Network Design

Thomas Stidsen

tks@imm.dtu.dk

Informatics and Mathematical Modeling
Technical University of Denmark

Learning Objectives

After this lecture you will:

- Have seen the basic fixed-charge network design
- Have seen two cases of different network planning problems:
  - Iceland Telecom (node placement)
  - Global Connect (network expansion)

Network Design

Network design is a very broad term, because many things can be designed:

- The location, size, and type of nodes (switches): Long term, > 5 years
- The location, size, and type of links (cables): Very long term > 10 years

Optimization

Long term planning is nice, we do not to the same degree worry about running times of the algorithms ... but getting reliable data is a **HUGE** problem: How will the network be used in 10 years ???

Fixed-Charge Network Design

The classical network design problem:

- An extension of the Multi Commodity Flow problem, taking into account long term investment costs
- Much harder problem
- Can column generation be used ?
- Can be extended in many ways ...

Fixed-Charge Network Design - terms

- x_{ij}^{kl}: Flow from node i to node j of demand kl
- $y_{\{ij\}}$: Should (bi-directional) link (cable) be established between node i and node j
- D_{kl}: Connection demand for demand kl
- $c_{\{ij\}}$: Cost pr. capacity unit for the (bi-directional) link between i and j
- $f_{\{ij\}}$: Static costs, (digging costs) for the (bi-directional) link between i and j

Fixed-Charge Network Design

Min:

$$\sum_{kl} \sum_{ij} c_{ij} \cdot D_{kl} \cdot x_{ij}^{kl} + \sum_{ij} f_{ij} \cdot y_{ij}$$

s.t.:

$$\sum_{j} x_{ij}^{kl} - \sum_{j} x_{ji}^{kl} = \begin{cases} 1 & i = k \\ -1 & i = l \\ 0 & \end{cases} \qquad \forall\, kl$$

$$\sum_{kl} (x_{ij}^{kl} + x_{ji}^{kl}) \leq M \cdot y_{ij} \qquad \forall\, \{ij\}$$

$$x_{ij}^{kl} \in R_{+} \qquad y_{ij} \in \{0,1\}$$

Fixed Charge Network design

This is a *hard* problem:

- The LP bound can be very bad, because of the big-M in the second constraint.
- The direct formulation has the same number of continuous variables as the Multi-Commodity flow problem (and I said that was a problem ...)

So what to do

It turns out that this problem rather hard to work with. We used Dantzig-Wolfe/Column Generation to deal with the Multi-Commodity Flow Problem, so why not do the same here ?

- Unfortunately, direct column generation (followed by branch-and-price) does *NOT* work well here. *WHY ?* (later slide)

Other approaches

A number of other approaches have been attempted:

- Lagrangian relaxation, but it more or less faces the same problems as Dantzig-Wolfe/column generation
- Benders decomposition, but this faces the usual Benders decomposition problems:
  - Slow solution of master problem
  - Weak cuts

Why does DZ/CG not work very well ?

The obvious decomposition where the variables are now the *paths* u_p^{kl} does not solve the bad LP relaxation !

DZ-master: Fixed-Charge Network Design

Min:

$$\sum_{kl} \sum_{ij} c_p \cdot D_{kl} \cdot u_p^{kl} + \sum_{ij} f_{ij} \cdot y_{ij}$$

s.t.:

$$\sum_{p} u_p^{kl} = 1 \quad \forall\, kl \;\; (\alpha^{kl} \geq 0)$$

$$\sum_{kl} D_{kl} \cdot u_p^{kl} \leq M \cdot y_{ij} \quad \forall\, \{ij\} \;\; (\beta_{ij} \leq 0)$$

$$u_p^{kl} \in R_+ \qquad y_{ij} \in \{0, 1\}$$

The problem stays there ...

As you can see, we did *NOT* get rid of the big-M notation. We did get rid of the $O(N^4)$ number of variables ... which was why this decomposition was a good idea for the Multi Commodity Flow problem. What about the quality of the bound we get from the DZ decomposed problem ?

