optimization problem, as shown by the following proposition, which follows directly from (3), Proposition 1, and the fact that \( \Delta V_{t-1}(x) \geq 0 \). This later fact follows from first principles and also from Lemma 5 below.

**Proposition 2.** An inefficient set is never an optimal solution to (1).

3.2. Characterization of the Optimal Policy

We next show that the efficient sets are used in a quite simple order. Indeed, let \( m \) denote the number of efficient sets. These sets can be indexed \( S_1, \ldots, S_m \) such that both the revenues and probabilities of purchase are monotone increasing in the index. The result follows directly from Lemma 1:
Proposition 3. If the collection of m efficient sets is indexed such that $Q(S_1) \leq Q(S_2) \leq \cdots \leq Q(S_m)$, then $R(S_1) \leq R(S_2) \leq \cdots \leq R(S_m)$ as well.

For Example 1, we see from Table 3 that there are $m = 3$ efficient sets $\{Y\}$, $\{Y, K\}$, and $\{Y, K, M\}$. These can be ordered $S_1 = \{Y\}$, $S_2 = \{Y, K\}$, and $S_3 = \{Y, K, M\}$, with associated probabilities of purchase $Q_1 = 0.3$, $Q_2 = 0.8$, and $Q_3 = 1$ and revenue $R_1 = $420, $R_2 = $465, and $R_3 = $505 as claimed.

Henceforth, we assume the efficient sets are denoted $S_1, \ldots, S_m$ and are indexed in increasing order of revenue and probability. Also, to keep notation simpler, we let $R_k = R(S_k)$ and $Q_k = Q(S_k)$, and note $R_k$ and $Q_k$ are both increasing in $k$. So the Bellman equation can be further simplified to

$$V_t(x) = \max_{k=1, \ldots, m} \left\{ (R_k - Q_k e^{UP \Delta V_t}) + V_{t-1}(x) \right\}. \tag{4}$$

We show next that the optimal policy has a quite simple form when expressed in terms of the sequence $S_1, \ldots, S_m$ of efficient sets. We first need some preliminary lemmas.

Lemma 2. Let $l > k$ be two indices of efficient sets and suppose that $R_l - Q_l v_0 \geq R_k - Q_k v_0$ for some nonnegative $v_0$. Then, $R_l - Q_l v \geq R_k - Q_k v$ for all $v \leq v_0$.

Proof. Restating the condition, we want to show that if

$$R_l - R_k \geq (Q_l - Q_k) v$$

holds for $v = v_0$, it holds for all $v \leq v_0$, but this follows trivially from the fact that if $l > k$, $R_l - R_k \geq 0$ and $Q_l - Q_k \geq 0$, so the RHS above is nondecreasing in $v$. \hfill $\square$

The next lemma shows that the index of the optimal efficient set in (4) is decreasing in the value $\Delta V_{t-1}(x)$.

Lemma 3. Let $k^*$ denote the index of the efficient set $S_{k^*}$ that maximizes (4) (or greatest such index if more than one efficient set maximizes (4)). Then, $k^*$ is decreasing in $\Delta V_{t-1}(x)$.

Proof. The proof is by contradiction. For ease of notation, let $v = \Delta V_{t-1}(x)$ and let $0 \leq v_1 < v_2$ denote any two values of the marginal capacity. Let $k_i$ be the largest index that solves $\max_i [R_i - Q_i v_i]$ for $i = 1, 2$. Then, by the optimality of $k_i$, we have

$$R_{k_i} - Q_{k_i} v_i \geq R_{k_i} - Q_{k_i} v_{k_i}.$$

Now suppose $k_i > k_j$. Then, because $v_1 < v_2$, by Lemma 2, we would have that

$$R_{k_i} - Q_{k_i} v_i \geq R_{k_j} - Q_{k_j} v_j,$$

which means $k_2$ maximizes $R_k - Q_k v_1$ as well. But by definition, $k_1$ is the largest such maximizer, which contradicts the fact that $k_2 > k_1$; hence, we must have $k_1 > k_2$. \hfill $\square$

Our next lemma shows that the marginal value is decreasing in the remaining capacity (see the appendix for a proof).

Lemma 4. $\Delta V_t(x) \leq \Delta V_t(x - 1)$, $t = 1, \ldots, T$, $x = 1, \ldots, C$.

Thus, marginal values are decreasing, which is intuitive. (Nonmonotonicity of the marginal values could occur if there is demand for multiple inventory units (group requests); see, for example, Kleywegt and Papastavrou (1998), Lee and Hersh (1993), and Young and Van Slyke (1994).

The marginal values are increasing in the remaining time as well (see the appendix for a proof).

Lemma 5. $\Delta V_t(x) \geq \Delta V_{t-1}(x)$, $t = 1, \ldots, T$, $x = 1, \ldots, C$.

By combining Lemmas 3, 4, and 5, we obtain our main theorem:

Theorem 1. An optimal policy for (1) is to select a set $k^*$ from among the $m$ efficient, ordered sets $\{S_k : k = 1, \ldots, m\}$ that maximizes (4). Moreover, for a fixed $t$, the largest optimal index $k^*$ is increasing in the remaining capacity $x$, and for any fixed $x$, $k^*$ is decreasing in the remaining time $t$.

This characterization is significant for several reasons. First, it shows that the optimal sets can be reduced to only those that are efficient, which in many cases significantly reduces the number of sets we need to consider. Moreover, it shows that this limited number of sets can be sequenced in a natural way and that the more capacity we have (or the less time remaining), the higher the set we should use in this sequence. For example, applying Theorem 1 to Example 1, we see that the efficient sets $S_1 = \{Y\}$, $S_2 = \{Y, K\}$, and $S_3 = \{Y, K, M\}$ would be used as follows: With very large amounts of capacity remaining, $S_3$ is optimal—that is, all three fare products are opened. As capacity is consumed, at some point we switch to offering only $S_2$—that is, class $M$ is closed and only $Y$ and $K$ are offered. As capacity is reduced further, at some point we close product $M$ and only offer class $Y$ (i.e., set $S_1$ is used).

3.3. Identifying Efficient Sets
Finding the efficient sets is, in general, computationally complex. The general approach is to enumerate all $2^n - 1$ subsets of $N$. For each set $T$, the direct approach for testing efficiency is to solve the linear program (in variables $\alpha(S), S \subseteq N$)

$$\max \sum_{S \subseteq N} \alpha(S) R(S)$$

$$\sum_{S \subseteq N} \alpha(S) Q(S) \leq \alpha(T),$$

$$\sum_{S \subseteq N} \alpha(S) = 1,$$

$$\alpha(S) \geq 0, \quad S \subseteq N.$$
If the optimal objective function value is strictly greater than \( R(T) \), then the set \( T \) is inefficient and can be removed. However, even for moderate \( n \) this is a large LP.

A more efficient alternative to this LP approach is to use the following “largest marginal revenue” procedure: First, let \( S_0 = \emptyset \). Then, successive sets can be found by the following recursion: Let \( S_i \) be the \( i \)th efficient set. Then the \((i+1)\)st efficient set, \( S_{i+1} \), is found by checking among the sets \( S \) with \( Q(S) \geq Q(S_i) \) and \( R(S) \geq R(S_i) \) for the one that maximizes the marginal revenue ratio

\[
\frac{R(S) - R(S_i)}{Q(S) - Q(S_i)}.
\]

(Note: This is simply the increase in expected revenue per unit increase in expected demand.) The procedure starts with \( i = 0 \) and stops once no sets \( S \) with \( Q(S) \geq Q(S_i) \) and \( R(S) \geq R(S_i) \) exist. This procedure returns the complete sequence \( S_1, \ldots, S_m \). Because there are \( O(2^m) \) subsets to check at each step, the recursion has complexity \( O(m^2) \), where \( m \) is the number of efficient sets (which in the worst case could be \( O(2^n) \) itself).

For small numbers of products, this largest marginal revenue procedure is practical, especially because it can be performed off line. But it is still exponential in the number of products \( n \). For large numbers of products, heuristic or analytic approaches could be used to reduce the complexity of identifying efficient sets. For example, one could enumerate a limited collection of subsets \( S \) rather than all \( 2^n - 1 \) subsets and apply the largest marginal revenue procedure to determine which subsets in the collection are efficient relative to other sets in the collection. In some special cases, as shown below, one can identify which subsets are efficient analytically, thus eliminating the need to enumerate all possible subsets.

