



Duality and Projections

– *What's the use ?*

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Outline

- Projections revisited ...
- Farka's lemma
- Proposition 2.22 and 2.23
- Duality theory (2.6)
- Complementary slackness (2.7)
- Sensitivity analysis (2.8)



Just to recap

From the last lecture we know the most important:
The so-called Fourier-Motzkin projections may be applied to systems of linear inequalities: To remove variables from the system, which is on the other hand transformed to include other linear inequalities.



Projection Purpose

These projections can be used for two purposes:

- To check if feasible solutions exists to the systems (Farkas lemma 2.16, summing up Lemma 2.6 and Lemma 2.7)
- Find optimality values of Linear programs, by projecting all other variables than the optimality variable z_0 !

But as you were told in the last decomposition lecture, the number of constraints in the end becomes huge.





Why are projections interesting ?

This (part of the) course is about **decomposition algorithms**, so why bother about projections ?

- They provide an excellent tool for understanding the decomposition algorithms
- As we will see in this lecture they give a different view on duality theory

I expect that all of you have heard about duality theory, so why repeat it ? Because duality theory is extremely important and projections give a different understanding about it.



Important readings

- Section 2.3 (entire section)
- Section 2.4.2
- Lemma 2.16 (page 50)
- Proposition 2.22 and 2.23 (page 53-54)
- Section 2.6 (Duality Theory, entire section)
- Section 2.7 (Complementary Slackness, Theorem 2.33 and 2.34 and example 2.38)
- Section 2.8 (Sensitivity Analysis, until (not including Lemma 2.40))



Linear systems

objective: minimise

$$c^T x$$

s.t.

$$\begin{aligned} Ax &\geq b \\ x &\geq 0 \end{aligned}$$



Just looking at feasibility

Given a set of constraints (a linear system) Ex. 2.9:

$$3x_1 - 6x_2 + 6x_3 \geq 12 \quad (E1)$$

$$-x_1 + 3x_2 - 2x_3 \geq 3 \quad (E2)$$

$$x_1 - 4x_2 + x_3 \geq -15 \quad (E3)$$

$$-x_1 + 0x_2 + x_3 \geq -15 \quad (E4)$$





Scaling

Multiply first constraint with $\frac{1}{3}$:

$$x_1 - 2x_2 + 2x_3 \geq 4 \quad \frac{1}{3}(E1)$$

$$-x_1 + 3x_2 - 2x_3 \geq 3 \quad (E2)$$

$$x_1 - 4x_2 + x_3 \geq -15 \quad (E3)$$

$$-x_1 + 0x_2 + x_3 \geq -15 \quad (E4)$$

Now the constants multiplied to the variables x_1 are either 1 or -1



Removing first variable

Now we can remove the variable from the constraint system by adding the constraints (E1) to (E2) and E(4) and further add (E3) to (E2) and (E4):

$$x_2 + 0x_3 \geq 7 \quad \frac{1}{3}(E1) + (E2)$$

$$-2x_2 + 3x_3 \geq -11 \quad \frac{1}{3}(E1) + (E4)$$

$$-x_2 - x_3 \geq -12 \quad (E2) + (E3)$$

$$-4x_2 + 2x_3 \geq -30 \quad (E3) + (E4)$$



Scaling again

$$x_2 + 0x_3 \geq 7 \quad \frac{1}{3}(E1) + (E2)$$

$$-x_2 + \frac{3}{2}x_3 \geq \frac{-11}{2} \quad \frac{1}{6}(E1) + \frac{1}{2}(E4)$$

$$-x_2 - x_3 \geq -12 \quad (E2) + (E3)$$

$$-x_2 + \frac{1}{2}x_3 \geq -\frac{15}{2} \quad \frac{1}{4}(E3) + \frac{1}{4}(E4)$$