DZ-sub: Fixed-Charge Network Design

Min:

$$\alpha^{kl} - \sum_{kl} \sum_{ij} (c_{ij} + \beta_{ij}) \cdot D_{kl} \cdot x_{ij}^{kl}$$

s.t.:

$$\sum_{j} x_{ij}^{kl} - \sum_{j} x_{ji}^{kl} = \begin{cases} 1 & i = k \\ -1 & i = l \\ 0 & \end{cases} \qquad \forall\, kl$$

$$x_{ij}^{kl} \in \{0, 1\}$$

How to solve the sub-problem ?

This is an *easy* problem:

- We can simply relax the binary variables to $x_{ij}^{kl} \in [0, 1]$, and use a standard LP solver. The variables will obtain integer (binary) values (or they can be corrected !)
- We can solve the problem with a simple shortest path algorithm.

But this is actually a big problem for us !

The strength of a bound

Read this carefully: *THE STRENGTH OF THE DZ BOUND IS THE SAME AS THE LP BOUND, IF THE LP RELAXED SUB-PROBLEM OBTAIN INTEGER SOLUTIONS*

Original problem LP

Max:

$$x + 2y$$

s.t.:

$$\begin{aligned} -x + y &\leq 2 \\ x + y &\leq 4 \\ -x - y &\leq 0 \\ -x - y &\leq -2 \\ -x + y &\leq 1 \\ 2x + y &\leq 13 \\ -x - 3y &\leq -7 \\ x, y \in R & \end{aligned}$$

Original problem IP

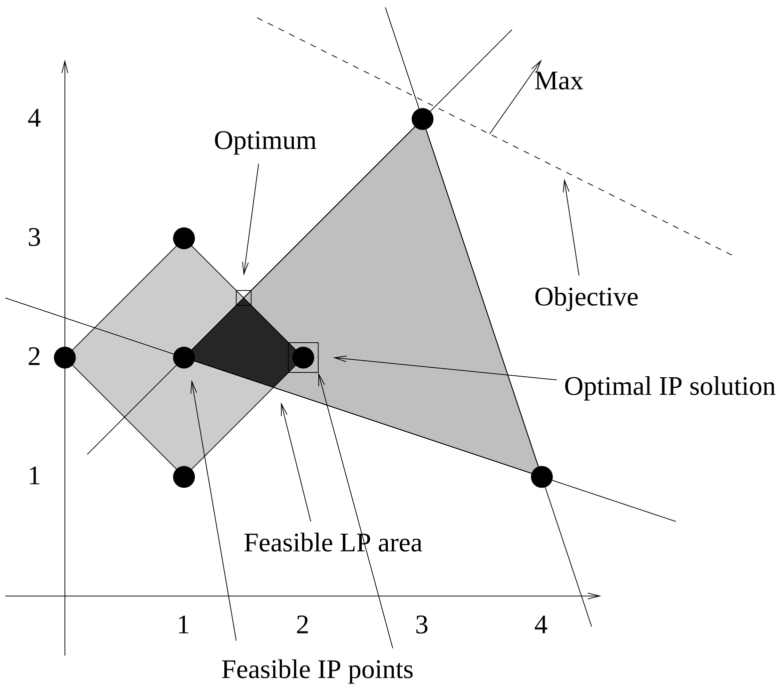
Max:

$$x + 2y$$

s.t.:

$$\begin{aligned} -x + y &\leq 2 \\ x + y &\leq 4 \\ -x - y &\leq 0 \\ -x - y &\leq -2 \\ -x + y &\leq 1 \\ 2x + y &\leq 13 \\ -x - 3y &\leq -7 \\ x, y &\in Z \end{aligned}$$

Graphically represented



DZ decomposition of Original problem

Max:

$$x + 2y$$

s.t.:

$$\begin{aligned}
-x + y &\leq 2 \quad master \\
x + y &\leq 4 \quad master \\
-x - y &\leq 0 \quad master \\
-x - y &\leq -2 \quad master \\
-x + y &\leq 1 \quad sub \\
2x + y &\leq 13 \quad sub \\
-x - 3y &\leq -7 \quad sub \\
x, y &\in Z
\end{aligned}$$

