Dantzig-Wolfe bound and Dantzig-Wolfe cookbook

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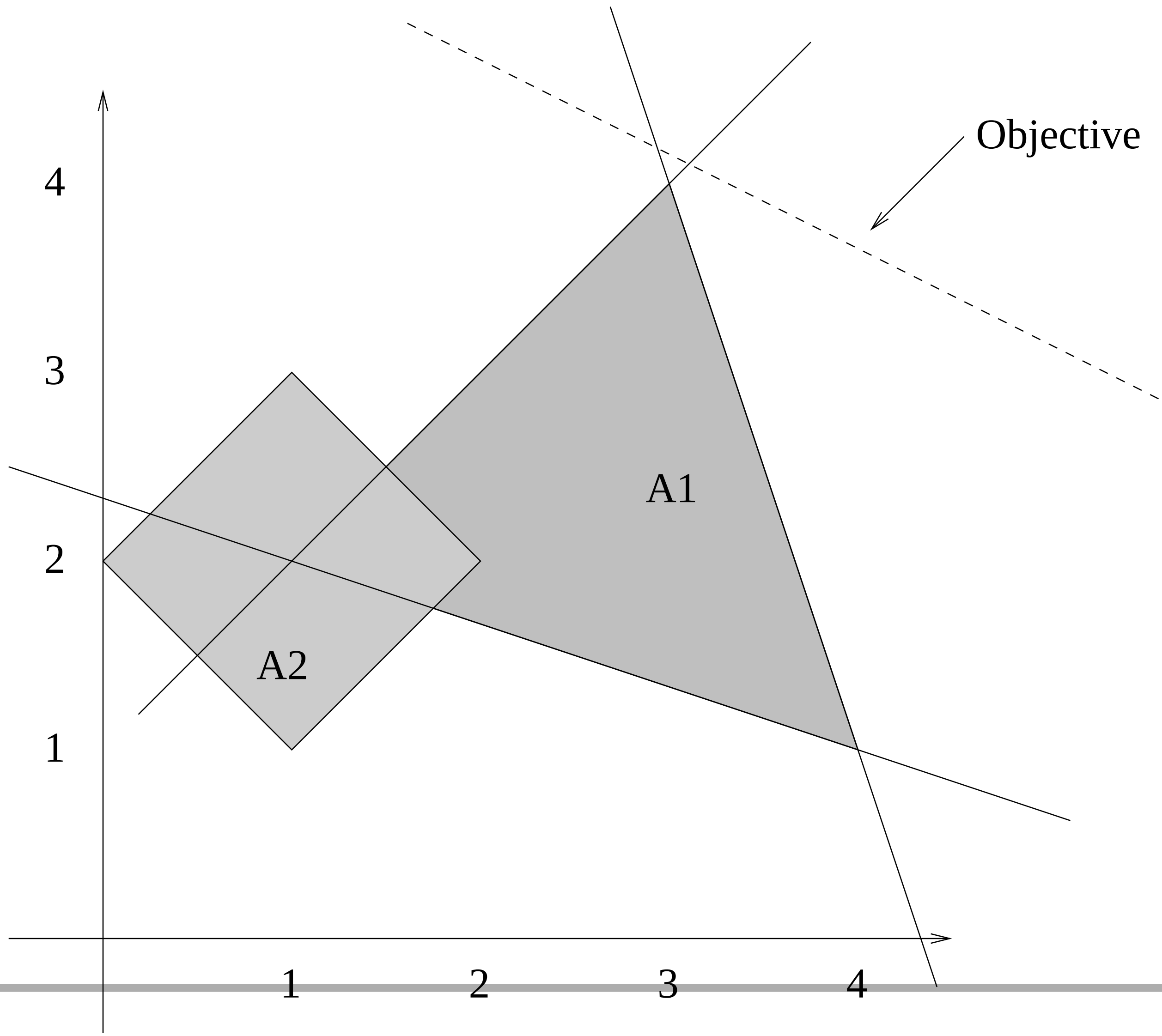
Outline

- LP strength of the Dantzig-Wolfe
- The exercise from last week ...
- The Dantzig-Wolfe cookbook