And we can again add out variable x_2



Removing second variable

After we have removed the second variable the system looks like:

$$\frac{3}{2}x_3 \geq \frac{3}{2} \quad \frac{1}{2}(E1) + (E2) + \frac{1}{2}(E4)$$

$$-x_3 \geq -5 \quad \frac{1}{3}(E1) + 2(E2) + (E3)$$

$$\frac{1}{2}x_3 \geq -\frac{1}{2} \quad \frac{1}{3}(E1) + (E2) + \frac{1}{4}(E3) + \frac{1}{4}(E4)$$



Scaling

$$\begin{aligned} x_3 &\geq 1 && \frac{1}{3}(E1) + \frac{2}{3}(E2) + \frac{1}{3}(E4) \\ -x_3 &\geq -5 && \frac{1}{3}(E1) + 2(E2) + (E3) \\ x_3 &\geq -1 && \frac{2}{3}(E1) + 2(E2) + \frac{1}{2}(E3) + \frac{1}{2}(E4) \end{aligned}$$

And we are ready to remove x_3 .



Final solution

After removal of all variables:

$$\begin{aligned} 0 &\geq -4 && \frac{2}{3}(E1) + \frac{8}{3}(E2) + (E3) + \frac{1}{3}(E4) \\ 0 &\geq -6 && (E1) + 4(E2) + \frac{3}{2}(E3) + \frac{1}{2}(E4) \end{aligned}$$

These constraint decides whether the system contains feasible solutions ! If all the lefthand sides are less or equal to 0 there is a solution otherwise not. $u_1 = \{\frac{2}{3}, \frac{8}{3}, 1, \frac{1}{3}\}$



Why “record” ?

Why did we “record” the scaling and additions ?
From Lemma 2.10 (p. 45) we have the following knowledge (for a given constraint k):

- $A^T u^k = 0$
- $b^T u^k = d_k$

Let's test that on our example:



What do we have?

$$A = \begin{vmatrix} 3 & -6 & 6 \\ -1 & 3 & -2 \\ 1 & -4 & 1 \\ -1 & 0 & 1 \end{vmatrix}$$

$$u_1 = \{\frac{2}{3}, \frac{8}{3}, 1, \frac{1}{3}\}$$



$$A^T \cdot u_1$$

$$A^T = \begin{vmatrix} 3 & -1 & 1 & -1 \\ -6 & 3 & -4 & 0 \\ 6 & -2 & 1 & 1 \end{vmatrix}$$

$$u_1 = \begin{vmatrix} \frac{2}{3} \\ \frac{8}{3} \\ 1 \\ \frac{1}{3} \end{vmatrix}$$

$$A^T \cdot u_1 = \{0, 0, 0\}$$



Farkas Lemma (p. 50)

Farkas Lemma states that either the system

$$Ax \geq b$$

or

$$A^T u = 0 \quad \wedge \quad b^T u > 0 \quad \wedge \quad u \geq 0$$

Proof follows in three parts ...



Farkas I: Either one or the other

Both systems cannot have the solution $(\bar{x}, \bar{u})$ because:

$$Ax \geq b \Rightarrow \bar{u}^T A\bar{x} \geq \bar{u}^T b > 0$$

But that contradicts that:

$$A^T u = 0$$

Hence at most one of the systems may be satisfied.



Farkas II: $Ax \geq b$ has a solution

Ok, we are done ... (that was easy ...)



Farkas III: $Ax \geq b$ has no solution

Ok, the original $Ax \geq b$ has no solution, but when we then use projections we will, because of Corollary 2.11 (just summarizing the effects of projection) end up with a system where:

$$0 \geq d_k > 0$$

Then by Lemma 2.10 we have a solution $\bar{u}$

$$A^T u = 0 \quad \wedge \quad b^T u > 0 \quad \wedge \quad u \geq 0$$



Application of projection to mixed systems

What if we have the following polyhedron:

$$P = \{(x, y) \in \mathfrak{R}^{n_1} \times \mathfrak{R}^{n_2} | Ax + By \geq d_k\}$$

We can then eliminate some of the variables:

$$proj_y(P) = \{y \in \mathfrak{R}^{n_2} | (u^k)^T By \geq (u^k)^T d_k, k = 1 \dots q\}$$

This is nothing fancy ! Just stop the projections before all variables are projected out.