Reformulation II

Max:

$$5\lambda_1$$

s.t.:

$$\begin{aligned}
-\lambda_1 + 2\lambda_1 &\leq 2 \\
\lambda_1 + 2\lambda_1 &\leq 4 \\
-\lambda_1 - 2\lambda_1 &\leq 0 \\
-\lambda_1 - 2\lambda_1 &\leq -2 \\
\lambda_1 &= 1 \\
\lambda_1 \in R &
\end{aligned}$$

I wonder what the optimal solution is ??? (using as

Sub-problem IP

Max:

$$
\begin{aligned}
c_r &= x + 2y - \alpha \cdot A1 \cdot (x, y)) - \beta \\
&= x + 2y - (-\alpha_1 x + \alpha_2 x - \alpha_3 x - \alpha_4 x) \\
&\quad -(\alpha_1 y + \alpha_2 y - \alpha_3 y - \alpha_4 y) - \beta \\
&= x + 2y - 5
\end{aligned}
$$

s.t.:

$$
\begin{aligned}
-x + y &\leq 1 \\
2x + y &\leq 13 \\
-x - 3y &\leq -7 \\
x, y \in Z
\end{aligned}
$$

Sub-problem LP

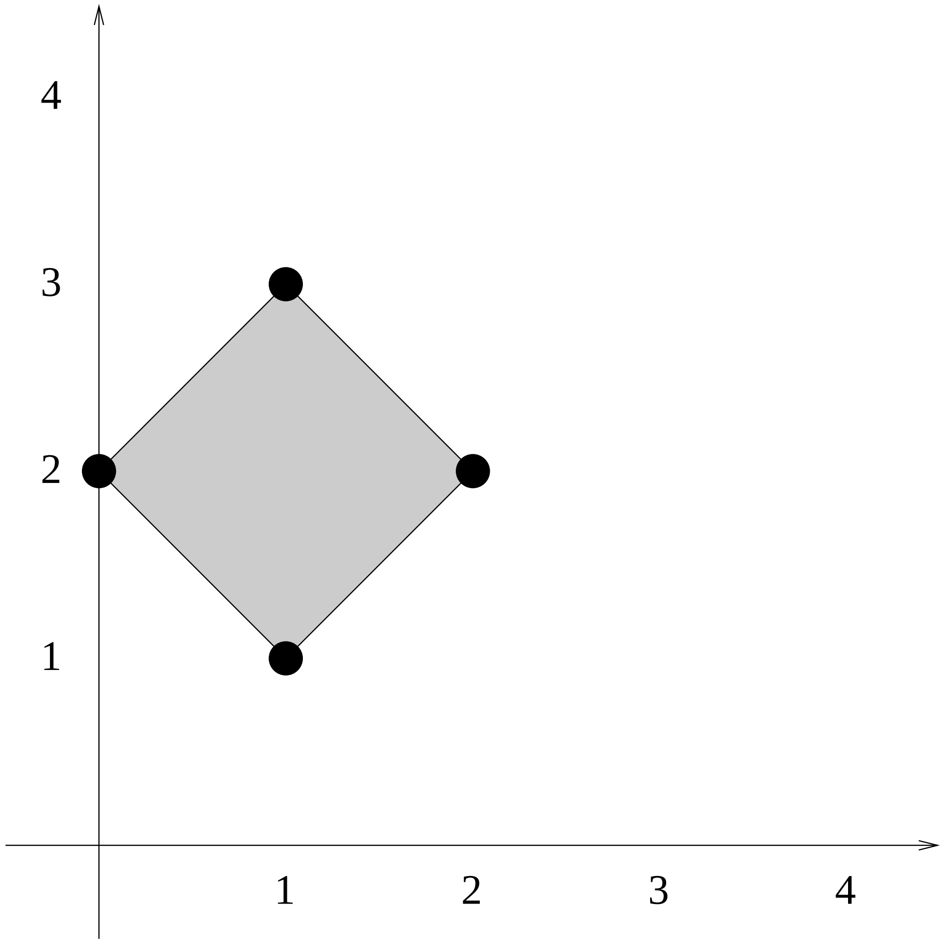
Max:

$$\begin{aligned} c_r &= x + 2y - \alpha \cdot A1 \cdot (x, y)) - \beta \\ &= x + 2y - (-\alpha_1 x + \alpha_2 x - \alpha_3 x - \alpha_4 x) \\ &\quad -(\alpha_1 y + \alpha_2 y - \alpha_3 y - \alpha_4 y) - \beta \\ &= x + 2y - 5 \end{aligned}$$

