

2. Single-facility location

2.1. Euclidean distance

2.2. Rectangular distance

2.3. ℓ_p distance

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2. Single-facility location – p1/14

2.1. Euclidean distance

Example fig. 2.1

$$\min. W(\mathbf{x}) = \sum_{j=1}^n w_j \ell_2(\mathbf{x} - \mathbf{a}_j) = \sum_j w_j \ell_2(\mathbf{x} - \mathbf{a}_j)$$

$$W'(\mathbf{x}) = \sum_j w_j (\mathbf{x} - \mathbf{a}_j) / \ell_2(\mathbf{x} - \mathbf{a}_j)$$

Varignon frame

Fixed point optimality condition:

$\mathbf{x} = \mathbf{a}_1$ is optimal
if and only if

$$w_1 \geq \ell_2 \left[\sum_{j=2}^n w_j (\mathbf{a}_1 - \mathbf{a}_j) / \ell_2(\mathbf{a}_1 - \mathbf{a}_j) \right]$$

etc.

2. Single-facility location – p2/14

Solution method

Check each existing facility for optimality

If none is optimal, start the Weiszfeld procedure:

$$\mathbf{x}^{k+1} = \frac{\sum_j w_j \mathbf{a}_j / \ell_2(\mathbf{x}^k - \mathbf{a}_j)}{\sum_j w_j / \ell_2(\mathbf{x}^k - \mathbf{a}_j)}$$

Starting point e.g. CoG: $\mathbf{x}^0 = \sum_j w_j \mathbf{a}_j / \sum_j w_j$

Stopping criterion?

Lower bounds on the optimal objective function value:

$$W(\mathbf{x}^*) \geq W(\mathbf{x}^k) - \ell_2[W'(\mathbf{x}^k)] \max_j \{\ell_2(\mathbf{x}^k - \mathbf{a}_j)\}$$

$$W(\mathbf{x}^*) \geq W(\mathbf{x}^k) - \max_j \{W'(\mathbf{x}^k)(\mathbf{x}^k - \mathbf{a}_j)\}$$

Example 2.2

2. Single-facility location – p3/14

Solution method exercises

1. Solve min. $\sum_j w_j \ell_2^2(\mathbf{x} - \mathbf{a}_j)$

2. Show that $LB2 \geq LB1$

$$LB1 = W(\mathbf{x}^k) - \ell_2[W'(\mathbf{x}^k)] \max_j \{\ell_2(\mathbf{x}^k - \mathbf{a}_j)\}$$

$$LB2 = W(\mathbf{x}^k) - \max_j \{W'(\mathbf{x}^k)(\mathbf{x}^k - \mathbf{a}_j)\}$$

$$W(\mathbf{x}) = \sum_j w_j \ell_2(\mathbf{x} - \mathbf{a}_j)$$

$$W'(\mathbf{x}) = \sum_j w_j (\mathbf{x} - \mathbf{a}_j) / \ell_2(\mathbf{x} - \mathbf{a}_j)$$

2. Single-facility location – p4/14

Details on optimality condition

$$\begin{aligned}
 W(\mathbf{x}) &= w_1 \ell_2(\mathbf{x} - \mathbf{a}_1) + \sum_{j=2}^n w_j \ell_2(\mathbf{x} - \mathbf{a}_j) \\
 f(\mathbf{x}) &= \ell_2(\mathbf{x} - \mathbf{a}_1) \Rightarrow f'_d(\mathbf{a}_1) = \ell_2(\mathbf{d}) \\
 \text{Let } \mathbf{z} &= \sum_{j=2}^n w_j (\mathbf{a}_1 - \mathbf{a}_j) / \ell_2(\mathbf{a}_1 - \mathbf{a}_j) \\
 W'_d(\mathbf{a}_1) &= \\
 w_1 \ell_2(\mathbf{d}) + \mathbf{z} \mathbf{d} &\geq \quad (\text{with equality for } \mathbf{d} = -\mathbf{z}) \\
 w_1 \ell_2(\mathbf{d}) - \ell_2(\mathbf{z}) \ell_2(\mathbf{d}) &= \\
 [w_1 - \ell_2(\mathbf{z})] \ell_2(\mathbf{d}) \\
 w_1 \geq \ell_2(\mathbf{z}) &\Rightarrow W'_d(\mathbf{a}_1) \geq 0 \quad \forall \mathbf{d} \\
 W'_{-\mathbf{z}}(\mathbf{a}_1) \geq 0 &\Rightarrow w_1 \geq \ell_2(\mathbf{z})
 \end{aligned}$$