Proposition 2.22

This is hard ! (I will not go through the proof), if:

$$P = \{(x, y) \in \mathfrak{R}^{n_1} \times \mathfrak{R}^{n_2} | Ax + By \geq b\}$$

Then the projection:

$$proj_y(P) = \{y \in \mathfrak{R}^{n_2} | (u^k)^T By \geq (u^k)^T b, k = 1 \dots q\}$$

Is correctly defined by the vectors $u^1, \dots, u^r$ from the projection cone:

$$C_x(P) = \{u \in \mathfrak{R}^m | A^T u = 0, u \geq 0\}$$



Proposition 2.23

Correspondingly: If we perform the projection and create the multipliers $u^i, i = 1, \dots, q$, then the extreme rays of the projection cone:

$$C_x(P) = \{u \in \mathfrak{R}^m | A^T u = 0, u \geq 0\}$$

are contained in this set.



Significance of Prop. 2.22 and 2.23

Why are these propositions 2.22 and 2.23 important? Because we want to use projection theory to project out **some** of the variables from our problem. The big problem is though that we know that this will create a huge set of constraints. What proposition 2.22 says is that we only need to look at a **subset** of these, the subset defined by the extreme rays, and that this subset will always be included into projection, proposition 2.23.



Duality in light of projections

Given the projection theory we can study duality theory from a different point of view. I assume all of you already know this, but it can be fruitful to look at it from a different point of view:

- Weak duality (Lemma 2.28)
- Strong duality (Theorem 2.29)



Duality in projections

$$\begin{aligned}
 z_0 &\leq 36 && 1(E0) + \frac{5}{2}(E1) + 6(E3) \\
 z_0 &\leq 45 && 1(E0) + \frac{3}{2}(E1) + \frac{9}{2}(E2) + \frac{9}{2}(E3) \\
 z_0 &\leq 42 && 1(E0) + 3(E2) + \frac{4}{3}(E1) + 4(E3) \\
 z_0 &\leq 54 && 1(E0) + 9(E2) + 6(E3) \\
 0 &\leq 4 && \frac{2}{9}(E1) + \frac{2}{3}(E3) + (E4) \\
 0 &\leq 6 && (E2) + (E3) + (E4)
 \end{aligned}$$

Our result is then: 36 and our duals: $u\{\frac{5}{2}, 0, 6, 0\}$.



The u multipliers

They are the scaling factors we use during projection. In the end, for each constraint in the system, these specify how the constraint is generated: (Lemma 2.10)

$$A^T u^k = 0 \quad b^T u^k = d_k$$

When we have the **most binding** constraint, we can hence now see how the objective changes depending on the original b values !



Weak Duality (Lemma 2.28)

Given: An optimal solution $\bar{x}$ to the system $\min\{c^T x | Ax \geq b\}$ and a solution $\bar{u}$ to the system $A^T u = c$, then $c^T \bar{x} \geq b^T \bar{u}$. Now we will use projection theory to prove this.



Weak Duality II

The proof is quite simple because of the projection theory:

1. $\bar{u} \geq 0 \Rightarrow u^T Ax \geq u^T b$, because this is an aggregate constraint system
2. If $Ax \geq b$ then $u^T Ax \geq u^T b$, if there is a solution, any generated constraint will also apply
3. We have by assumption $A^T u = c$, so when $u^T Ax \geq u^T b \Rightarrow c^T x \geq u^T b$

In our projection, this means that each of our constraints in the final projection system gives a lower bound on the z_0 value.



Strong duality

As you should know, there is further a strong duality theorem stating:

The optimal solution to $\min\{c^T x | Ax \geq b\}$ $\bar{x}$ then the optimal value $c^T \bar{x}$ is equal to the optimal value $b^T \bar{u}$ of the dual program $\max\{b^T u | A^T u = c, u \geq 0\}$.



