

## 4. Multifacility location

### 4.1. Rectangular distances

### 4.2. $\ell_p$ distances

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## Example 4.1

(table error p. 79:  $w_{121} = 0$ )

$$W(\mathbf{x}_1, \mathbf{x}_2) = .8\ell(\mathbf{x}_1 - \mathbf{a}_1) + .8\ell(\mathbf{x}_1 - \mathbf{a}_2) + 0 + .8\ell(\mathbf{x}_2 - \mathbf{a}_2) + .4\ell(\mathbf{x}_2 - \mathbf{a}_3) + 4\ell(\mathbf{x}_1 - \mathbf{x}_2)$$

$$W(\mathbf{x}) = W(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m) = \sum_i \sum_j w_{1ij} \ell(\mathbf{x}_i - \mathbf{a}_j) + \sum \sum_{i < r} w_{2ir} \ell(\mathbf{x}_i - \mathbf{x}_r)$$

$m = 2$  new facilities at  $\mathbf{x}_1$  and  $\mathbf{x}_2$

$$W(\mathbf{x}_1, \mathbf{x}_2) = \sum_i \sum_j w_{ij} \ell(\mathbf{x}_i - \mathbf{a}_j) + v\ell(\mathbf{x}_1 - \mathbf{x}_2)$$

$W(\mathbf{x}_1, \mathbf{x}_2)$  is a convex function of  $(\mathbf{x}_1, \mathbf{x}_2)$

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## 4.1. Rectangular distances

The problem is separable along the coordinate axes

One-dimensional problem:

$$W(x_1, x_2) = \sum_i \sum_j w_{ij} |x_i - a_j| + v|x_1 - x_2|$$

The natural solution attempt fails:

1. Set  $k = 0$ , guess  $x_2^0$
2. Keeping  $x_2$  fixed at  $x_2^k$ , solve for  $x_1$ :  $x_1 = x_1^k$
3. Keeping  $x_1$  fixed at  $x_1^k$ , solve for  $x_2$ :  $x_2 = x_2^{k+1}$
4. Augment  $k$  and loop until no change

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## Example 4.2

(the example has been slightly changed)

| $a_j$    | 1 | 2 | 3 | 5   |         |
|----------|---|---|---|-----|---------|
| $w_{1j}$ | 0 | 2 | 1 | 1.1 |         |
| $w_{2j}$ | 3 | 1 | 0 | 2.1 | $v = 2$ |

$$x_2 = 5 \rightarrow x_1 = 5 \rightarrow x_2 = 5$$

$$W(5, 5) = 23 \quad \text{not optimal}$$

$$x_1 = x_2 \rightarrow x_1 = x_2 = 2$$

$$x_2 = 2 \rightarrow x_1 = 2 \rightarrow x_2 = 2$$

$$W(2, 2) = 13.6 \quad \text{optimal}$$

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## Linear programm. formulation

$m = 1$ , one dimension:

$$\begin{array}{ll}\min & \sum_j w_j (d_j^+ + d_j^-) \\ \text{s.t.} & x - d_j^+ + d_j^- = a_j \quad \forall j \\ & d_j^+, d_j^- \geq 0 \quad \forall j\end{array}$$

### Exercises

1. Show that  $\ell(\mathbf{x}_1 - \mathbf{x}_2)$  is convex
2. Write the dual of the LP problem
3. Solve the dual without Simplex

## 4.2. $\ell_p$ distances

$$W(\mathbf{x}_1, \mathbf{x}_2) = \sum_i \sum_j w_{ij} \ell_p(\mathbf{x}_i - \mathbf{a}_j) + v \ell_p(\mathbf{x}_1 - \mathbf{x}_2)$$

Hyperbolic approximation:

$$L_p(\mathbf{y}) = L_p(y_1, y_2) = [(y_1^2 + \epsilon)^{p/2} + (y_2^2 + \epsilon)^{p/2}]^{1/p}$$

For  $p = 2, \epsilon = 0$ , differentiating and 'solving' yields

$$\mathbf{x}_1^{k+1} = \frac{\sum_j w_{1j} \mathbf{a}_j / \ell_2(\mathbf{x}_1^k - \mathbf{a}_j) + v \mathbf{x}_2^k / \ell_2(\mathbf{x}_1^k - \mathbf{x}_2^k)}{\sum_j w_{1j} / \ell_2(\mathbf{x}_1^k - \mathbf{a}_j) + v / \ell_2(\mathbf{x}_1^k - \mathbf{x}_2^k)}$$

$$\mathbf{x}_2^{k+1} = \frac{\sum_j w_{2j} \mathbf{a}_j / \ell_2(\mathbf{x}_2^k - \mathbf{a}_j) + v \mathbf{x}_1^k / \ell_2(\mathbf{x}_1^k - \mathbf{x}_2^k)}{\sum_j w_{2j} / \ell_2(\mathbf{x}_2^k - \mathbf{a}_j) + v / \ell_2(\mathbf{x}_1^k - \mathbf{x}_2^k)}$$

Table 4.6:  $n = 5, m = 3, p = 1.8, \epsilon = .001$

## Sufficient optimality conditions

If  $w_{11} \geq \sum_{j=2}^n w_{1j} + v$ , then  $\mathbf{x}_1^* = \mathbf{a}_1$  etc.

If  $v \geq \sum_j w_{1j}$  or  $v \geq \sum_j w_{2j}$ , then  $\mathbf{x}_1^* = \mathbf{x}_2^*$

### Exercises

1. Verify the last optimality condition
2. Solve the problem in example 4.1  
Hint: Use sufficient optimality conditions