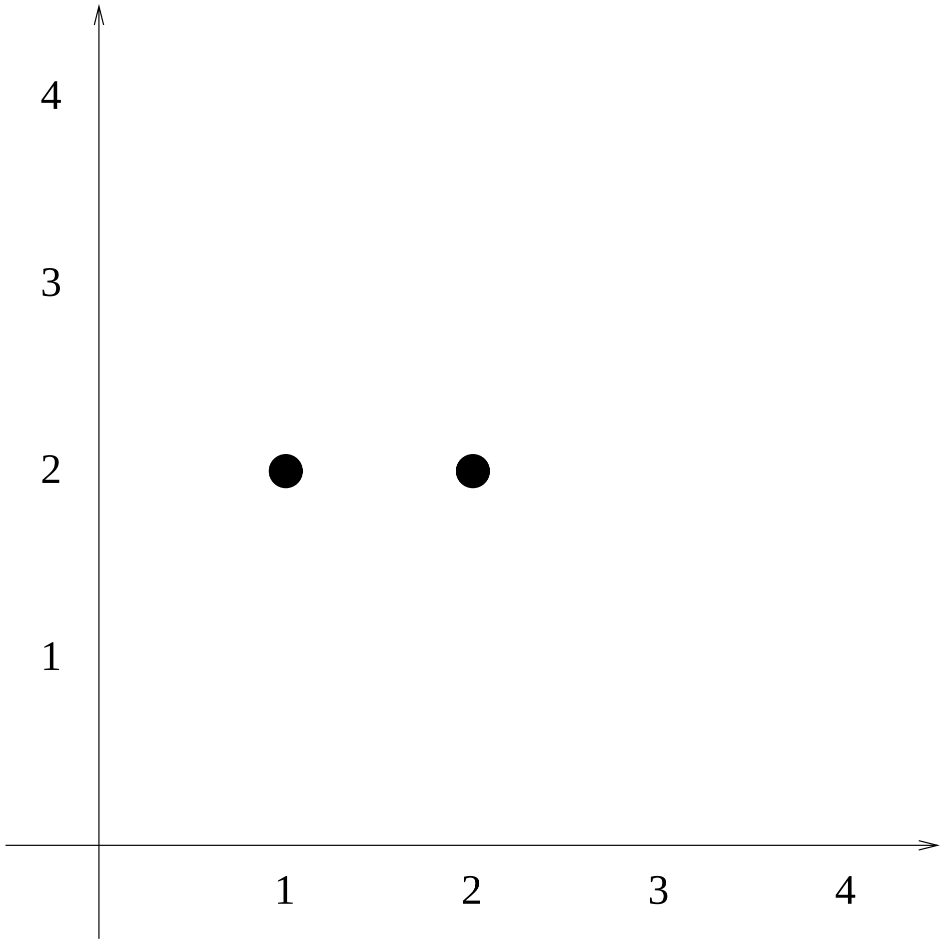
s.t.:

$$\begin{aligned} -x + y &\leq 1 \\ 2x + y &\leq 13 \\ -x - 3y &\leq -7 \\ x, y &\in R \end{aligned}$$