### 4. Optimality of Nested Allocation Policies

The optimization results above have significant implications for the optimality of nested allocation policies. The notion of efficiency from Definition 1 and Theorem 1 can be used to provide a quite complete characterization of cases in which nested allocation policies are optimal. They also can be used to provide conditions under which the optimal nesting is by fare order.

#### 4.1. General Nested Policies

To begin, we first need to precisely define a nested allocation policy.

**Definition 2.** A control policy for (1) is called a nested policy if there is an increasing family of subsets \( S_1 \subseteq S_2 \subseteq \cdots \subseteq S_m \) and an index \( k_i(x) \) that is increasing in \( x \), such that set \( S_{k_i(x)} \) is chosen at time \( t \) when the remaining capacity is \( x \).

Though this is a somewhat abstract definition of a nested policy, it is in fact the natural generalization of nested allocations from the traditional single-leg models. In particular, it implies an ordering of the products based on when they first appear in the increasing sequence of sets \( S_i \). That is, Product \( i \) is considered “higher” than Product \( j \) in the nesting order if Product \( i \) appears earlier in the sequence. Returning to Example 1, we see that the efficient sets are indeed nested according to this definition because \( S_1 = \{Y\} \), \( S_2 = \{Y, K\} \), and \( S_3 = \{Y, K, M\} \) are increasing. Class \( Y \) would be considered the highest in the nested order, followed by class \( K \), and then class \( M \).

We will say a policy is nested by fare order if the nesting order is the same as the fare order. Note that in Example 1 the nested order is not the fare order, because \( M \) is the lowest-ranked fare product in nested order but \( K \) has the lowest revenue.

If the optimal policy is nested, one can use a nested allocation policy to implement it. A nested allocation policy is defined as follows: For each set \( S_k \) we can define a set of nested protection levels, \( p_k, k = 1, \ldots, m \), with \( p_1 \leq p_2 \leq \cdots \leq p_m \) such that products lower than \( S_k \) are closed if the remaining capacity is less than \( p_k \). The protection levels are defined by

\[
p_k = \max\{x: R_k - Q_k \Delta V_{i-1}(x) > R_k - Q_k \Delta V_{i-1}(x)\},
\]

where for notational convenience we define \( p_0 = 0 \) and \( p_m = C \). The set \( S_k \) should be used if and only if \( p_k \geq x > p_{k-1} \), and the products \( S_m - S_k \) should be closed if \( x \leq p_k \). That is, there is a critical threshold of capacity below which we close off fares in \( S_m - S_k \).

We can also define nested booking limits for Product \( i \), \( b_i \), as follows: Let \( k(i) \) denote the index of the first set in which Product \( i \) appears in the sequence of efficient sets. Then, the booking limit is

\[
b_i = C - p_{k(i)-1}, \quad i = 1, \ldots, n,
\]

where \( p_{k(i)-1} \) is the protection limit for the set \( S_{k(i)-1} \).

We again return to Example 1 to illustrate this concept. Table 4 shows the objective function value \( R_k - Q_k \Delta V_{i-1}(x) \) for each of the three efficient sets \( k = 1, 2, 3 \) (recall \( S_1 = \{Y\}, S_2 = \{Y, K\}, \) and \( S_3 = \{Y, K, M\} \) for a particular marginal value function \( \Delta V_{i-1}(x) \) (the marginal value function is assumed given here). Capacities are in the range \( x = 1, 2, \ldots, 20 \). The last column of Table 4 gives the index, \( k_i(x) \), of the efficient set that is optimal for each capacity \( x \). Note

---

1 This idea was suggested to us by Guillermo Gallego and Bob Phillips.
that for capacities 1, 2, and 3, the set \( S_1 = \{ Y \} \) is the optimal set, so class \( Y \) is the only open fare. Once we reach four units of remaining capacity, set \( S_2 = \{ Y, K \} \) becomes optimal and we open class \( K \) in addition to class \( Y \). When the remaining capacity reaches 13, set \( S_3 = \{ Y, K, M \} \) becomes optimal, and we open \( M \) in addition to \( Y \) and \( K \). As a result, the protection level for set \( S_1 \) is \( p_1 = 3 \), and the protection level for set \( S_2 \) is \( p_2 = 12 \). (\( S_3 \) has a protection level equal to capacity, \( p_3 = C \), by definition.) Assuming the capacity is \( C = 20 \), the booking limit for \( M \) is \( b_1 = C - p_2 = 20 - 12 = 8 \); the booking limit for \( K \) is \( b_2 = C - p_1 = 20 - 3 = 17 \); and the booking limit for \( Y \) is \( b_3 = C - p_3 = C - 0 = C \).

It is worth emphasizing at this point that the sequence of efficient sets (and whether they are nested, and if so their nesting order) are all a function of the particular choice model and the product revenue values. So if these inputs change, the efficient sets could certainly change. Hence, if consumer behavior or revenues change, one needs to recompute the efficient sets. Indeed, while thus far we have assumed the model parameters are the same for all times \( t \), nothing in the analysis prevents us from allowing different choice models and different revenues at each time \( t \)—i.e., a choice model \( P_j(t, S) \) and revenues \( r_j(t) \) that vary with \( t \). They only difference in this time-varying case is that the efficient sets could change with time—and if the sets are nested, the nesting order could change as well. That is, for each \( t \) we could potentially have a different sequence of efficient sets. However, within each period \( t \) the first part of Theorem 1 still holds; the optimal index \( k^* \) will still be monotone in the remaining capacity \( x \). (Both Lemmas 4 and 5 also continue to hold in the time-varying case.) The fact that the efficient sets could change over time would clearly make the implementation more complex, but the basic structure of the optimal policy within each period remains the same.

### 4.2. Nesting by Fare Order

Fares provide a natural nesting ordering, and traditionally this is how most revenue management systems have been conceived and implemented. From a practical standpoint, therefore, it is important to understand when a particular choice model leads to nesting by fare order. Yet, Example 1 makes clear that nesting by fare order need not be the optimal policy in general. Some choice models have this property; others do not. Next, we derive conditions that guarantee a given model will always have this property.

Recall that the set of products \( N = \{ 1, \ldots, n \} \) is assumed to be indexed so that \( r_1 \geq r_2 \geq \cdots \geq r_n \geq 0 \). We will say a set is *complete* if it is of the form \( A_k = \{ 1, \ldots, k \} \) for some \( k \) and *incomplete* otherwise. If \( A_k \) is used, then it means that Product \( k \) and all products with revenues higher than \( k \) are offered for sale.

As an aside, note that completeness of the efficient sets is an important property if one wants to use a *bid price control*. In bid price controls, we set a threshold price that can depend on \( t \) and \( x \), denoted \( \pi(t, x) \), such that \( j \) is open for sale if and only if \( r_j \geq \pi(t, x) \). Indeed, the following proposition is easily seen to hold:

**Proposition 4.** A bid price policy is optimal if and only if the efficient sets \( S_1, \ldots, S_m \) are complete.

In other words, if the efficient sets are complete, then a bid price control can be used to implement it because a threshold price can be used to separate the fares in \( A_k = \{ 1, \ldots, k \} \) from those in \( N - A_k = \{ k + 1, \ldots, n \} \). If some efficient sets are incomplete, this simple procedure fails.

We next examine the implications of complete sets for nesting by fare order. Recall that Proposition 1 (slightly rearranged) states that for any value \( \nu > 0 \), efficient sets are the only solutions to

\[
\max_{S \subseteq N} \sum_{j=1}^{n} (r_j - \nu) P_j(S).
\]

Let \( x_t = r_t - \nu \) and note that \( x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \) if \( r_1 \geq r_2 \geq \cdots \geq r_n \geq 0 \). The following is a definition of the nested-by-fare-order property:

**Definition 3.** A choice model \( P_j(S), j = 1, \ldots, n \), \( S \subseteq N \), has the nesting-by-fare-order property if it satisfies two properties:

(i) The probability of purchase, \( Q(S) = \sum_{j \in S} P_j(S) \), is increasing in \( S \), that is, \( Q(T) \geq Q(S) \) if \( S \subseteq T \).
set of weights in Table 6. is available. Then, the resulting mixture probability will be as given both 1 and 2 are available and fare Product 1 if only fare Product 1 Type B can afford only Product 3. Type A will buy fare Product 2 if of Type A is eligible for fare Products 1 and 2 only and customer models for which the nested-by-fare-order property 2 Thisspecificchoicemodelmayappearodd, but it canariseifthere order property does not hold for this choice model.2

Thus, by Theorem 1, the optimal policy will be nested by fare order.