Details on lower bound $LB2$

$\mathbf{x}^*$ is in the convex hull of the existing facilities, C

$$\begin{aligned}
 W'(\mathbf{x}^k)(\mathbf{x}^* - \mathbf{x}^k) &\geq \\
 \min\{W'(\mathbf{x}^k)(\mathbf{y} - \mathbf{x}^k) : \mathbf{y} \in C\} &= \\
 \min_j\{W'(\mathbf{x}^k)(\mathbf{a}_j - \mathbf{x}^k)\} &= \\
 -\max_j\{W'(\mathbf{x}^k)(\mathbf{x}^k - \mathbf{a}_j)\} \\
 W(\mathbf{x}^*) &\geq \\
 W(\mathbf{x}^k) + W'(\mathbf{x}^k)(\mathbf{x}^* - \mathbf{x}^k) &\geq \\
 W(\mathbf{x}^k) - \max_j\{W'(\mathbf{x}^k)(\mathbf{x}^k - \mathbf{a}_j)\}
 \end{aligned}$$

Majority theorem

$\mathbf{x} = \mathbf{a}_1$ is optimal if $w_1 \geq \sum_{j=2}^n w_j$ etc.

$$\begin{aligned}
 W(\mathbf{a}_1) &= \\
 w_1 \ell_2(\mathbf{a}_1 - \mathbf{a}_1) + \sum_{j=2}^n w_j \ell_2(\mathbf{a}_1 - \mathbf{a}_j) &= \\
 0 + \sum_{j=2}^n w_j \ell_2(\mathbf{a}_1 - \mathbf{x} + \mathbf{x} - \mathbf{a}_j) &\leq \quad \text{by triangle ineq.} \\
 \ell_2(\mathbf{a}_1 - \mathbf{x}) \sum_{j=2}^n w_j + \sum_{j=2}^n w_j \ell_2(\mathbf{x} - \mathbf{a}_j) &\leq \\
 \ell_2(\mathbf{x} - \mathbf{a}_1) w_1 + \sum_{j=2}^n w_j \ell_2(\mathbf{x} - \mathbf{a}_j) &= \\
 W(\mathbf{x})
 \end{aligned}$$

Exercise

Use the fixed point optimality condition to prove the majority theorem

MATLAB example

```

w=[1 1 2]; % w_j's
A=[1 0; 0 1; 100 100]; % a_j's
n=length(w);
x=w*A/sum(w) % CoG [50.25 50.25]
for k=1:1000 % 1000 Weiszfeld iter.s
    for j=1:n
        s(j)=w(j)/norm(x-A(j,:));
    end
    x=s*A/sum(s);
end
x % [51.48 51.48]

```

2.2. Rectangular distance

One dimension: $W(x) = \sum_j w_j |x - a_j|$

Number the existing facilities, so

$$a_1 < a_2 < \dots < a_n$$

For $a_i < x < a_{i+1}$:

$$W(x) = \sum_{j=1}^i w_j (x - a_j) + \sum_{j=i+1}^n w_j (a_j - x)$$

$$W'(x) = \sum_{j=1}^i w_j - \sum_{j=i+1}^n w_j = 2 \sum_{j=1}^i w_j - \sum_j w_j$$

The median is optimal:

$$x^* = a_{i^*}, \quad i^* = \min \{i : \sum_{j=1}^i w_j \geq \sum_j w_j / 2\}$$

Two dimensions

$$W(\mathbf{x}) = \sum_j w_j \ell_1(\mathbf{x} - \mathbf{a}_j) = \sum_j w_j (|x_1 - a_{j1}| + |x_2 - a_{j2}|)$$

Decomposable into two one-dimensional problems:

$$W_1(x_1) = \sum_j w_j |x_1 - a_{j1}|$$

$$\text{solution } x_1^*, \quad W_1^* = W_1(x_1^*)$$

$$W_2(x_2) = \sum_j w_j |x_2 - a_{j2}|$$

$$\text{solution } x_2^*, \quad W_2^* = W_2(x_2^*)$$

$$\mathbf{x}^* = (x_1^*, x_2^*), \quad W(\mathbf{x}^*) = W_1^* + W_2^*$$

Example, figure 2.3

Exercises

Consider the Euclidean problem:

min. $W(\mathbf{y}) = \sum_j w_j \ell_2(\mathbf{y} - \mathbf{a}_j)$ with solution $\mathbf{y}^*$

1. Show that $W(\mathbf{y}) \geq \ell_2(W_1^*, W_2^*) \forall \mathbf{y}$
2. Verify the strongest lower bound: $W(\mathbf{y}^*) \geq$

$$\min_{y_1} \left\{ \sum_j w_j \frac{|x_1^k - a_{j1}|}{\ell_2(\mathbf{x}^k - \mathbf{a}_j)} |y_1 - a_{j1}| \right\} +$$

$$\min_{y_2} \left\{ \sum_j w_j \frac{|x_2^k - a_{j2}|}{\ell_2(\mathbf{x}^k - \mathbf{a}_j)} |y_2 - a_{j2}| \right\}$$