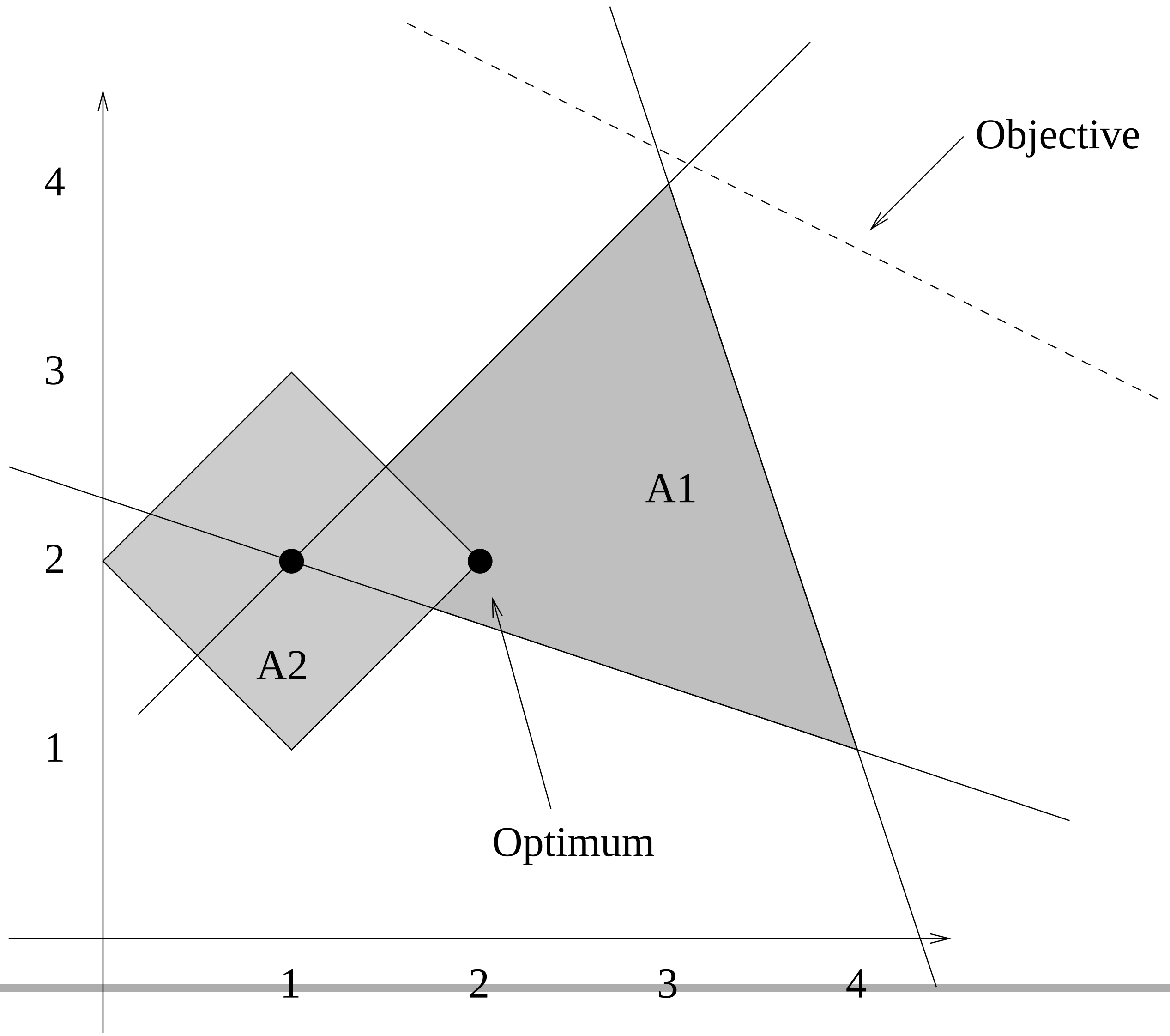
Dantzig-Wolfe bound strength

I have previously stated that the main reason for using DZ is speed. This is not entirely true: The LP bound can be strengthened using Dantzig-Wolfe !