A choice model will fail to have the nested-by-fare-class property if there is a vector of fares r that makes an incomplete set efficient. Here is a simple example where this is the case:

**Example 2.** Let N = {1, 2, 3}, r1 = $410, r2 = $110, and r3 = $60 and take v = 10 so that x1 = $400, x2 = $100, and x3 = $50. Let the choice probabilities conditioned on the choice set be given as in Table 5. Table 5 shows that the set that maximizes \( \sum x_j P_j(S) \) is \{1, 3\}, an incomplete set. Thus, the nesting-by-fare-order property does not hold for this choice model.

The next theorem gives a characterization of choice models for which the nested-by-fare-order property holds (see the appendix for a proof):

**Theorem 2.** A choice model has the nested-by-fare-order property if the following two conditions hold:

(i) The probability of purchase, Q(S), is increasing in S.

(ii) For every incomplete set T, there exists a convex set of weights \( \alpha_j, j = 1, \ldots, n \), satisfying \( \alpha_j \geq 0 \) and \( \sum_{j=1}^n \alpha_j = 1 \), such that the probabilities defined by

\[
\overline{P}_j(\alpha) = \sum_{k=1}^n \alpha_k P_k(A_k), \quad j = 1, \ldots, n,
\]

where \( A_k = \{1, \ldots, k\} \) is the kth complete set, satisfy

\[
\sum_{j=1}^i \overline{P}_j(\alpha) \geq \sum_{j=1}^i P_j(T), \quad i = 1, \ldots, n - 1
\]

and

\[
\sum_{j=1}^n \overline{P}_j(\alpha) = \sum_{j=1}^n P_j(T).
\]

Readers familiar with the theory of majorization (see Marshall and Olkin 1979) will note that the above conditions are equivalent to saying that the vector \((\overline{P}_1(\alpha), \ldots, \overline{P}_n(\alpha))\) majorizes the vector \((P_1(T), \ldots, P_n(T))\). In words, it says that the nested-by-fare-order property holds if we can find a convex combination of complete sets that has the same probability of purchase as the incomplete set \(T\) (i.e., \( \sum_{j=1}^n \overline{P}_j(\alpha) = \sum_{j=1}^n P_j(T) = Q(T) \)). Yet, the convex combination produces at least as high a probability of purchasing each of the nested sets \{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n - 1\}. This property is sufficient to ensure that the expected revenue is at least as large, because it implies that for all \( i = 1, \ldots, n \),

\[
P_i(T) - \overline{P}_i(\alpha) \leq \sum_{j=1}^{i-1} \overline{P}_j(\alpha) - \sum_{j=1}^{i-1} P_j(T).
\]

So, if we use \( T \) rather than the convex combination of complete set, then any increase in the probability of selling Product \( i \) (the left-hand side) is always less than the loss in probability of selling one of the higher-revenue Products \( 1, \ldots, i - 1 \) (the right-hand side). Stated another way, if we sell less of Product \( i \) using the convex combination, it is only because we are selling more of products with even higher revenues. Thus, the convex combination produces at least as much expected revenue and for the same probability of purchase.

Although verification of the conditions of the theorem in general involves testing all incomplete subsets \( T \), verifying it for a specific model or functional form is often simpler. Below we apply the basic theory to three special cases—the traditional independent demand model, the multinomial logit model, and a model where a customer purchases the lowest open fare—and show that the optimal policy in each case is nested by fare order.

Finally, note that while Theorem 2 is stated in terms of the family of complete sets \( A_1, A_2, \ldots, A_n \), the proof does not rely on the fact that the sets are complete. Indeed, the theorem holds for any specified family of sets. In particular, if one conjectures that the efficient set always comes from a particular family, one can use Theorem 2 to try to verify this conjecture.
4.3. Independent Demand Model

In the independent demand model, customers arrive at random and only want one of \( n \) available fare products. This is the traditional yield management model of Lee and Hersh (1993) and corresponds to the following choice model in our formulation:

\[
P_j(S) = \begin{cases} 
q_j, & j \in S \\
0, & \text{otherwise}
\end{cases} \quad \forall S \subseteq N.
\]

That is, an arriving customer chooses Product \( j \) with probability \( q_j \) independent of the choice set offered, and if \( j \) is not offered, the customer does not purchase. We will show:

**Proposition 5.** For the independent model, the only efficient sets are the complete sets \( A_k, k = 1, \ldots, n \). Moreover, the optimal policy is a nested allocation policy where the nesting is by fare order.

**Proof.** First, it is clear that the independent demand model is increasing, because \( Q(S) = \sum_{i \in S} q_i \). Verifying Condition (ii) of Theorem 2 is also quite easy: Let \( T \) be an incomplete set and define the index \( k^* \) by

\[
\sum_{i=1}^{k^*} q_i \leq \sum_{i \in T} q_i < \sum_{i=1}^{k^*+1} q_i.
\]

Note that \( k^* \) is less than the largest index in \( T \), that is, \( \max\{j : j \in T\} > k^* \) (otherwise, \( T = A_k \), and it is complete). Next, define \( \lambda \) by

\[
\lambda \sum_{i=1}^{k^*} q_i + (1 - \lambda) \sum_{i \in T} q_i = \sum_{i=1}^{k^*+1} q_i
\]

and define the convex weights

\[
\alpha_k = \begin{cases} 
\lambda, & k = k^*, \\
1 - \lambda, & k = k^* + 1, \\
0, & \text{otherwise}.
\end{cases}
\]

Using these weights, we have that for \( j \leq k^* \),

\[
\bar{P}_j(\alpha) = \lambda P_j(A_{k^*}) + (1 - \lambda) P_j(A_{k^*+1}) = \lambda q_j + (1 - \lambda) q_{k^*+1} = q_j \geq P_j(T).
\]

Summing for all values \( i \leq j \), we obtain that for \( j \leq k^* \), \( \sum_{i=1}^{j} \bar{P}_i(\alpha) \geq \sum_{i=1}^{j} P_i(T) \) as required by Theorem 2. For \( j > k^* \), we have by (6)

\[
\sum_{i=1}^{j} \bar{P}_i(\alpha) = \sum_{i \in T} q_i \geq \sum_{i=1}^{j} P_i(T),
\]

with equality holding throughout when \( j = n \). This completes the proof of Condition (ii) of Theorem 2. \( \square \)

Note that one can also show that the complete sets \( A_k \) are the only efficient sets directly without using Theorem 2. Indeed, by Proposition 1, the only efficient sets are those that maximize \( R(S) - vQ(S) \) over \( S \subseteq N \) for some nonnegative \( v \). For the independent model,

\[
R(S) = \sum_{j \in S} r_j q_j \quad \text{and} \quad Q(S) = \sum_{j \in S} q_j,
\]

so this optimization can be written as

\[
\max \sum_{j=1}^{n} (r_j - v) q_j z_j, \quad z_j \in \{0, 1\}, \quad j = 1, \ldots, n.
\]

An optimal solution to this problem is easily seen to be

\[
z_j^* = \begin{cases} 
1, & r_j > v, \\
0, & \text{otherwise}.
\end{cases}
\]

Because \( r_1 \geq r_2 \geq \cdots \geq r_n \), this solution corresponds to the set \( A_k = \{1, 2, \ldots, k\} \), where \( k \) is the largest value \( j \) such that \( r_j > v \).

As a result of the nested-by-fare-order property, we recover the Lee and Hersh (1993) result as we should. But more importantly, we see exactly why the independent choice model leads to an optimal policy of this form.

4.4. Multinomial Logit Choice Model

We now turn to a slightly more complex choice model—the multinomial logit (MNL) model. The MNL model is used widely in travel demand forecasting and marketing (see Ben-Akiva and Lerman 1985). Still, the MNL model has certain deficiencies in representing choice among alternatives with shared attributes (the independence from irrelevant alternatives (IIA) property—or “red-bus, blue-bus” problem) and hence should be used with caution.