Hint: Use the Schwarz inequality

$$|x_1^k - a_{j1}| |y_1 - a_{j1}| + |x_2^k - a_{j2}| |y_2 - a_{j2}| \leq$$

$$\ell_2(\mathbf{x}^k - \mathbf{a}_j) \ell_2(\mathbf{y} - \mathbf{a}_j) \forall \mathbf{y}$$

2.3. ℓ_p distance

$$W(\mathbf{x}) = \sum_j w_j \ell_p(\mathbf{x} - \mathbf{a}_j)$$

$$\ell_p(\mathbf{y}) = (|y_1|^p + |y_2|^p)^{1/p}, \quad p > 1$$

$$\partial \ell_p(\mathbf{y}) / \partial y_1 = \text{sign}(y_1) |y_1|^{p-1} [\ell_p(\mathbf{y})]^{1-p} \quad \text{etc.}$$

Optimality condition:

$\mathbf{x} = \mathbf{a}_1$ is optimal if and only if

$$w_1 \geq \ell_q \left[\sum_{j=2}^n w_j \ell_p'(\mathbf{a}_1 - \mathbf{a}_j) \right], \quad q = \frac{p}{p-1} \quad \text{etc.}$$

Hyperbolic approximation: $|y_1| \rightarrow (y_1^2 + \epsilon)^{1/2}, \quad \epsilon > 0$

$$L_p(\mathbf{y}) = [(y_1^2 + \epsilon)^{p/2} + (y_2^2 + \epsilon)^{p/2}]^{1/p} > \ell_p(\mathbf{y})$$

$$\partial L_p(\mathbf{y}) / \partial y_1 = [L_p(\mathbf{y})]^{1-p} (y_1^2 + \epsilon)^{p/2-1} y_1 \quad \text{etc.}$$

$$W_H(\mathbf{x}) = \sum_j w_j L_p(\mathbf{x} - \mathbf{a}_j) \quad \text{solution } \mathbf{x}_H$$

W_H is differentiable and strictly convex

Weiszfeld type procedure

$$x_1^{k+1} = \frac{\sum_j w_j a_{j1} [L_p(\mathbf{x}^k - \mathbf{a}_j)]^{1-p} [(x_1^k - a_{j1})^2 + \epsilon]^{p/2-1}}{\sum_j w_j [L_p(\mathbf{x}^k - \mathbf{a}_j)]^{1-p} [(x_1^k - a_{j1})^2 + \epsilon]^{p/2-1}}$$

$$L_p(\mathbf{y}) \leq \ell_p(\mathbf{y}) + 2^{1/p} \sqrt{\epsilon} \quad [\Leftarrow y_i^2 + \epsilon \leq (|y_i| + \sqrt{\epsilon})^2]$$

$$W_H(\mathbf{x}) - W(\mathbf{x}) \leq 2^{1/p} \sqrt{\epsilon} \sum_j w_j \quad \forall \mathbf{x}$$

$$W(\mathbf{x}_H) - W(\mathbf{x}^*) < W_H(\mathbf{x}_H) - W(\mathbf{x}^*) \leq$$

$$W_H(\mathbf{x}^*) - W(\mathbf{x}^*) \leq 2^{1/p} \sqrt{\epsilon} \sum_j w_j$$

Examples, tables 2.5, 2.6a, 2.6b

ℓ_p exercises

1. Show that in one dimension W_H is strictly convex

2. Show that $W(\mathbf{y}) \geq \ell_p(W_1^*, W_2^*) \quad \forall \mathbf{y}$

3. Verify the lower bound $W(\mathbf{y}^*) \geq$

$$\min_{y_1} \left\{ \sum_j \frac{w_j |x_1^k - a_{j1}|}{\ell_q(\mathbf{x}^k - \mathbf{a}_j)} |y_1 - a_{j1}| \right\} +$$

$$\min_{y_2} \left\{ \sum_j \frac{w_j |x_2^k - a_{j2}|}{\ell_q(\mathbf{x}^k - \mathbf{a}_j)} |y_2 - a_{j2}| \right\}$$

Stopping criterion: Stop when

$$W(\mathbf{x}^k) \leq (1 + \epsilon) LB \rightarrow$$

$$W(\mathbf{x}^k)/(1 + \epsilon) \leq W(\mathbf{x}^*) \leq W(\mathbf{x}^k) \text{ or}$$

$$0 \leq \frac{W(\mathbf{x}^k) - W(\mathbf{x}^*)}{W(\mathbf{x}^*)} \leq \epsilon$$