From projection theory

Again we have (for the primal system):

$$\begin{aligned} z_0 &\geq d_k, & k = 1, \dots, q \\ 0 &\geq d_k, & k = q + 1, \dots, r \end{aligned}$$



Hence ...

- Because there is an optimal solution to the primal problem, $d_k \leq 0$ for $k = q + 1, \dots, r$
- Further, $q \geq 1$ because the problem otherwise would be unbounded and hence not have an optimal solution
- Then we know that the optimal solution to the primal problem is $z_0 = \max\{d_k | k = 1, \dots, q\}$
- For this maximal solution we know $b^T u = d$ (Lemma 2.10)



Finally

Then we know that this dual solution u is optimal, because it has the same objective value as the primal solution and optimality is given by the weak duality theorem.



What are the dual values u Really ???

The objective changes by the marginal value times ϵ if the RHS is increased by ϵ (improvement if maximizing, decrease if minimizing). Notice that when we “created” the u vector we **ONLY** considered the A matrix ! All the calculations were made without considering the actual value of the b ! Hence, if we change b , it will not change the u values ! But then the u values gives us a direct link between the original right hand sides (RHS) and the optimal value:



What are the dual values u Really ??? II

- If one of the b values is changed ϵ we can calculate the change in the optimal value by simply multiplying with the corresponding u value.
- If we change the value “too much” another constraint in the final tableau may be come binding: **This is why the dual values are only of value for marginal changes !**



Complementary Slackness

Given a system $\min\{c^T x \mid Ax \geq b\}$ we may use projection to get a constraint system complementary slackness states: (Theorem 2.33)

$$(b - A\bar{x})^T \bar{u} = 0$$

This is a necessary and sufficient condition for the pair $(\bar{x}, \bar{u})$ to be optimal



Optimality $\Rightarrow (b - A\bar{x})^T \bar{u} = 0$

Assume optimality, then using the possibility to make aggregate constraints we have:

$$c^T \bar{x} = (\bar{u}^T A) \bar{x} \geq \bar{u}^T b$$

If $A_{i\bullet} \bar{x} > b_i$ and $u_i > 0$ then the above inequality is strict, which is not possible for optimal values because then we know $c^T \bar{x} = \bar{u}^T b$ for the corresponding constraint in the aggregate constraint set.



$(b - A\bar{x})^T \bar{u} = 0 \Rightarrow \text{optimality}$

$$(b - A\bar{x})^T \bar{u} = 0$$

$\Downarrow$

$$b^T \bar{u} = (A\bar{x})^T \bar{u}$$

But since $\bar{u}$ is feasible, $A^T \bar{u} = c$ hence $b^T \bar{u} = c^T \bar{x}$ and then because of weak duality, the pair $(\bar{x}, \bar{u})$ is optimal.



But ...

One weakness with complementary slackness is that $\bar{u} = 0$ we can conclude nothing about $b - A\bar{x}$. Correspondingly, $b - A\bar{x} = 0$ does not imply that $\bar{u} > 0$.

Strict Complementary slackness deals with this problem.



Sensitivity Analysis

Given an optimal solution, how much can we change the parameters without changing the solution (that is, the current optimum is not any longer the optimum) ?

- b : This we can analyze now.
- c : Closely connected with the **reduced costs**, which you will hear much more about when we get to Dantzig-Wolfe/Column generation.
- A : Much harder ...



Sensitivity Analysis

Again, assuming we have a feasible solution:

$$z_0 \geq d_k, \quad k = 1, \dots, q$$

We know $b^T u^k = d_k$, hence if we change the righthand side b of one constraint in the original problem, we will change each of the constraints in the projected system by an amount u^k .



