

6. Minimax location

6.1 Euclidean distances

6.2 Rectangular distances

6.3 Addenda: Maximin location

Henrik Juel

OR, IMM, DTU

6. Minimax location – p. 1/13

6.1 Euclidean distances

minimize $M(\mathbf{x}) = \max_j \{w_j \ell(\mathbf{x} - \mathbf{a}_j)\}$

M is convex

Unweighted case:

$$w_j = 1 \quad \forall j$$

$$\min. \max_j \{\ell_2^2(\mathbf{x} - \mathbf{a}_j)\}$$

Quadratic programming formulation:

$$\min. x_1^2 + x_2^2 + x_3$$

$$\text{s.t. } 2a_{j1}x_1 + 2a_{j2}x_2 + x_3 \geq a_{j1}^2 + a_{j2}^2 \quad \forall j$$

6. Minimax location – p. 2/13

Smallest circle encl. all points

0. Pick any two points
1. Draw the minimal circle for the two points; pick any point outside the circle (if none, stop)
2. If the triangle has an angle of at least 90° , drop this point and go to step 1
3. Draw the minimal circle for the three points; pick any point outside the circle (if none, stop); drop a suitable old point and go to step 2

Converges to optimal solution

Extensible to weighted case

6. Minimax location – p. 3/13

Euclidean exercises

1. Show that $\max_j \{w_j \ell(\mathbf{x} - \mathbf{a}_j)\}$ is convex
2. Verify the graphical solution procedure

Weighted, constrained case:

$$\min. \max_j \{w_j \ell_2(\mathbf{x} - \mathbf{a}_j)\} \quad \text{s.t. } \mathbf{x} \in F$$

Define $C_j(x_0) = \{\mathbf{x} \in \mathbf{R}^2 : w_j \ell_2(\mathbf{x} - \mathbf{a}_j) \leq x_0\} \quad \forall j$
and $C(x_0) = \cap_j C_j(x_0)$

6. Minimax location – p. 4/13

Computer graphics solution

0. Pick some x_0 and display F
1. Display $C(x_0)$
2. If $C(x_0)$ intersects F , decrease x_0 ;
else increase x_0 ;
go to step 1

Example 6.1

$\mathbf{a}_j$	(3, 2)	(6, 2)	(4, 4)	(8, 1)	(5, 7)
w_j	1	2	2	1	1

Figure 6.1

Non-linear programming

(chapter 13 in Hillier and Lieberman 7th Ed.)

minimize $f(\mathbf{y})$ s.t. $g_j(\mathbf{y}) \geq 0 \forall j$

An optimal solution satisfies
the Karush-Kuhn-Tucker conditions:

$$\sum_j v_j g'_j(\mathbf{y}) = f'(\mathbf{y})$$

$$v_j \geq 0, g_j(\mathbf{y}) \geq 0, v_j g_j(\mathbf{y}) = 0 \forall j$$

If f is convex and the g_j 's are concave,
the KKT conditions are also sufficient for optimality

Weighted case

$$\min. M(\mathbf{x}) = \max_j \{w_j \ell_2(\mathbf{x} - \mathbf{a}_j)\}$$

$$\min. z \quad \text{s.t.} \quad z - w_j^2 \ell_2^2(\mathbf{x} - \mathbf{a}_j) \geq 0 \forall j$$

KKT conditions:

derivative wrt. z :

$$\sum_j v_j = 1$$

derivatives wrt. $\mathbf{x}$:

$$\sum_j v_j w_j^2 (\mathbf{x} - \mathbf{a}_j) = \mathbf{0} \rightarrow$$

$$\mathbf{x} = \sum_j v_j w_j^2 \mathbf{a}_j / \sum_j v_j w_j^2$$

Solution procedure

Start: $v_j^{(0)} = 1/n \forall j$

In iteration t compute:

$$\mathbf{x}^{(t)} = \frac{\sum_j v_j^{(t)} w_j^2 \mathbf{a}_j}{\sum_j v_j^{(t)} w_j^2}$$

$$v_j^{(t+1)} = \frac{v_j^{(t)} w_j \ell_2(\mathbf{x}^{(t)} - \mathbf{a}_j)}{\sum_i v_i^{(t)} w_i \ell_2(\mathbf{x}^{(t)} - \mathbf{a}_i)} \forall j$$