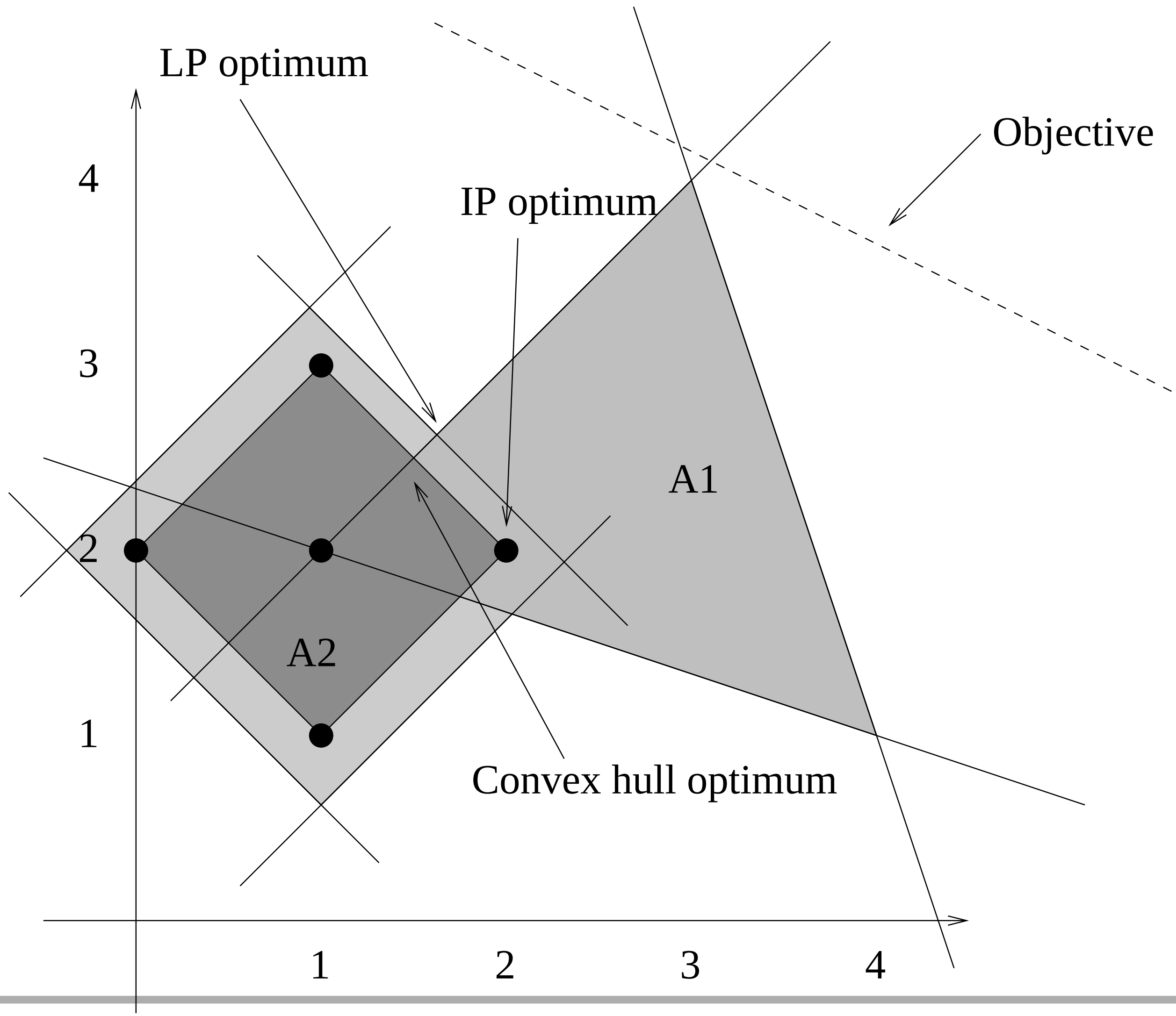
Example from last time



Example from last time: MIP version



Example from last time: MIP version



Condition for bound improvement

So, besides improving the *speed* of LP solution, we can get an improved bound if:

- The sub-problem (A_2) has non-integral solution property
- So we have to solve an integer problem which is hard in the sub-problem ...
- We may need specialized algorithms ...
- We may be able to improve solution speed if a number of smaller sub-problems are created.