In the MNL model, consumers are utility maximizers and the utility of each choice is a random variable. Formally, the utility of each alternative \( j \) is assumed to be of the form

\[
U_j = u_j + \xi_j,
\]

where \( u_j \) is the mean utility of choice \( j \) and \( \xi_j \) is an i.i.d., Gumbel random noise term with mean zero and scale parameter one for all \( j \). (The Gumbel, or double-exponential, distribution with zero mean and scale parameter is \( F(x) = \exp(-e^{-(x+y)}) \), \( x \geq 0 \), where \( y \approx 0.577 \) is Euler’s constant.) Because utility is an ordinal measure, the assumption of zero mean and a scale parameter of one are without loss of generality (see Ben-Akiva and Lerman 1985). Similarly, there is a no-purchase option where the no-purchase utility is assumed to be

\[
U_0 = u_0 + \xi_0.
\]
where and $\xi_0$ is also Gumbel with mean zero and scale parameter one. Again, because utility is ordinal, without loss of generality we can assume $u_0 = 0$.

Under this utility model, one can show (see Ben-Akiva and Lerman 1985 for a derivation) that the choice probabilities are given by

$$P_j(S) = \frac{\sum_{i \in S} e^{vi}}{\sum_{i \in S} e^{vi} + e^{u_j}}, \quad j \in S \text{ or } j = 0 \quad (7)$$

and zero otherwise. For notational convenience, we define “weights” $w_j = e^{u_j}$, $j = 0, 1, \ldots, n$, so that the choice probabilities can be expressed as

$$P_j(S) = \frac{w_j}{\sum_{i \in S} w_i + 1}, \quad j \in S \text{ or } j = 0. \quad (8)$$

That is, the probability of choosing $j$ is its “weight” $w_j$ divided by the weights of all other choices in $S$. (Recall that $u_0 = 0$ so $w_0 = 1$.) Note that because $e^x$ is monotone increasing in $x$, higher values of $u_j$ imply higher values of $w_j$.

We next show that for this choice model, the optimal policy is again nested by fare order (the proof is similar to the independent demand model, albeit more involved, and is provided in the appendix):

**Proposition 6.** For the MNL choice model, the only efficient sets are the complete sets $A_k$, $k = 1, \ldots, n$. Moreover, the optimal policy is a nested allocation policy where the nesting is by fare order.

Thus, for the MNL model, a nested allocation policy where the nesting is by fare order is again optimal. However, the nested allocations are not necessarily the same as in the independent model.

### 4.5. Lowest Open Fare Model

In this third model of customer purchase behavior, the seller offers a set of products and the buyer only considers the lowest priced product and the no-purchase option. Assume a fixed probability of purchase $q_j$, if $j$ is the lowest fare offered, and $q_1 \leq q_2 \leq \cdots \leq q_n$. These probabilities are assumed to be independent of the offer set. This model is indeed different from the independent class model because if the seller offers a set $T$ in which $j$ is the lowest price product, then under the independent demand model, the buyer has a probability of purchase $q_j$ for all $i \in T$, whereas in the lowest open fare model, the buyer has a zero probability of purchase for all $i \in T$, $i \neq j$ (all products that are not the lowest priced product), and the purchase probability for the lowest price Product $j$ is $q_j$.

The choice probabilities $q_j$ themselves can be any arbitrary numbers decreasing with $j$, but in practice are likely to be derived from a choice model such as a binary logit (choice between $j$ and no-purchase).

Using Theorem 2, it is very easy to show that this choice model has the nested-by-fare-order property.

For a set $T$ of Theorem 2, if $j$ is the product with the lowest fare, just choose $\alpha_j = 1$ and set all other $\alpha$’s to 0. One can easily verify that the conditions of Theorem 2 are satisfied by this choice of $\alpha$’s.

### 4.6. Comparisons to Traditional Optimality Conditions

The optimality conditions in the nested-by-fare-order cases provide some intuition into how choice-based controls differ from traditional controls. Indeed, it is not hard to show that if the efficient sets are the complete sets (nesting by fare order is optimal), then it is optimal to open fare $k + 1$ if and only if

$$P_{k+1}(A_{k+1})(r_{k+1} - \Delta V_{k-1}(x)) \geq \sum_{j=1}^k \Delta P_j(A_j)(r_j - \Delta V_{t-1}(x)), \quad (9)$$

where

$$\Delta P_j(A_j) = P_j(A_j) - P_j(A_{k+1})$$

is the change (usually an increase for most choice models) in purchase probability for Product $j$ as the result of not offering Product $k + 1$. This condition is intuitive: The left-hand side is the probability of selling Product $k + 1$ times the “net gain” from selling it—that is, the revenue we get from Product $k + 1$ minus the opportunity cost, $\Delta V_{k-1}(x)$, of using a unit of capacity. The right-hand side is the net gain (loss) among the other products caused by eliminating (adding) Product $k + 1$ (i.e., the sum over all the other Products $j$ in $A_k$ of the change in purchase probability times the net gain from selling $j$). Therefore, condition (9) simply says that if the expected gain on Product $k + 1$ exceeds the incremental loss on the other products caused by adding $k + 1$, then it pays to open $k + 1$; otherwise, $k + 1$ should be closed.

Condition (9) should be compared to the optimality condition for the independent demand model; namely, it is optimal to open Product $k + 1$ if and only if

$$r_{k+1} - \Delta V_{k-1}(x) \geq 0.$$
5. Estimation

Next, we consider the problem of how to estimate our demand model from historical data. To do so, we assume the choice probabilities are given by a parametric model of the form

\[ P_j(S) = P_j(\gamma, \beta, S), \quad j = 0, 1, \ldots, n, \quad (10) \]

where \( z = (z_1, \ldots, z_n) \), \( z_i \) is a vector of known attributes of choice \( i \), and \( \beta \) is a vector of weights on these attributes. For example, these choice probabilities could be determined by the MNL model (7), in which the mean utility is modeled as a linear function of known attributes (e.g., price, indicator variables for product restrictions, etc.), \( u_i = \beta^T z_i \). However, other parametric models could be used as well.

We are interested in estimating the weights \( \beta \) as well as the arrival probability \( \hat{\lambda} \) from historical data.

Given complete observations of arrivals, purchases, and no-purchase outcomes, one can estimate \( \lambda \) and \( \beta \) using maximum likelihood methods. In particular, let \( D \) denote a set of intervals, indexed by \( t \), in which independent arrival events and choice decisions have been observed. The set \( D \) could combine intervals from many flight departures, and, deviating somewhat from our notational convention thus far, \( t \) here does not necessarily represent the time remaining for a particular flight. For each observation period \( t \in D \), let

\[ a(t) = \begin{cases} 1 & \text{if customer arrives in period } t, \\ 0 & \text{otherwise}. \end{cases} \]

Let \( A \) denote the set of periods \( t \) with arrivals \((a(t) = 1)\) and \( \bar{A} = D - A \) denote the periods with no arrivals. If \( t \in A \), let \( j(t) \) denote the choice made by the arriving customer. (For \( t \in \bar{A} \), define \( j(t) \) arbitrarily.) Finally, let \( S(t) \) denote the set of open fare products in interval \( t \). The likelihood function is then

\[ \prod_{t \in D} \left[ \lambda P_{j(t)}(z, \beta, S(t)) \right]^{a(t)} (1 - \lambda)^{1 - a(t)}. \]

Taking logs, we obtain the log-likelihood function

\[ \mathcal{L} = \sum_{t \in D} \left[ a(t) \ln(P_{j(t)}(z, \beta, S(t))) + a(t) \ln(\lambda) \right. \\
\left. + (1 - a(t)) \ln(1 - \lambda) \right]. \quad (11) \]

Note that \( \mathcal{L} \) is separable in \( \beta \) and \( \lambda \). Maximizing \( \mathcal{L} \) with respect to \( \lambda \), we obtain the estimate

\[ \hat{\lambda} = \frac{1}{|D|} \sum_{t \in D} a(t) = \frac{|A|}{|D|}, \]

where \(|D|\) (resp. \(|A|\)) denotes the cardinality of \( D \) (resp. \(|A|\)). This is intuitive; the estimate of \( \lambda \) (the arrival probability) is simply the number of periods with arrivals divided by the total number of periods.

The maximum likelihood estimate (MLE), \( \hat{\beta} \), is then determined by solving

\[ \max_{\beta \in \mathcal{A}} \sum_{t \in D} \ln(P_{j(t)}(z, \beta, S(t))). \quad (12) \]

This is simply the usual maximum likelihood problem for the choice model applied to those periods with customer arrivals. For most parametric choice models, this problem is relatively easy to solve numerically. For example, the log-likelihood function for the MNL model has closed-form first and second partial derivatives and is jointly concave in most cases (see McFadden 1974, Ben-Akiva and Lerman 1985). Hence, standard nonlinear programming methods (e.g., Newton’s method) can be used. Combining these two estimates gives the MLE for the complete model.