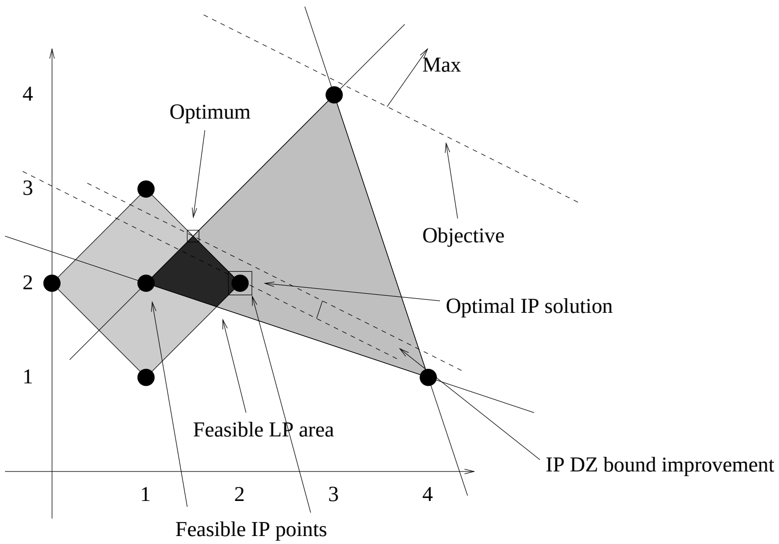
Sub-problem LP



Sub-problem IP



The resulting IP improvement in the bound



The trade-off

For this reason there exists an important trade-off:

- We want fast solution of the sub-problem
- We want sub-problems where the LP relaxation does *NOT* lead to *INTEGER* solutions
- We can use a MIP solver to solve the sub-problem, it is slow, but it is actually applied.
- Often a special algorithm is designed which solves the IP sub-problem faster than general MIP solvers (the classic example: Constrained shortest path).

Then what

Then what can we do ?

- What about the other decomposition ? (the short answer: I don’t know !) The sub-problem becomes "strange"
- Probably, Branch & Cut will perform best (but this is another long story).

Extending the Fixed-Charge Network Design Mo

It is quite easy to add a number of extra features:

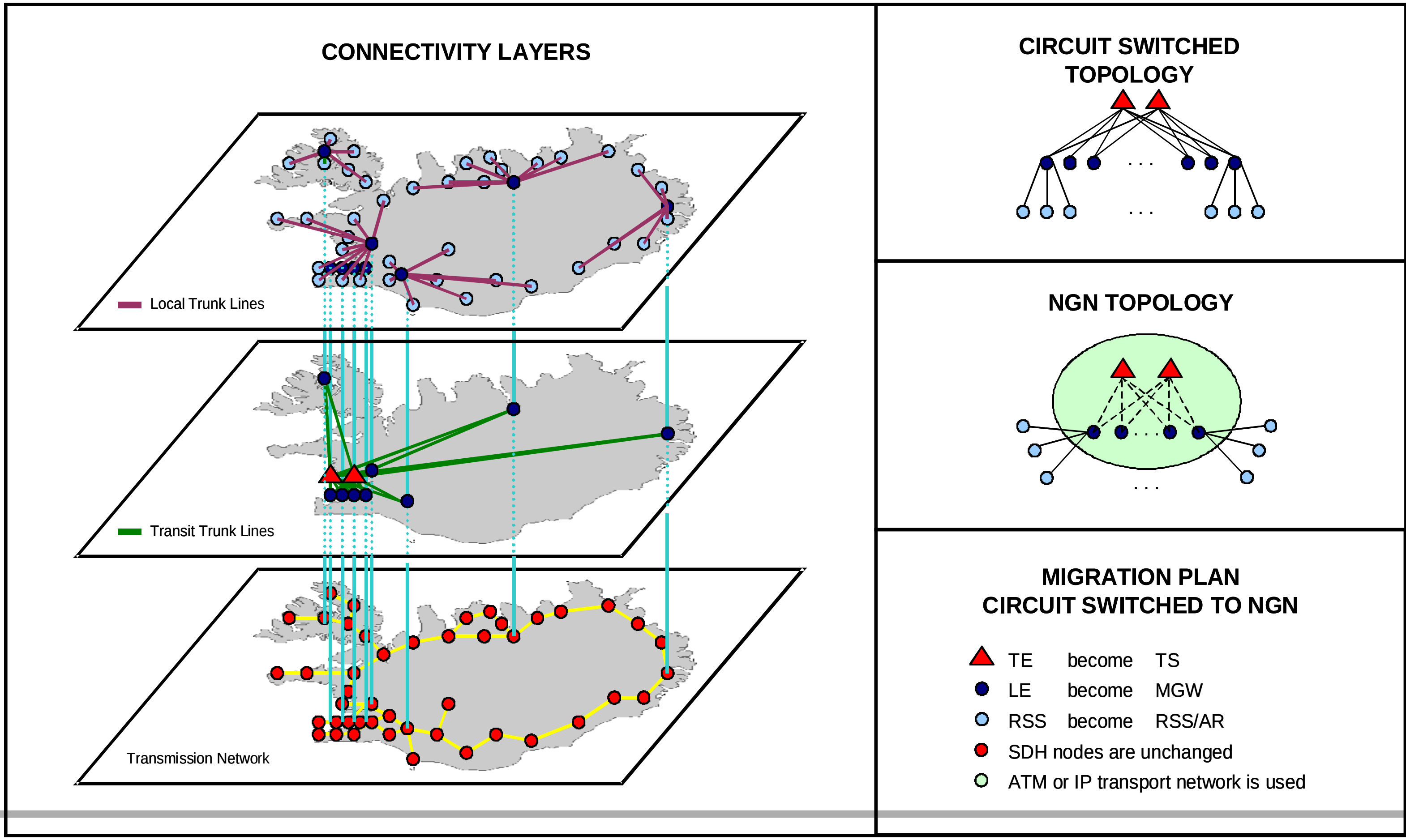
- Modularities to the link sizes: $\sum_{kl}(x_{ij}^{kl} + x_{ji}^{kl}) \leq \sum_l CAP_l \cdot y_{ij}^l \quad \forall \{ij\}$
- Nodes sizes can easily be added as a constraint
- Transmission time limitations
- Demands (stochastic !)
- Step-wise cost functions
- Protection requirements (rings, p-cycles, path protection)

