Projections
– *Fourier-Motzkin Elimination*

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Outline

Introduction

If you have started reading the RKM (RKM stands for Richard Kipp Martin, who have written the book from which most of the chapters are copied) you will see that the mathematical requirements are significant. Dont despair we will gradually work our way into the matter ... You should start by reading:

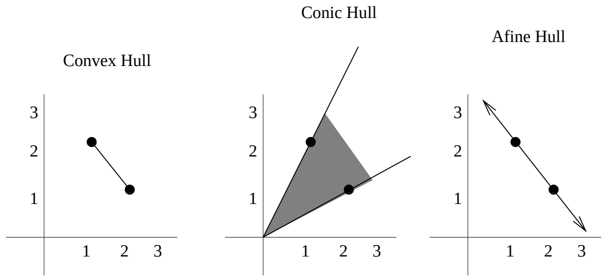
- Appendix A2
- Chap. 2.1 to (including) Farkas Lemma, 2.16 (p. 50).

Appendix A2, I

Contains many definitions in a very compact form: **Convex Hull**,**Conic Hull** and **Affine Hull**, defined by:

- $\lambda_1 x^1 + \lambda_2 x^2 + \ldots + \lambda_k x^k$, where $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i \geq 0$ (convex hull)
- $\lambda_1 x^1 + \lambda_2 x^2 + \ldots + \lambda_k x^k$, where $\lambda_i \geq 0$ (conic hull)
- $\lambda_1 x^1 + \lambda_2 x^2 + \ldots + \lambda_k x^k$, where $\sum_{i=1}^k \lambda_i = 1$ (affine hull)

Appendix A2, II

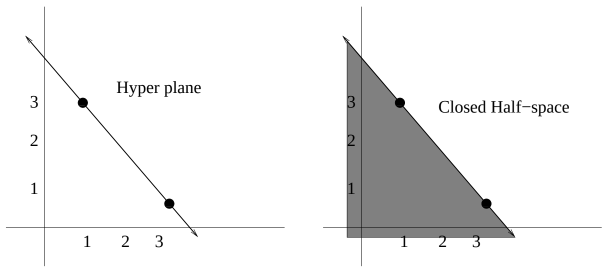


Appendix A2, III

Particularly p. 638 gives some important definitions:

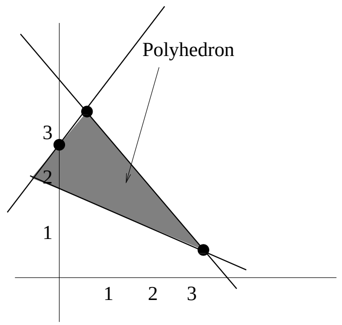
- $\{x \in \mathbf{R}^n | a^T x = \beta\}$, defines a *hyperplane*.
- $\{x \in \mathbf{R}^n | a^T x \leq \beta\}$ (or $\{x \in \mathbf{R}^n | a^T x \geq \beta\}$), defines a *closed half-space*.

Appendix A2, IIII



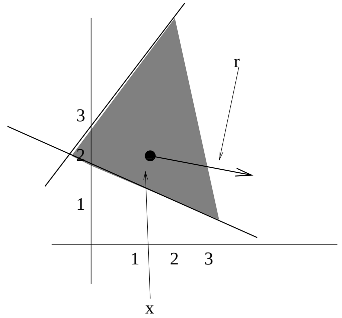
Appendix A2, IIIII

The **polyhedron** is the intersection of a finite number of closed half-spaces, i.e. $P = \{x \in \mathbf{R}^n | Ax \leq b\}$, where A is a $m \times n$ matrix (m rows and n columns).



Appendix A2, IIIIII

Given a polyhedron $P = \{x \in \mathbf{R}^n | Ax \leq b\}$, r is a ray of P if and only if, given $x \in P$, the set $\{y \in \mathbf{R}^n | y = x + \lambda r, \lambda \in \mathbf{R}\}$.



Appendix A2, IIIIIII

A very important result is given at the bottom of p. 638:
$P = conv(\{x^1, \ldots, x^q\}) + cone(\{x^{q+1}, \ldots, x^r\})$
This we will use a lot for the Benders algorithm.