Job planning under budget constraint

- i: Jobs to perform
- j: Persons to perform job
- $C_{i,j}$: Cost if person j performs job i
- $T_{i,j}$: Training cost to train j for job i

Direct model

Min:

$$\sum_{i,j} C_{i,j} x_{i,j}$$

s.t.:

$$\begin{aligned}
\sum_i x_{i,j} &= 1 \quad \forall j \\
\sum_j x_{i,j} &= 1 \quad \forall i \\
\sum_{i,j} T_{i,j} \cdot x_{i,j} &\leq 18
\end{aligned}$$

$$x_{i,j} \in [0,1]$$

The possibilities of decomposition

Which constraint should be “removed” ?

- The assignment constraints ?
  - The sub-problem creates full schedules assigning all jobs to all persons
  - The master problem selects combinations of the schedules.
- The budget constraint ???

1 Sub-problem (only the constraints)

Min: $c^r_s \quad = \quad ???$

s.t.:

$$\sum_i x_{i,j} \quad = \quad 1 \quad \forall j$$

$$\sum_j x_{i,j} \quad = \quad 1 \quad \forall i$$

$$x_{i,j} \in \{0, 1\}$$

Notice: $\alpha \leq 0$ and $\beta \in R$

1 Master problem

Min:

$$\sum_s C_s \cdot y_s$$

s.t.:

$$\begin{aligned} \sum_s T_s \cdot y_s &\leq 18 \quad (\alpha \leq 0) \\ \sum_s y_s &= 1 \quad (\beta \in R) \end{aligned}$$

$$y_s \in [0, 1]$$

1 Sub-problem

Min:
$$c_s^r = \sum_{i,j} C_{i,j} x_{i,j} - \alpha \sum_{i,j} T_{i,j} \cdot x_{i,j} - \beta$$
$$= \sum_{i,j} (C_{i,j} - \alpha \cdot T_{i,j}) x_{i,j} - \beta$$

s.t.:
$$\sum_i x_{i,j} = 1 \quad \forall j$$
$$\sum_j x_{i,j} = 1 \quad \forall i$$
$$x_{i,j} \in \{0,1\}$$

Notice: $\alpha \leq 0$ and $\beta \in R$

2 decomposition

- The budget constraint:
  - The “schedules” are now job selections which obeys the training budget constraint. The master problem job is then to mix (fractionally) the different schedules
  - The subproblem is then a *knapsack* problem

...

2 Sub-problem (without objective function)

Min:

$$c_{i,j} \quad = \quad ???$$

s.t.:

$$\sum_{i,j} T_{i,j} \cdot x_{i,j} \quad \leq \quad 18$$

$$x_{i,j} \in \{0,1\}$$

Notice: $\alpha_j \in R$ and $\beta_i \in R$

2 Master problem

Min:

$$\sum_s C_s \cdot y_s$$

s.t.:

$$\begin{aligned}
\sum_s A_j^s y_s &= 1 \quad \forall j \quad (\alpha_j \in R) \\
\sum_s B_i^s y_s &= 1 \quad \forall i \quad (\beta_i \in R) \\
\sum_s y_s &= 1 \qquad\quad (\gamma \in R)
\end{aligned}$$

$$y_s \in [0, 1]$$

2 Sub-problem

Min:

$$
\begin{aligned}
c_{i,j} &= \sum_{i,j} C_{i,j} \cdot x_{i,j} - \sum_{j} \alpha_j \sum_{i} x_{i,j} - \sum_{i} \beta_i \sum_{j} x_{i,j} - \gamma \\
&= \sum_{i,j} (C_{i,j} - \alpha_j - \beta_i) x_{i,j} - \gamma
\end{aligned}
$$

s.t.:

$$
\begin{aligned}
\sum_{i,j} T_{i,j} \cdot x_{i,j} &\leq 18 \\
x_{i,j} &\in \{0, 1\}
\end{aligned}
$$

Notice: $\alpha_j \in R$ and $\beta_i \in R$

Dantzig-Wolfe cookbook

- Write up the original MIP problem (carefully !)
- Write up dual variables for each type of constraints and set dual variable bounds
- Move set of constraints (A_2) to sub-problem (ignore the sub-problem objective for now).
- **WHAT IS THE SUB-PROBLEM INTUITIVELY ???** What does one solution correspond to ?

Dantzig-Wolfe cookbook

- Write up the master problem, with the new variables, both constraints and objective function
- Finally, write up the sub-problem objective: $min c_{red} = (c - \pi \cdot A_1)x - \alpha$, i.e. the original cost minus the column times the dual variables.

Open question: How should the sub-problem be solved ?