The difficulty with this MLE approach in practice is that one rarely observes all arrivals. Typically, only purchase transaction data are available. Thus, it is impossible to distinguish a period without an arrival from a period in which there was an arrival but the arriving customer did not purchase. With this incompleteness in the data, the above procedure cannot be used. One could always try to write down the likelihood function with incomplete data, but typically the function becomes very complex and difficult to maximize.

To overcome this problem, we propose using the EM method of Dempster et al. (1977). This method works by starting with arbitrary initial estimates, \( \hat{\beta} \) and \( \hat{\lambda} \). These estimates are then used to compute the conditional expected value of \( \mathcal{L} \): \( E[\mathcal{L} | \hat{\beta}, \hat{\lambda}] \) (the expectation step). The resulting expected log-likelihood function is then maximized to generate new estimates \( \hat{\beta} \) and \( \hat{\lambda} \) (the maximization step), and the procedure is repeated until it converges. While technical convergence problems can arise, in practice the EM method is a robust and efficient way to compute maximum likelihood estimates for incomplete data. (See McLachlan and Krishnan 1996 for a comprehensive reference on the EM method.) It has also been used in other yield management contexts, in particular by McGill (1995), to estimate multivariate normal demand data with censoring.

To apply the EM method in our case, let \( P \) denote the set of periods in which customers purchase and \( \bar{P} = D - P \) denote the period in which there are no-purchase transactions. We can then write the complete log-likelihood function as

\[ \mathcal{L} = \sum_{t \in P} \left[ \ln(\lambda) + \ln(P_{j(t)}(z, \beta, S(t))) \right] \\
+ \sum_{t \in \bar{P}} \left[ a(t) \ln(\lambda) + \ln(P_0(z, \beta, S(t))) \right. \\
\left. + (1 - a(t)) \ln(1 - \lambda) \right]. \quad (13) \]
where \( P_0(z, \hat{\beta}, S(t)) \) is the no-purchase probability for observation \( t \) given \( \hat{\beta} \).

The unknown data are the values \( a(t), t \in \tilde{P} \), in the second sum. However, given estimates \( \hat{\beta} \) and \( \hat{\lambda} \), we can determine their expected values (denoted \( \hat{a}(t) \)) easily via Bayes’s rule:

\[
\hat{a}(t) = E[a(t) | t \in \tilde{P}, \hat{\beta}, \hat{\lambda}] = P(a(t) = 1 | t \in \tilde{P}, \hat{\beta}, \hat{\lambda}) = \frac{P(t \in \tilde{P} | a(t) = 1, \hat{\beta}, \hat{\lambda})P(a(t) = 1 | \hat{\beta}, \hat{\lambda})}{P(t \in \tilde{P} | \hat{\beta}, \hat{\lambda})} = \frac{\hat{\lambda}P_0(z, \hat{\beta}, S(t))}{\hat{\lambda}P_0(z, \hat{\beta}, S(t)) + (1 - \hat{\lambda})}.
\]

Substituting \( \hat{a}(t) \) into (13), we obtain the expected log-likelihood for the incomplete data

\[
E[\mathcal{L} | \hat{\beta}, \hat{\lambda}] = \sum_{t \in \tilde{P}} [\ln(\lambda) + \ln(P_j(\beta, S(t)))]
+ \sum_{t \in \tilde{P}}[\hat{a}(t)(\ln(\lambda) + \ln(P_0(z, \beta, S(t)))) + (1 - \hat{a}(t)) \ln(1 - \lambda)].
\]

As in the case of the complete log-likelihood function, this function is separable in \( \beta \) and \( \lambda \). Maximizing with respect to \( \lambda \), we obtain the updated estimate

\[
\lambda^* = \frac{|P| + \sum_{t \in \tilde{P}} \hat{a}(t)}{|P| + |\tilde{P}|}.
\]

This is again intuitive; our estimate of \( \lambda \) is the number of observed arrivals, \( |P| \), plus the estimated number of arrivals from unobservable periods, \( \sum_{t \in \tilde{P}} \hat{a}(t) \), divided by the total number of periods \( |P| + |\tilde{P}| = |D| \). We can then maximize the first two sums in (15) to obtain the updated estimate \( \beta^* \). Note that this expression is of the same functional form as the complete data case (12). The entire procedure is then repeated.

Summarizing the algorithm:

**Step 0.** Initialize \( \hat{\beta} \) and \( \hat{\lambda} \).

**Step 1.** Expectation step

For \( t \in \tilde{P} \), use the current estimates \( \hat{\beta} \) and \( \hat{\lambda} \) to compute \( \hat{a}(t) \) from (14).

**Step 2.** Maximization step

Compute \( \lambda^* \) using (16).

Compute \( \beta^* \) by solving

\[
\max_\beta \left\{ \sum_{t \in \tilde{P}} \ln(P_j(\beta, S(t))) + \sum_{t \in \tilde{P}} \hat{a}(t) \ln(P_0(z, \beta, S(t))) \right\}.
\]

**Step 3.** Convergence test

IF \( \| (\lambda, \beta) - (\lambda^*, \beta^*) \| < \epsilon \), THEN STOP;
ELSE \( \lambda \leftarrow \lambda^*, \beta \leftarrow \beta^* \), and GO TO Step 1.

If the expected log-likelihood \( E[\mathcal{L} | \hat{\beta}, \hat{\lambda}] \), given in (15) is continuous in both \( (\beta, \lambda) \) and \( (\hat{\beta}, \hat{\lambda}) \), a result by Wu (1983), shows that if the sequence of estimates converges, the resulting value will be a stationary point of the incomplete log-likelihood function. Whether the sequence diverges—or converges to something other than the global maximum—is more difficult to determine. In practice, the method has proved to be very robust in other contexts, and this has been our experience in simulated experiments for our problem. (See McLachlan and Krishnan 1996 for further discussion of convergence properties of the EM algorithm.)

One interesting fact is that there can be multiple pairs \( (\beta, \lambda) \) that produce the same probabilities of sales. In this case, the EM estimates will only find one such pair. To take a trivial case, consider the MNL model and suppose there is only \( n = 1 \) fare product and that \( x_1 \) and \( \beta \) are scalars. The probability that we observe a sale if this fare product is open is

\[
p = \lambda \frac{e^{\beta x_1}}{e^{\beta x_1} + 1}.
\]

It is clear that there are a continuum of values \( (\beta, \lambda) \) that will produce the same value \( p \). However, the MLE will identify only one such pair. This difficulty is not a fault of the EM or MNL method per se; it is rather a reflection of the fact that—as in this simple example—there may be more than one probability model that produces the same purchase probabilities. In such cases, it is simply not possible to uniquely identify the model from observed data; there is in effect a degree of freedom that we cannot resolve.

### 6. Numerical Examples

In this section, we provide results of a small simulation study to compare our choice-based method to a traditional single-leg method. We consider two examples: our running three-product (Simulation Example 1) and a second 10-product example based on the MNL choice model (Simulation Example 2). In both cases we tested our method against a traditional forecasting/unconstraining and optimization method, the buy-up variation of the Expected Marginal Seal Revenue-b (EMSR-b) (Belobaba and Weatherford 1996), a very popular heuristic in practice. We first describe the EMSR-b heuristic and its modification to account for buy-up. We then discuss the two examples.

#### 6.1. EMSR-b with Buy-up

EMSR-b is a fixed protection level policy, which sets a static set of protection levels for fare Products 1 through \( n - 1 \) given by the vector \( x = (x_1, \ldots, x_{n-1}) \), where \( x_1 \leq x_2 \leq \cdots \leq x_{n-1} \). (There is no protection level for the lowest fare Product, \( n \).) Protection levels are nested in the sense that \( x_i \) represents the number of inventory units to reserve (protect) for all of
fare Products 1, 2, . . . , j. Reservations for fare Product \( j + 1 \) are accepted if and only if the number of inventory units remaining is strictly greater than the protection limit \( x_j \). (Such policies are optimal when low-fare products book strictly before higher-fare products and fare product demands are mutually independent (Brumelle and McGill 1993).)