Sensitivity Analysis

Given the optimally binding constraint k' , we check all the other constraints for the allowable increase before one of these becomes binding:

$$\min_{k \neq k'} \left\{ \frac{b_{k'} - b_k}{u_k - u_{k'}} \right\}$$



Sensitivity Analysis example

In eq. 2.34-2.40:

$$\begin{array}{ll} z_0 \geq -22 & (E0) + 2(E2) + (E4) \\ z_0 \geq -24 & (E0) + 3(E2) + 0.5(E5) \\ z_0 \geq -44 & (E0) + 4(E2) + 2(E3) + (E4) \\ z_0 \geq -35 & (E0) + 4(E2) + (E3) + 0.5(E5) \\ z_0 \geq -30 & (E0) + (E1) + 4(E2) \\ z_0 \geq -32 & (E0) + 4(E2) + (E6) \\ 0 \geq -11 & (E2) + (E3) \end{array}$$

What if we raise the righthandside of (E2) b_2 ?





Sensitivity Analysis example

$$\min\left\{\frac{-22 - (-24)}{3 - 2}, \frac{-22 - (-44)}{4 - 2}, \frac{-22 - (-35)}{4 - 2}, \frac{-22 - (-30)}{4 - 2}, \frac{-22 - (-32)}{4 - 2}, 11\right\}$$

$$= \min\left\{\frac{2}{1}, \frac{22}{2}, \frac{13}{2}, \frac{8}{2}, \frac{10}{2}, 11\right\}$$

Error in book on page 68 in the denominator in the first part of the *min* expression (wrong order of subtraction).



SOB

The mnemonic rule SOB: Sensible Odd Bizarre is taken from Hillier & Lieberman (p. 249) but originates from Prof. Arthur T. Benjamin.



SOB II

- Formulate the primal problem in either maximization form or minimization form, and then the dual problem automatically will be in the other form.
- Label the different forms of functional constraints and of constraints on individual variables (domain constraints) as being sensible, odd or bizarre. The labeling of the functional constraints depends on whether the problem is a *maximization* or a *minimization* problem.



SOB III

- For each constraint on an *individual variable* in the dual problem, use the form that has the same label as for the functional constraint in the primal problem that corresponds to this dual variable.
- For each *functional constraint* in the dual problem, use the form that has the same label as for the constraint on the corresponding individual variable in the primal problem.





Table (p. 250 H & L.)

Label	Primal Problem (or Dual Problem)	Dual Problem (or Primal Problem)
	Maximize Constraint i :	Minimize Variable y_i (or x_i):
Sensible	$\leq$ form	$y_i \geq 0$
Odd	$=$ form	Unconstrained
Bizarre	$\geq$ form	$y'_i \leq 0$
	Variable x_j (or y_i):	Constraint j
Sensible	$x_j \geq 0$	$\geq$ form
Odd	Unconstrained	$=$ form
Bizarre	$x'_i \leq 0$	$\leq$ form



Radiation Therapy (02709)

minimize:

$$z = 0.4 \cdot x + 0.5 \cdot y$$

constrained to:

$$0.3 \cdot x + 0.1 \cdot y \leq 2.7 \quad \text{Bizzare}$$

$$0.5 \cdot x + 0.5 \cdot y = 6 \quad \text{Odd}$$

$$0.6 \cdot x + 0.4 \cdot y \geq 6 \quad \text{Sensible}$$

$$x \geq 0 \quad \text{Sensible}$$

$$y \geq 0 \quad \text{Sensible}$$



(Dual) Radiation Therapy (02709)

maximize:

$$z = 2.7 \cdot \alpha + 6 \cdot \beta + 6 \cdot \gamma$$

constrained to:

$$0.3 \cdot \alpha + 0.5 \cdot \beta + 0.6 \cdot \gamma \leq 0.4 \quad \text{Sensible}$$

$$0.1 \cdot \alpha + 0.5 \cdot \beta + 0.4 \cdot \gamma \leq 0.5 \quad \text{Sensible}$$

$$\alpha \leq 0 \quad \text{Bizzare}$$

$$\beta \quad \text{Odd}$$

$$\gamma \geq 0 \quad \text{Sensible}$$