

3. Variations

3.1 Location on a sphere

3.2 Point and area destinations with ℓ_1

3.3 Linear facility

3.4 Dynamic location

3.5 Probabilistic location

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3.1 Location on a sphere

Coordinate system:

x_1 latitude (angle from the Equator),

$-90^\circ \leq x_1 \leq 90^\circ$

x_2 longitude (angle from Greenwich),

$-180^\circ \leq x_2 \leq 180^\circ$

Equator: $x_1 = 0^\circ$, Greenwich meridian: $x_2 = 0^\circ$

Antipode to $(x_1, x_2) : (-x_1, x_2 \pm 180^\circ)$

Great circle distance $A(\mathbf{x}, \mathbf{a}) = A[(x_1, x_2), (a_1, a_2)]$

$\cos A = \cos x_1 \cos a_1 \cos(x_2 - a_2) + \sin x_1 \sin a_1$

$\sin A \cdot \partial A / \partial x_1 =$

$\sin x_1 \cos a_1 \cos(x_2 - a_2) - \cos x_1 \sin a_1$

$\sin A \cdot \partial A / \partial x_2 = \cos x_1 \cos a_1 \sin(x_2 - a_2)$

Objective function

$$W(\mathbf{x}) = \sum_j w_j A(\mathbf{x}, \mathbf{a}_j)$$

$$\text{Define } u_j(\mathbf{x}) = w_j / \sin A(\mathbf{x}, \mathbf{a}_j)$$

$$\partial W / \partial x_1 =$$

$$\sum_j u_j(\mathbf{x}) [\sin x_1 \cos a_{j1} \cos(x_2 - a_{j2}) - \cos x_1 \sin a_{j1}]$$

$$[= \sum_j f_{j1}(\mathbf{x})]$$

$$\partial W / \partial x_2 =$$

$$\sum_j u_j(\mathbf{x}) \cos x_1 \cos a_{j1} \sin(x_2 - a_{j2})$$

$$[= \sum_j \cos x_1 f_{j2}(\mathbf{x})]$$

$\mathbf{x} = \mathbf{a}_1$ is a local minimum if and only if

$$w_1 \geq \ell_2[\sum_{j=2}^n f_{j1}(\mathbf{a}_1), \sum_{j=2}^n f_{j2}(\mathbf{a}_1)] \quad \text{etc.}$$

Directional derivative

$$A'_d(\mathbf{a}, \mathbf{a}) = \ell_2(d_1, d_2 \cos a_1)$$

$$W'_d(\mathbf{a}_1) =$$

$$w_1 \ell_2(d_1, d_2 \cos a_{11})$$

$$+ d_1 \sum_{j=2}^n f_{j1}(\mathbf{a}_1) + d_2 \sum_{j=2}^n \cos a_{11} f_{j2}(\mathbf{a}_1)$$

$$\geq$$

$$w_1 \ell_2(d_1, d_2 \cos a_{11})$$

$$- \ell_2[\sum_{j=2}^n f_{j1}(\mathbf{a}_1), \sum_{j=2}^n f_{j2}(\mathbf{a}_1)] \ell_2(d_1, d_2 \cos a_{11}) \quad \forall \mathbf{d}$$

Solution procedure

This Weiszfeld type procedure produces a point and its antipode:

$$\tan x_2^{k+1} = \frac{\sum_j w_j \cos a_{j1} \sin a_{j2} / \sin A(\mathbf{x}^k, \mathbf{a}_j)}{\sum_j w_j \cos a_{j1} \cos a_{j2} / \sin A(\mathbf{x}^k, \mathbf{a}_j)}$$

$$\frac{\tan x_1^{k+1}}{\sin x_2^{k+1}} = \frac{\sum_j w_j \sin a_{j1} / \sin A(\mathbf{x}^k, \mathbf{a}_j)}{\sum_j w_j \cos a_{j1} \sin a_{j2} / \sin A(\mathbf{x}^k, \mathbf{a}_j)}$$