Linear systems

In RKM the following notation is used:

objective: minimise

$$c^T x$$

s.t.

$$\begin{aligned} Ax &\geq b \\ x &\geq 0 \end{aligned}$$

Dual variables are called u. (Notice this is for Linear Programs)

Fourier-Motzkin Elimination

How can we check if feasible values of $\vec{x} = \{x_1, \ldots, x_n\}$ exists ? Given a system of constraints (2.5, 2.6 and 2.7):

$$\begin{aligned} x_1 + a_{i2}x_2 + \ldots + a_{in}x_n &\geq b_i \; i = 1, \ldots, m_1 \\ -x_1 + a_{i2}x_2 + \ldots + a_{in}x_n &\geq b_i \; i = m_1 + 1, \ldots, m_1 + m_2 \\ 0 * x_1 + a_{i2}x_2 + \ldots + a_{in}x_n &\geq b_i \; i = m_1 + m_2 + 1, ., m \end{aligned}$$

For a point $\vec{x} = \{x_1, \ldots, x_n\}$ to be a feasible point in the above system (polyhedron) it **must** satisfy all the equations (i.e. there is an implicit AND between all constraints ...)

Developing Projections

Why can we do this ? (see p. 39): Given a system of n variables and m constraints:

$$
\begin{aligned}
x_1 + a_{i2}x_2 + . + a_{in}x_n &\geq b_i \; i = 1, ., m_1 \\
-x_1 + a_{i2}x_2 + . + a_{in}x_n &\geq b_i \; i = m_1 + 1, ., m_1 + m_2 \\
a_{i2}x_2 + . + a_{in}x_n &\geq b_i \; i = m_1 + m_2 + 1, ., m
\end{aligned}
$$

Hence we have m_1 $+1$ inequalities, m_2 -1 inequalities and $m - m_1 - m_2$ 0 inequalities.

Ignoring the 0 inequalities

$$
\begin{aligned}
x_1 &\geq b_i - a_{i2}x_2 - . - a_{in}x_n \; i = 1, ., m_1 \\
x_1 &\leq a_{i2}x_2 + . + a_{in}x_n - b_i \; i = m_1 + 1, ., m_1 + m_2
\end{aligned}
$$

Putting it together (2.10)

$$
\begin{array}{ccc}
b_i - a_{i2}x_2 - . - a_{in}x_n & \leq x_1 \leq & a_{i2}x_2 + . + a_{in}x_n - b_i \\
 & \Downarrow & \\
b_i - a_{i2}x_2 - . - a_{in}x_n & \leq & a_{i2}x_2 + . + a_{in}x_n - b_i
\end{array}
$$

Notice that to check that there is a common value which is legal for ALL pairs of $\leq$ and $\geq$ constraints. How many pairs of these constraints do we have to check ? $m_1 \cdot m_2$!

Re-adjusting

$$
\begin{aligned}
(a_{k2} + a_{i2})x_2 + . + (a_{kn} + a_{in})x_n &\geq (b_k + b_i) \\
i = 1, ., m_1, k = m_1 + 1, ., m_1 + m_2 & \\
a_{i2}x_2 + . + a_{in} &\geq b_i \\
i = m_1 + m_2 + 1, ., m &
\end{aligned}
$$

Intuitively I

The combination of the constraints simply means that all the constraints which "push" the value of x_1 upwards (the $\geq$) constrains should be checked for conflicts with the constraints which "drag" the value of x_1 downwards (the $\leq$) constraints.

Intuitively II

- This is the explanation of the possible large growth in the number of equations.
- If only 1 or -1 equations exists, that variable is only bounded from one side ! This means two things:
  - That variable is unbounded !
  - The inequalities which contain the variables are guaranteed feasible and we can simply remove these constraints (key observation 4, p. 44)

Projection correctness

The Fourier-Motzkin Elimination is correct:

- **Lemma 2.6**: If P is not empty then the FM projection of P is neither.
- **Lemma 2.7**: If FM projection of P is not empty then neither is P.

Hence, by removing the variables we may test if feasible points of a polyhedron exists: **Proposition 2.8**.

Projections

Given a set of constraints (a linear system) Ex. 2.9:

$$\begin{aligned} 3x_1 - 6x_2 + 6x_3 &\geq 12 \quad (E1) \\ -x_1 + 3x_2 - 2x_3 &\geq 3 \quad (E2) \\ x_1 - 4x_2 + x_3 &\geq -15 \quad (E3) \\ -x_1 + 0x_2 + x_3 &\geq -15 \quad (E4) \end{aligned}$$

We can now perform projections by removing the variables !

