Facilities Location

42121

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DTU Management

Textbook

LMW

Love, R.F., J.G. Morris, and G.O. Wesolowsky:
Facilities Location: Models & Methods
North-Holland 1988

(out of print!)

1. Nature of facility location

Where should we locate something?

- Warehouse
- Levels of warehouses
- Fire station
- Missile detection system
- Mine shaft
- Machines in factory
- Nuclear waste dump (obnoxious)

Economics, Geography, Planning

Operations Research: Facility Location

Model typology

Location models

- Continuous
  Location is possible everywhere
- Network
  Location is possible anywhere on a network
- Discrete
  Location is possible only at a finite number of sites

Continuous models

LMW chapters

- 2. Single-facility location
- 3. Variations
  Sphere, Linear, Probabilistic, etc.
- 4. Multi-facility location
- 6. Minimax location
- 7. Continuous location-allocation
- 8. Discrete location-allocation
  Plant location

10. Models of travel distances

Distance measures:
distance between points,
$\mathbf{x} = (x_1, x_2)$ and $\mathbf{a} = (a_1, a_2)$
length of vector, $\mathbf{x} - \mathbf{a}$
$d(\mathbf{x}, \mathbf{a}) = \ell(\mathbf{x} - \mathbf{a})$

Euclidean distance: $\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}$
Rectangular distance: $|x_1 - a_1| + |x_2 - a_2|$
ℓ_p distance: $(|x_1 - a_1|^p + |x_2 - a_2|^p)^{1/p}$, for $p \geq 1$

ℓ_p norm: $\ell_p(\mathbf{y}) = (|y_1|^p + |y_2|^p)^{1/p}$

$p = 1$: Rectangular, $p = 2$: Euclidean

Norm $N(\mathbf{x})$

1. $N(\mathbf{0}) = 0$
2. $N(\mathbf{x}) > 0$ if $\mathbf{x} \neq \mathbf{0}$
3. $N(\mathbf{x} + \mathbf{y}) \leq N(\mathbf{x}) + N(\mathbf{y})$ triangle inequality
4. $N(\alpha\mathbf{x}) = |\alpha| N(\mathbf{x})$ positive homogeneity

$\ell_p(\mathbf{x})$ is a norm for $p \geq 1$

Scalar product: $\mathbf{xy} = x_1 y_1 + x_2 y_2$

Schwarz inequality: $|\mathbf{xy}| \leq \ell_2(\mathbf{x})\ell_2(\mathbf{y})$

Hölder ineq.: $|\mathbf{xy}| \leq \ell_p(\mathbf{x})\ell_q(\mathbf{y})$ where $\frac{1}{p} + \frac{1}{q} = 1$

Norm exercises

1. Investigate the ℓ_∞ norm
   $\ell_\infty(\mathbf{y}) = \lim_{p \to \infty} \ell_p(\mathbf{y})$
2. Is $\alpha_1 \ell_1(\mathbf{x}) + \alpha_2 \ell_\infty(\mathbf{x})$ a norm?
   For what values of the scalars α_1, α_2?
3. Is squared Euclidean a norm?
   $f(\mathbf{y}) = \ell_2^2(\mathbf{y}) = \mathbf{yy}$

If norm property 4 is relaxed to
$G(\alpha\mathbf{x}) = \alpha G(\mathbf{x})$ for $\alpha \geq 0$,
the measure is called a Gauge

Convexity

$\alpha_1 \geq 0,\ \alpha_2 \geq 0,\ \alpha_1 + \alpha_2 = 1$

Convex combinations: $\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2$

Convex set S:
$\mathbf{x}_1, \mathbf{x}_2 \in S \Rightarrow \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 \in S$

Convex function $f(\mathbf{x})$:
$f(\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2) \leq \alpha_1 f(\mathbf{x}_1) + \alpha_2 f(\mathbf{x}_2)$

A local minimum of a convex function is also global

$\ell_p(\mathbf{x})$ is a convex function, for $p \geq 1$

Convexity exercises

Consider a convex function f and define the level sets:
$L(\beta) = \{\mathbf{x} : f(\mathbf{x}) \leq \beta\}$

1. Show that the level sets are convex sets
2. If all level sets are convex, is the function convex?
3. Show that the minimizers of f form a convex set
4. Show that $f(\mathbf{x} - \mathbf{a})$ is a convex function
5. Show that $f_1(\mathbf{x}) + f_2(\mathbf{x})$ is convex, when f_1 and f_2 are convex functions
6. Is a convex function of a convex function convex?
7. Is the unit disk of a norm a convex set?

Differentiability

Derivative: $f'(\mathbf{x}) = (\partial f(\mathbf{x})/\partial x_1, \partial f(\mathbf{x})/\partial x_2)$

$f(\mathbf{y}) \geq f(\mathbf{x}) + f'(\mathbf{x})(\mathbf{y} - \mathbf{x}) \quad \forall\, \mathbf{y}$

Subdifferential at $\mathbf{x}$: the set of vectors $\mathbf{z}$ such that
$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{z}(\mathbf{y} - \mathbf{x}) \quad \forall\, \mathbf{y}$
Each such vector is a subgradient

Directional derivative at $\mathbf{x}$ in direction $\mathbf{d}$:

$$f'_{\mathbf{d}}(\mathbf{x}) = \lim_{\alpha \to 0+} \frac{f(\mathbf{x} + \alpha \mathbf{d}) - f(\mathbf{x})}{\alpha}$$

$f'_{\mathbf{d}}(\mathbf{x}) = f'(\mathbf{x})\mathbf{d}$ if f is differentiable at $\mathbf{x}$

Differentiability and exercises

Almost equivalent statements:
$\mathbf{x}^*$ minimizes f
$f'(\mathbf{x}^*) = \mathbf{0}$
$\mathbf{0}$ is in the subdifferential at $\mathbf{x}^*$
$f'_{\mathbf{d}}(\mathbf{x}^*)$ is nonnegative in all directions

Consider $f(\mathbf{x}) = \sum_{j=1}^{n} w_j \ell_2(\mathbf{x} - \mathbf{a}_j)$ with $w_j > 0\ \forall\, j$

1. Show that f is convex
2. Find the derivative of f
3. Where is f nondifferentiable?
4. Find the directional derivative at these points