Example 3.1

Spherical location exercises

1. Show that a point is a local minimum if and only if its antipode is a local maximum
2. With $f(\mathbf{x}) = A(\mathbf{x}, \mathbf{a})$, show that $f'_d(\mathbf{a}) = \ell_2(d_1, d_2 \cos a_1)$
Hint: $\cos y \approx 1 - \frac{1}{2}y^2$ for small y
3. Verify the solution procedure

3.2 Point and area destinations

Some customers are uniformly distributed on a rectangular region R with area A and total weight uA

Transportation cost:

$$T(\mathbf{x}) = u \iint_R (|x_1 - z_1| + |x_2 - z_2|) dz_1 dz_2$$

$$W(\mathbf{x}) = \sum_j w_j \ell_1(\mathbf{x} - \mathbf{a}_j) + \sum_i T_i(\mathbf{x})$$

Decomposable and solvable almost like the pure point model with ℓ_1

Example figure 3.7

3.3 Linear facility

Simple linear regression minimizes the sum of squared vertical distances
We want to minimize the weighted sum of perpendicular distances

$$\text{Linear facility: } x_2 = A + Sx_1, \quad S = \tan \theta$$

Perpendicular distance between the line and a point (a_1, a_2) above the line: $f(\theta)$

$$f(\theta) = [a_2 - (A + S a_1)] \cos \theta = (a_2 - A) \cos \theta - a_1 \sin \theta$$

$$f''(\theta) = -f(\theta) < 0$$

$$1 / \cos^2 \theta = \tan^2 \theta + 1 = S^2 + 1$$

$$W(A, S) = \sum_j w_j |A + S a_{j1} - a_{j2}| / \sqrt{S^2 + 1}$$

Solution procedure

Rotating the line with one point fixed yields a concave W for a sufficiently small rotation, $\theta_1 \leq \theta \leq \theta_2$. The minimum of a concave function on an interval is achieved at an endpoint. It suffices to consider lines through two existing facilities

Translating the line with slope fixed: It suffices to consider median lines

Simple combined procedure: Example figure 3.13

[Linear facility exercise:](#)

Solve problem 3.12ab page 70

3. Variations – p.9/11

One-dim. prob. loc. problem

Stochastic customer locations: Y_j

$$W(x) = E(\sum_j w_j |x - Y_j|) = \sum_j w_j E|x - Y_j|$$

$$W'(x) = 2 \sum_j w_j P\{Y_j \leq x\} - \sum_j w_j$$

x^* with $W'(x^*) = 0$ can be found by interval bisection

Example in tables 3.18-19

[Probabilistic location exercises:](#)

1. How would you solve a two-dimensional problem with rectangular distances and stochastic customer locations?
2. Solve problem 3.23 on page 72
Hint: interval bisection is unnecessary

3. Variations – p.11/11

3.5 Probabilistic location

Minimize expected transportation costs

Stochastic weights: W_j

$$W(\mathbf{x}) = E[\sum_j W_j \ell_p(\mathbf{x} - \mathbf{a}_j)] = \sum_j E(W_j) \ell_p(\mathbf{x} - \mathbf{a}_j)$$

Let Y be a random variable with pdf f

$$P\{Y \leq x\} = \int_{-\infty}^x f(y) dy = F(x)$$

$$g(x) = E|x - Y| = \int_{-\infty}^{\infty} |x - y| f(y) dy = \int_{-\infty}^x (x - y) f(y) dy + \int_x^{\infty} (y - x) f(y) dy$$

$$\int_{-\infty}^x (x - y) f(y) dy + \int_x^{\infty} (y - x) f(y) dy$$

$$\text{Leibniz: } g'(x) = 2F(x) - 1$$

$$g''(x) = 2f(x) \geq 0$$

3. Variations – p.10/11