EMSR-b sets the protection levels \( x_j \) as follows: Given estimates of the means, \( \hat{\mu}_j \), and standard deviations, \( \hat{\sigma}_j \), for each fare Product \( j \), the EMSR-b heuristic sets \( x_j \) so that

\[
r_{j+1} = \bar{r}_j P(\bar{X}_j > x_j),
\]

where \( \bar{X}_j \) is a normal random variable with mean \( \sum_{i=1}^j \hat{\mu}_i \) and variance \( \sum_{i=1}^j \hat{\sigma}_j^2 \), and \( \bar{r}_j \) is a weighted average revenue, given by

\[
\bar{r}_j = \frac{\sum_{i=1}^j r_j \hat{\mu}_i}{\sum_{i=1}^j \hat{\mu}_i}.
\]

The idea behind this approximation is to reduce the complexity of the fully nested problem by aggregating fare products 1, 2, . . . , \( j \) into a single-fare product. Then, one treats the problem as a simple, two-fare-class problem and applies Littlewood’s rule (1972).

The variation of EMSR-b with buy-up factors proposed by Belobaba and Weatherford (1996) is as follows: Consider the simple two-class static model. Littlewood’s rule (slightly restated) is to accept demand from class 2 if and only if \( r_2 - r_1 P(X_1 \geq x_1) \geq 0 \). Now suppose that there is a probability \( q \) that a customer for class 2 will buy a class 1 fare if class 2 is closed. The net benefit of accepting the request is still the same, but now rather than losing the request, if we reject it, there is some chance the customer will sell up to class 1. If so, we earn a net benefit of \( r_1 - r_1 P(X_1 \geq x_1) \) (the class 1 revenue minus the marginal cost). Thus, it is optimal to accept class 2 now if \( r_2 - r_1 P(X_1 \geq x_1) \geq qr_1 (1 - P(X_1 \geq x_1)) \), or equivalently if

\[
r_2 \geq (1 - q) r_1 P(X_1 \geq x_1) + qr_1.
\]

Note that the right-hand side above is strictly larger than the right-hand side of Littlewood’s rule, which means that the above rule is more likely to reject class 2 demand.

Using this reasoning for the two-fare case, EMSR-b can then be extended to allow for a buy-up factor by modifying the equation for determining the protection level \( x_j \) as follows:

\[
r_{j+1} = (1 - q_{j+1}) \tilde{r}_j P(\bar{X}_j > x_j) + q_{j+1} \tilde{r}_j,
\]

where \( q_{j+1} \) is the probability that a customer of class \( j + 1 \) buys up to one of the classes \( j, j - 1, \ldots, 1 \); \( \tilde{r}_j \) is an estimate of the average revenue received given that they buy-up; and \( \bar{X}_j \) and \( \tilde{r}_j \) are defined as before.

There is no standard method for converting given choice model parameters into buy-up factors for the EMSR-b method. Hence, we selected what we felt were reasonable (albeit ad hoc) approaches, as discussed in detail for each example below.

### 6.2. Simulation Example 1

This set of simulations is based on our running three-product example, Example 1. We compared the optimal control given by the choice-based dynamic program to the traditional EMSR-b (buy-up) recommendations. (Recall that for Example 1 the optimal DP policy uses the fare products in the order \( Y, K, M \), whereas the EMSR-b uses them in the fare order \( Y, M, K \).)

We tested assuming a capacity \( C = 20 \) and three population sizes: 15, 20, and 25. The fares, restrictions, and customer segments are as given in Table 2. For a population of 20 this results in an unconstrained mean and variance, as shown in Table 6. These statistics were used to create inputs for EMSR-b.

The buy-up factors were computed as shown in Table 7. While there is not one correct way to convert a choice model into a traditional independent model, our translation tries to be as reasonable as possible. To this end, Table 6 lists the unconstrained choices and

### Table 6  Segment Choices in Example 1 If All Products Are Open and Resulting Demand for Population Size of 20

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Bus. 1</td>
<td>0.1</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>N.E.</td>
<td>N.E.</td>
<td>Y</td>
<td>2</td>
<td>1.8</td>
</tr>
<tr>
<td>Bus. 2</td>
<td>0.2</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>N.E.</td>
<td>M</td>
<td>4</td>
<td>3.2</td>
</tr>
<tr>
<td>Leis. 1</td>
<td>0.2</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>N.E.</td>
<td>M</td>
<td>4</td>
<td>3.2</td>
</tr>
<tr>
<td>Leis. 2</td>
<td>0.2</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>K</td>
<td>4</td>
<td>3.2</td>
</tr>
<tr>
<td>Leis. 3</td>
<td>0.3</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>K</td>
<td>6</td>
<td>4.2</td>
</tr>
</tbody>
</table>

**Note.** N.E. signifies the segment is not eligible.

### Table 7  Inputs to the EMSR-b Model

<table>
<thead>
<tr>
<th>Class</th>
<th>Mean</th>
<th>SD</th>
<th>Buy-up factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>2</td>
<td>1.34</td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>8</td>
<td>2.52</td>
<td>0.33</td>
</tr>
<tr>
<td>K</td>
<td>10</td>
<td>2.72</td>
<td>0.40</td>
</tr>
</tbody>
</table>
demands (i.e., demand when all classes are open) that result. This roughly mimics the traditional practice of unconstraining and forecasting demand in each class. Table 7 sums the unconstrained demands into each fare class.

The buy-up factors are estimated as follows: The buy-up factor for $K$ is given by the percentage $K$ customers who would buy-up to $M$ if we go from offering $Y, M, K$ to $Y, M$. Similarly, the buy-up factor for $M$ is the fraction who buy-up to $Y$ when going from offering $Y, M$ to $Y$. Table 7 shows these demands and buy-up factors for a population of 20, and Table 8 shows the computed EMSR-b protection levels for this population size.

We simulated arrivals and applied the choice DP and EMSR-b methods to control fare product availability. The arrivals were uniformly distributed over the booking period. The EMSR-b heuristic was reoptimized prior to the decision on each arrival. Because the arrivals are distributed uniformly over the booking period, the demand mean and variances at the time of reoptimization were prorated based on the remaining time till departure.

The results for the three load factors are summarized in Table 9. Note that choice DP shows significant improvements on this example, achieving an 11.48% improvement in revenue in the high-demand case. Part of this improvement can be attributed to the fact that the choice DP uses a different sequence of fares (i.e., only the efficient sets $\{Y\}, \{Y, K\}, \{Y, M, K\}$).

### 6.3. Simulation Example 2

In this set of simulations we used the MNL choice model to test both the estimation and optimization methods. We do not address the validity of the MNL model itself; we use it simply to illustrate another example of the basic behavior of the optimal policy relative to traditional heuristic approaches.

The capacity is $C = 185$ and there are $n = 10$ fare classes with fares as shown in Table 10. For simplicity, we assumed the random utility had only one attribute, denoted $x$, which was simply the price. The coefficient, $\beta$, on the price attribute was taken to be either $\beta^L = -0.0015$ (low price sensitivity, denoted $L$) or $\beta^H = -0.005$ (high price sensitivity, denoted $H$). The values $w_j^L = e^{\beta^L x}$ and $w_j^H = e^{\beta^H x}$ are shown in Table 10 as well.

Arrivals over the booking period were generated by simulating a homogeneous Poisson process with a mean of 205. (Thus, if the booking period is broken up into intervals of size $\Delta$, then $\lambda = 205\Delta$ in the choice-based DP. We used $\Delta = 1/410$ or $\lambda = 0.5$ in our computations.) The choice parameters were estimated using the EM method as described in §5. The training set consisted of 50 simulated days during which the available classes were controlled using EMSR-b, as described above. The EM method produced estimates of $\hat{\beta}^L = -0.0014$ and $\hat{\beta}^H = -0.0048$, which are very close to the actual values of $\beta^L = -0.0015$ and $\beta^H = -0.005$. To mimic the real-world combination of forecasting and optimization, these estimated values were used in the choice DP algorithm.

Bookings were generated for a period of 100 simulated days and the controls of each method were applied. The results of 15 simulated flights are shown in Table 11. In the case where the price sensitivity is low, the choice DP has significantly higher revenues; 68,373 versus 61,049 for EMSR. This represents a 12% improvement in revenue, which is very large when compared to the typical 1%–2% differences in revenues that one finds when comparing optimization methods. In the case of high price sensitivity, the revenue difference between the two methods is essentially identical within the simulation error of our test. This is not surprising, because buy-up in particular can best be exploited when customers are not very price sensitive.