Scaling

Multiply first constraint with $\frac{1}{3}$:

$$\begin{array}{rcll} x_1 - 2x_2 + 2x_3 & \geq & 4 & \frac{1}{3}(E1) \\ -x_1 + 3x_2 - 2x_3 & \geq & 3 & (E2) \\ x_1 - 4x_2 + x_3 & \geq & -15 & (E3) \\ -x_1 + 0x_2 + x_3 & \geq & -15 & (E4) \end{array}$$

Now the constants multiplied to the variables x_1 are either 1 or -1

Removing first variable

Now we can remove the variable from the constraint system by adding the constraints (E1) to (E2) and E(4) and further add (E3) to (E2) and (E4):

$$\begin{array}{rcll} x_2 + 0x_3 & \geq & 7 & \frac{1}{3}(E1) + (E2) \\ -2x_2 + 3x_3 & \geq & -11 & \frac{1}{3}(E1) + (E4) \\ -x_2 - x_3 & \geq & -12 & (E2) + (E3) \\ -4x_2 + 2x_3 & \geq & -30 & (E3) + (E4) \end{array}$$

Removing second variable

After we have removed the second variable the system looks like:

$$\begin{array}{rcll} \frac{3}{2}x_3 & \geq & \frac{3}{2} & \frac{1}{2}(E1) + (E2) + \frac{1}{2}(E4) \\ -x_3 & \geq & -5 & \frac{1}{3}(E1) + 2(E2) + (E3) \\ \frac{1}{2}x_3 & \geq & -\frac{1}{2} & \frac{1}{3}(E1) + (E2) + \frac{1}{4}(E3) + \frac{1}{4}(E4) \end{array}$$

Final solution

After removal of all variables:

$$\begin{array}{rcll} 0 & \geq & -4 & \frac{2}{3}(E1) + \frac{8}{3}(E2) + (E3) + \frac{1}{3}(E4) \\ 0 & \geq & -6 & (E1) + 4(E2) + \frac{3}{2}(E3) + \frac{1}{2}(E4) \end{array}$$

These constraint decides whether the system contains feasible solutions ! If all the righthand sides are less or equal to 0 there is a solution otherwise not.

Key observations 1

For each variable in the system:

- Scale the variable constant to ± 1 *using positive numbers only*
- Add all constraints with $+1$ to all constraints with -1, ignoring all constraints with constant 0

Key observations 2

The coefficient when scaling the constraints u generates the new aggregated constraints: $(u^t B)x \geq u^T b$.
These multipliers are the solutions to the following system: $A^T u = 0$.

Key observations 3,5

The method is **not** an efficient computational method: Given a system with m constraints and n variables there is a worstcase number of constraints of: $(\frac{1}{2})^{(2^{n+1}-2)} m^{2^n}$. This is a **huge** growth.

Key observations 4

What if only positive (negative) coefficients to the "project out" variable exists ? The only projection possible is $u = 0$ and the polyhedron $Ax \geq b$ is unbounded.

To recap

The projection method is **essential** in RKM:

- The method is called the Fourier-Motzkin elimination method
- It is a method to reduce the number of variables in a system ....
- .... but the number of constraints may be increased
- When all the variables have been replaced, it is trivial to check for feasibility

Lemma 2.16 (Farkas)

The main theorem regarding projection of systems:
Either $Ax \geq b$ has a solution
or $A^T u = 0$ and $b^T u > 0$ and $u \geq 0$
Hence projections can uniuqely tell us whether a system has a solution or not ...

Optimizing using projections

How can we apply projections when we have an objective function ?

$$\begin{aligned} z_0 - c^T x &\geq 0 \\ Ax &\geq b \\ x_j &\geq 0 \end{aligned}$$

Optimizing using projections

In the end:

$$\begin{aligned} z_0 &\geq d_k \\ 0 &\geq d_{k'} \end{aligned}$$

Now simply find the most limiting constraint i.e. $max(d_k)$ and this is the optimal value of z_0. If any of $d_{k'} > 0$ the system is infeasible.