Case stories

- Iceland Telecom (node placement): Article on this published: H.M. Sigurdsson, S.E. Thorsteinsson and T. Stidsen : Cost optimization methods in the design of next generation networks, in IEEE Communications Magazine", 42(9), pp. 118-122, 2004
- Global Connect (network expansion): Master thesis project being worked out currently

The Iceland Case

Iceland Telecom wants to upgrade their network, see below:



Node design

Which size and type of switch should be positioned where ? We want to minimize the price, maximize the robustness of the network and be flexible and prepared for (un-certain future developments). The going trend today is to create **ONE** network, using off-the-shelf components .... This may lead to worse quality but much cheaper solutions ...

The Constants

Find the right number and the right places for the new switches. Given a demand:

- BL_i: Number of requested 2 Mb/s lines from local exchanges to one new switch.
- $LLC_{i,n}$: Monthly cost of leasing a 2 Mb/s line from local exchange i to one new switch n
- CTF_n: Fixed costs for establishing a new switch in location n
- G^{max}: Maximal number of 2 Mb/s lines which can be handled by a new switch

The Variables

- x_n: Should a new switch be established in node n, $x_n \in \{0, 1\}$
- $u_{i,n}$: How many 2 Mb/s lines from switch i should be connected to the new switch n

The Model

Min:

$$\sum_n (CTF_n \cdot x_n + \sum_i LLC_{i,n} \cdot u_{i,n})$$

s.t.:

$$\begin{aligned} \sum_n u_{i,n} &\geq BL_i \quad \forall i \\ \sum_i u_{i,n} &\leq G^{max} \cdot x_n \quad \forall n \end{aligned}$$

$$x_n \in \{0, 1\} \qquad u_{i,n} \in N_0$$

What kind of model is this ???