In terms of qualitative behavior, the choice DP frequently closes lower-fare products to force consumers to buy-up to higher fares. EMSR-b, even with the

### Table 8 Protections for the EMSR-b Model Without and with Buy-up Factors

<table>
<thead>
<tr>
<th>Class</th>
<th>EMSR-b</th>
<th>EMSR-b with buy-up</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$</td>
<td>1.57</td>
<td>2.20</td>
</tr>
<tr>
<td>$Y + M$</td>
<td>7.55</td>
<td>8.71</td>
</tr>
</tbody>
</table>

### Table 9 Simulation Results Comparison Between the Choice Dynamic Program and the EMSR-b Model with Buy-up

<table>
<thead>
<tr>
<th>Population size</th>
<th>EMSR-b revenue</th>
<th>Choice DP revenue</th>
<th>% gain from choice DP</th>
<th>99% conf. int. error on % gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>7,466</td>
<td>7,466</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>9,825</td>
<td>10,129</td>
<td>3.10</td>
<td>±0.0050</td>
</tr>
<tr>
<td>25</td>
<td>10,142</td>
<td>11,301</td>
<td>11.48</td>
<td>±0.0079</td>
</tr>
</tbody>
</table>

### Table 10 Fares and $w_j$ Values for Example 2

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_j$</td>
<td>600</td>
<td>550</td>
<td>475</td>
<td>400</td>
<td>300</td>
<td>280</td>
<td>240</td>
<td>200</td>
<td>185</td>
<td>175</td>
</tr>
<tr>
<td>$w_j^L$</td>
<td>0.407</td>
<td>0.438</td>
<td>0.490</td>
<td>0.549</td>
<td>0.638</td>
<td>0.657</td>
<td>0.698</td>
<td>0.741</td>
<td>0.758</td>
<td>0.769</td>
</tr>
<tr>
<td>$w_j^H$</td>
<td>0.050</td>
<td>0.064</td>
<td>0.093</td>
<td>0.135</td>
<td>0.223</td>
<td>0.247</td>
<td>0.301</td>
<td>0.368</td>
<td>0.397</td>
<td>0.417</td>
</tr>
</tbody>
</table>
buy-up heuristic, opens up more fare classes and allows customers to buy at lower prices. This difference shows up in the load factors, which are lower for the choice-based DP. While this drop in load factors may at first be worrisome, it is not unexpected given that the choice-based DP is deliberately increasing the probability that customers will not purchase by restricting discounts. However, the revenue increases from the higher fares more than compensate for the lower volume of sales in the low-price-sensitivity case and produce essentially the same revenues in the high-price-sensitivity case. While the magnitude of these results is specific to this particular set of numbers and choice probabilities, the results do show how it may be possible to increase revenue by exploiting choice behavior.

7. Conclusion

Our analysis provides a quite complete characterization of optimal policies under a general choice model of demand. The fact that the optimal policy consists of selecting a set from a sequence of efficient sets—and that the optimal set to select is further along the sequence the more capacity (or less time) one has available—is strikingly simple given the prima facia complexity of the problem. Moreover, the analysis based on efficient sets provides insight into when nested and nested-by-fare-class policies are optimal, which is useful in understanding the traditional independent demand model, as well as other possible demand models such as the MNL. Finally, we provided a general method to estimate choice models when no-purchase outcomes are unobservable.

To practitioners, our choice-based approach to revenue management may be a departure from current thinking, but we believe it represents an attractive alternative for incorporating consumer behavior into revenue management—a long-standing goal for many industry practitioners.

We see several topics worthy of further study. One is to try to identify other classes of probability models for which nesting—or nesting by fare order—is optimal. Another worthwhile extension would be to model choice among a set of flights. This was one of the topics investigated by Andersson at SAS (1998) and Algers and Besser (2001), and it would be interesting to see if the estimation and optimization methods proposed here could be extended to model control of a set of related flights. Similar but even more complex would be to extend the model to networks. In both these cases, exact dynamic programming will most likely be impractical, so it would be interesting to see what approximation methods could be developed.

More generally, the choice-based approach presented here can provide a basis for other advanced practices, for example, a customer-relationship-management-level approach to revenue management, where availability controls are exerted at the individual customer level, or as a way to integrate and optimize both pricing and product attribute design within revenue management. The approach may also prove useful in analyzing competitive interactions if the choice set included competitors’ products as well as a firm’s own products.

Acknowledgments

The authors thank two anonymous referees and the associate editor for many constructive suggestions that greatly improved the manuscript. In particular, the proofs related to efficient sets were considerably simplified as a result of their feedback. The authors also thank several colleagues for their helpful comments on the paper, especially Guillermo Gallego, Bob Phillips, and Yimin Wang.

Appendix. Proofs

Proof of Proposition 1. We first show the “if” part: Suppose for some \( v > 0 \), \( T \) satisfies \( R(T) - vQ(T) \geq R(S) - vQ(S) \) for all \( S \subseteq N \). This is equivalent to

\[
(R(T) - R(S)) - v(Q(T) - Q(S)) \geq 0 \quad \forall S \subseteq N.
\]

Multiplying each inequality above by \( \alpha(S) \) where \( \{\alpha(S) : S \subseteq N\} \) are any set of probabilities with \( \sum_{S \subseteq N} \alpha(S) = 1 \) and adding the inequalities, we then have that

\[
\left(R(T) - \sum_{S \subseteq N} \alpha(S)R(S)\right) - v\left(Q(T) - \sum_{S \subseteq N} \alpha(S)Q(S)\right) \geq 0.
\]

Now if \( v > 0 \) and \( T \) is inefficient, there exist probabilities \( \alpha(S) \) such that \( R(T) < \sum_{S \subseteq N} \alpha(S)R(S) \) and \( Q(T) \geq \sum_{S \subseteq N} \alpha(S)Q(S) \), which contradicts the above inequality. Therefore, \( T \) must be efficient. This proves the “if” part.

To show the “only if” part, note that if \( T \) is efficient, then it lies on the efficient frontier \( \hat{R}(q) \) defined in Lemma 1. Because from Lemma 1 this efficient frontier is concave, there exists a supporting hyperplane \((u, v)\) such that

\[
R(T) = vQ(T) + u
\]

and

\[
R(S) \leq vQ(S) + u \quad \forall S \subseteq N.
\]

Hence, it follows that \( R(T) - vQ(T) \geq R(S) - vQ(S) \) for all \( S \subseteq N \). Furthermore, because by Lemma 1 the efficient frontier \( \hat{R}(q) \) is increasing, it must be that \( v \geq 0 \). This completes the “only if” proof. □
Proof of Lemma 4. The proof is by induction on $t$. First, the statement is trivially true for $t = 0$ by the boundary conditions (2). Assume it is true for period $t - 1$. Let $S_t(x)$ denote the optimal solution to (1) and note that

$$
\Delta V_t(x) - \Delta V_t(x - 1) = \Delta V_{t-1}(x) - \Delta V_{t-1}(x - 1)
$$

$$
+ \sum_{j \in S_t(x)} \alpha_P(S_t(x))(r_j - \Delta V_{t-1}(x))
$$

$$
- \sum_{j \in S_t(x)} \alpha_P(S_t(x))(r_j - \Delta V_{t-1}(x - 1))
$$

$$
- \sum_{j \in S_t(x-2)} \alpha_P(S_t(x-2))(r_j - \Delta V_{t-1}(x - 2)).
$$

(A.1)

From the optimality of the set defined by $S_t(\cdot)$, the following inequalities hold:

$$
\sum_{j \in S_t(x)} \alpha_P(S_t(x))(r_j - \Delta V_{t-1}(x - 1)) \geq 0
$$

and

$$
\sum_{j \in S_t(x-2)} \alpha_P(S_t(x-2))(r_j - \Delta V_{t-1}(x - 2)) \geq 0.
$$

Substituting into (A.1) we obtain

$$
\Delta V_t(x) - \Delta V_t(x - 1) \leq \Delta V_{t-1}(x) - \Delta V_{t-1}(x - 1)
$$

$$
+ \sum_{j \in S_t(x)} \alpha_P(S_t(x))(r_j - \Delta V_{t-1}(x))
$$

$$
- \sum_{j \in S_t(x)} \alpha_P(S_t(x))(r_j - \Delta V_{t-1}(x - 1))
$$

$$
- \sum_{j \in S_t(x-2)} \alpha_P(S_t(x-2))(r_j - \Delta V_{t-1}(x - 2)).
$$

Rearranging and canceling terms yields

$$
\Delta V_t(x) - \Delta V_t(x - 1) \leq \left(1 - \sum_{j \in S_t(x)} \alpha_P(S_t(x))(\Delta V_{t-1}(x) - \Delta V_{t-1}(x - 1))\right)
$$

$$
+ \sum_{j \in S_t(x-2)} \alpha_P(S_t(x-2))(\Delta V_{t-1}(x - 1) - \Delta V_{t-1}(x - 2)).
$$