Lower bound

$\mathbf{x} = \mathbf{x}^{(t)}$ is the optimal solution to the minimax location problem

$$\min. \sum_j v_j^{(t+1)} w_j \ell_2(\mathbf{x} - \mathbf{a}_j)$$

Lower bound:

$$M(\mathbf{x}^*) \geq \sum_j v_j^{(t+1)} w_j \ell_2(\mathbf{x}^{(t)} - \mathbf{a}_j)$$

Solution procedure is extensible to multi-facility minimax problems

Example 6.1

$$\begin{array}{c|ccccc} \mathbf{a}_j & (3, 2) & (6, 2) & (4, 4) & (8, 1) & (5, 7) \\ w_j & 1 & 2 & 2 & 1 & 1 \end{array}$$

t	1	2	10	20	50	78
$M(\mathbf{x}^{(t)})$	3.91	3.82	3.52	3.51	3.49	3.477
LB	3.21	3.29	3.44	3.45	3.47	3.474

$$\mathbf{x}^{(78)} = (5.69, 3.60)$$

$$v_j^{(78)} : .000, .002, .144, .519, .334$$

Exercise

Verify the lower bound

6.2 Rectangular distances

$$x_0 \geq |y_1| + |y_2| \Leftrightarrow x_0 \geq \pm y_1 \pm y_2 \text{ (4 inequalities)}$$

$$\min. x_0 \quad \text{s.t. } x_0 \geq w_j \ell_1(\mathbf{x} - \mathbf{a}_j) \quad \forall j$$

may be formulated as an LP problem with $4n$ constraints

Unweighted case:

$$\min. x_0 \quad \text{s.t.}$$

$$x_0 + x_1 + x_2 \geq \max_j \{+a_{j1} + a_{j2}\} = m_1$$

$$x_0 - x_1 + x_2 \geq \max_j \{-a_{j1} + a_{j2}\} = m_2$$

$$x_0 + x_1 - x_2 \geq \max_j \{+a_{j1} - a_{j2}\} = m_3$$

$$x_0 - x_1 - x_2 \geq \max_j \{-a_{j1} - a_{j2}\} = m_4$$

Unweighted case

Adding two constraints yields

$$2x_0 \geq m_1 + m_4 \text{ and}$$

$$2x_0 \geq m_2 + m_3 \Rightarrow$$

$$2x_0 \geq \max\{m_1 + m_4, m_2 + m_3\} = m_5$$

$$2\mathbf{x} = (m_1 - m_2, m_1 + m_2 - m_5) \text{ and}$$

$$2\mathbf{x} = (m_3 - m_4, -m_3 - m_4 + m_5)$$

are both feasible with $2x_0 = m_5$ and hence optimal

Example 6.1, unweighted case

$$\begin{array}{c|ccccc} \mathbf{a}_j & (3, 2) & (6, 2) & (4, 4) & (8, 1) & (5, 7) \\ w_j & 1 & 1 & 1 & 1 & 1 \end{array}$$

$$m_1 = 12, m_2 = 2, m_3 = 7, m_4 = -5; m_5 = 9$$

$$\mathbf{x}^* = (5, 2.5) \text{ or } \mathbf{x}^* = (6, 3.5)$$

or any point on their convex combination

Exercise: 6.9 on page 137

Obnoxious facility location:

$$\max. \min_j \{w_j \ell(\mathbf{x} - \mathbf{a}_j)\} \quad \text{s.t. } \mathbf{x} \in F$$

may be solved using computer graphics