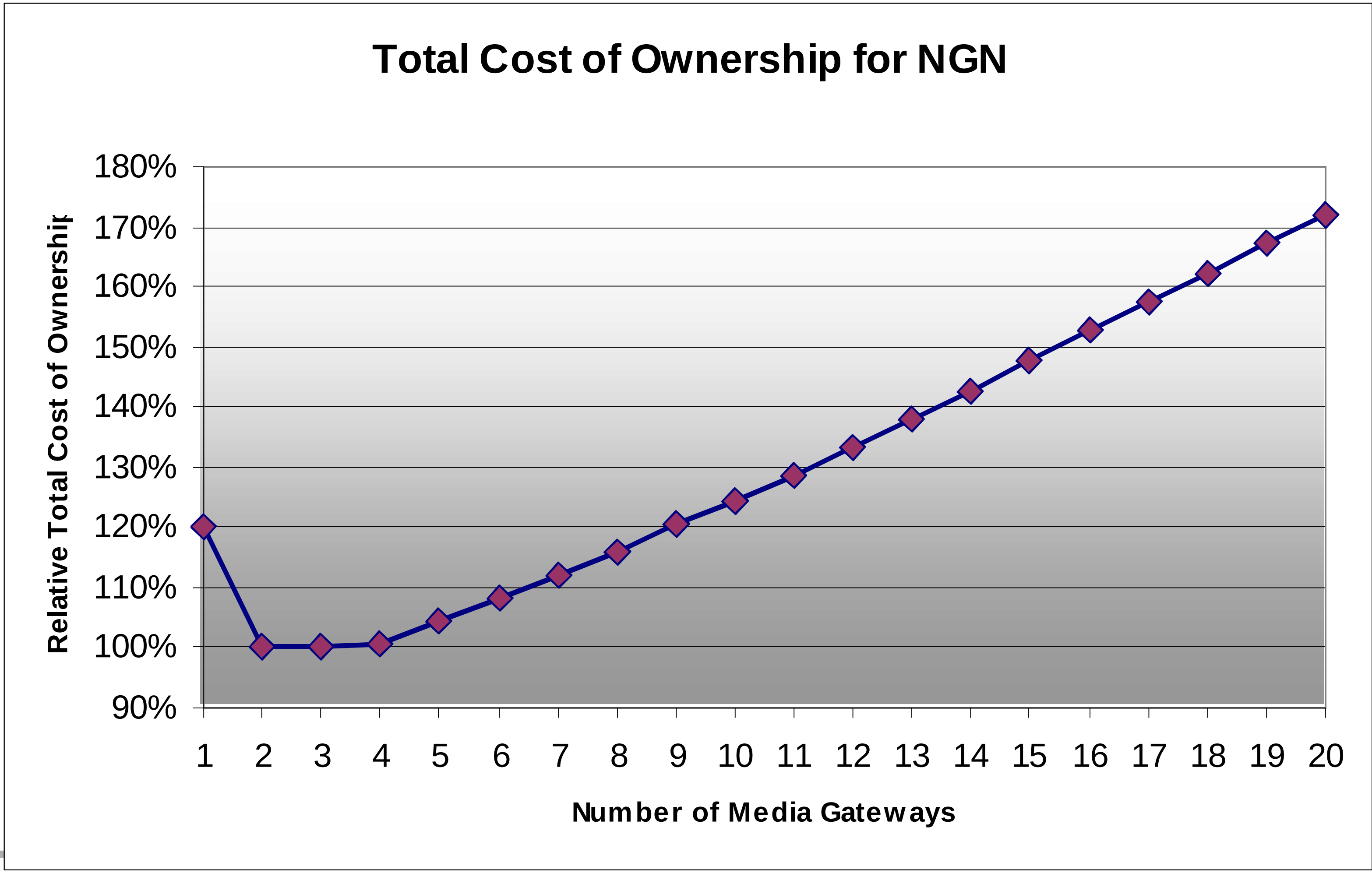
(This is an actual question !)

The Capacitated Facility Location Problem

- A very well known problem
- The switches are depots
- The local telephone exchanges are the customers

The result: 2-4

Look at the following graph:



Global Connect

- Danish company
- Sells "connections" to who-ever might want them: Dark fiber, and different kinds of circuit-switched connections
- See the network (not in slides)

Master project

Whenever a customer request one or more connections, the network is expanded (if necessary), both in terms of digging down more cables and of expanding the switches. Given the number of offers that Global Connect currently handles, it is a significant work, hence it would be a big advantage for the company if optimization models could replace time-consuming engineering work.