By induction, $\Delta V_{t-1}(x) - \Delta V_{t-1}(x - 1) \leq 0$ and $\Delta V_{t-1}(x - 1) - \Delta V_{t-1}(x - 2) \leq 0$. Therefore, $\Delta V_t(x) - \Delta V_t(x - 1) \leq 0$. □

Proof of Proposition 5. Note that by using (4), we have

$$
\Delta V_t(x) = V_t(x) - V_t(x - 1)
$$

$$
= \max_k \{\lambda(R_k - Q_k \Delta V_{t-1}(x))\}
$$

$$
- \max_k \{\lambda(R_k - Q_k \Delta V_{t-1}(x - 1))\} + \Delta V_{t-1}(x).
$$

From Lemma 4, $\Delta V_{t-1}(x) \leq \Delta V_{t-1}(x - 1)$, and therefore for any value $k$,

$$
\lambda(R_k - Q_k \Delta V_{t-1}(x)) - \lambda(R_k - Q_k \Delta V_{t-1}(x - 1)) \geq 0.
$$

Hence,

$$
\max_k \{\lambda(R_k - Q_k \Delta V_{t-1}(x))\} - \max_k \{\lambda(R_k - Q_k \Delta V_{t-1}(x - 1))\} \geq 0
$$

as well, and it follows that $\Delta V_t(x) \geq \Delta V_{t-1}(x)$. □

Proof of Theorem 2. For Part (i), the increasing property of the probability model is a requirement from the definition of the nested-by-fare-order property.

For Part (ii), let $T$ be an incomplete set. If the probability model has the nested-by-fare-order property, then there does not exist a set of values $x_1 \geq x_2 \geq \cdots \geq x_p$ such that $T$ is the unique solution to (5). Thus, the following linear system of inequalities (in the variables $x$ and the scalar $u$) has no solution:

$$
u - \sum_{k=1}^n P_k(T)x_j < 0, \quad (A.2)
$$

$$
\alpha_k : u - \sum_{k=1}^n P_k(A_k)x_j \geq 0, \quad k = 1, \ldots, n, \quad (A.3)
$$

$$
z_j : x_j - x_{j+1} \geq 0, \quad j = 1, \ldots, n - 1.
$$

But by Farkas' lemma, the system (A.3) has no solution if and only if the dual system in variables $\alpha_k$, $k = 1, \ldots, n$, and $z_j$, $j = 0, 1, \ldots, n$ is solvable:

$$
\sum_{k=1}^n \alpha_k P_k(A_k) - z_j = \sum_{j=1}^n \alpha_k P_k(A_k) = P_k(T), \quad j = 1, \ldots, n,
$$

$$
\sum_{k=1}^n \alpha_k P_k(A_k) \geq 0, \quad 0 \geq \sum_{k=1}^n \alpha_k P_k(A_k) \geq 0.
$$

where we define $z_0 = 0$ and $z_n = 0$ to simplify the notation.

We can next eliminate the $z$ variables. Note that for $j = 1$ from (A.2) together with $z_j \geq 0$, we have $\sum_{k=1}^n \alpha_k P_k(A_k) - P_k(T) = z_j \geq 0$, which implies

$$
\sum_{k=1}^n \alpha_k P_k(A_k) = P_k(T).
$$

For $j = 2, \ldots, n - 1$ using $z_j \geq 0$, we obtain $\sum_{k=1}^n \alpha_k P_k(A_k) - P_k(T) + z_{j-1} = z_j \geq 0$. Substituting for $z_{j-1}$ recursively we obtain

$$
\sum_{k=1}^n \alpha_k P_k(A_k) \geq \sum_{k=1}^n P_k(T).
$$

Similarly, for $j = n$, because $z_n = 0$ by definition, this same step gives $\sum_{k=1}^n \alpha_k P_k(A_k) = \sum_{k=1}^n P_k(T)$. Thus, there exist weights $\alpha$ satisfying the conditions of the theorem if and only if the probability model has the nested-by-fare-order property. □

Proof of Proposition 6. Before proving the result, we need a simple lemma:

**Lemma 6.** Consider the functions $f(y) = y/(y + 1)$ and $g(y) = 1/(y + 1)$. Suppose $y_1 \leq y_2 \leq y_3$ and $\lambda$ are nonnegative numbers satisfying $\lambda f(y_1) + (1 - \lambda)f(y_3) = f(y_2)$. Then, $\lambda g(y_1) + (1 - \lambda)g(y_3) = g(y_2)$ as well.
The proof follows easily from the fact that \( g(y) = 1 - f(y) \).

We now turn to the proof of Proposition 6. First, is clear from the definition of the MNL probabilities (8) that the total probability of purchase \( Q(S) = \sum_{s \in S} P(S) \) is increasing in \( S \).

To show the second set of conditions of Theorem 2, let \( T \) be an incomplete set and again define the index \( k^* \) by

\[
\sum_{i=1}^{k^*} w_i \leq \sum_{i \in T} w_i < \sum_{i=1}^{k^*+1} w_i.
\]

Note that \( k^* \) is less than the largest index in \( T \), that is, \( \max\{j : j \in T\} > k^* \) as before. Next, define \( \lambda \) by

\[
\lambda \sum_{i=1}^{k^*} w_i + 1 + (1 - \lambda) \sum_{i=1}^{k^*+1} w_i + 1 = \sum_{i \in T} w_i + 1 \tag{A.4}
\]

and note by Lemma 6 that

\[
\lambda \frac{1}{\sum_{i=1}^{k^*} w_i + 1} + (1 - \lambda) \frac{1}{\sum_{i=1}^{k^*+1} w_i + 1} = \frac{1}{\sum_{i \in T} w_i + 1} \tag{A.5}
\]

as well. Also define the convex weights

\[
\alpha_k = \begin{cases} 
\lambda, & k = k^* , \\
1 - \lambda, & k = k^* + 1 , \\
0, & \text{otherwise}.
\end{cases}
\]

Using these weights and the MNL probabilities (8), we have that for \( j \leq k^* \),

\[
P_j^T(\alpha) = \lambda P_j(A_{k^*}) + (1 - \lambda) P_j(A_{k^*+1})
\]

\[
= \lambda \frac{w_j}{\sum_{i=1}^{k^*} w_i + 1} + (1 - \lambda) \frac{w_j}{\sum_{i=1}^{k^*+1} w_i + 1}
\]

\[
= \frac{w_j}{\sum_{i \in T} w_i + 1} \geq P_j(T),
\]

where the second equality follows from (A.5) and the last inequality follows from the fact that \( P_j(T) = w_j/(\sum_{i \in T} w_i + 1) \) if \( j \in T \) and \( P_j(T) = 0 \) otherwise. Summing for all values \( i \leq j \), we obtain that for \( j \leq k^* \), \( \sum_{i=1}^{j} P_j(\alpha) = \sum_{i=1}^{j} P_j(T) \) as required by Theorem 2. For \( j > k^* \) we have by (A.4)

\[
\sum_{i=1}^{j} P_i(\alpha) = \sum_{i=1}^{k^*} \left( \lambda \frac{w_i}{\sum_{i=1}^{k^*} w_i + 1} + (1 - \lambda) \frac{w_i}{\sum_{i=1}^{k^*+1} w_i + 1} \right)
\]

\[
+ (1 - \lambda) \frac{w_{k^*+1}}{\sum_{i=1}^{k^*} w_i + 1}
\]

\[
= \lambda \sum_{i=1}^{k^*} \frac{w_i}{\sum_{i=1}^{k^*} w_i + 1} + (1 - \lambda) \sum_{i=1}^{k^*+1} \frac{w_i}{\sum_{i=1}^{k^*+1} w_i + 1}
\]

\[
= \sum_{i \in T} w_i + 1 \geq \sum_{i=1}^{j} P_i(T).
\]

Finally, evaluating the above for \( j = n \) shows that \( \sum_{i=1}^{n} P_i(\alpha) = \sum_{i=1}^{n} P_i(T) \), which completes the proof. □


