



Benders (once more ...)

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Outline

- The algorithm
- The assignment from last week
- GAMS Implementation



Definitions

- RBMP: Relaxed Benders Master Problem
- BPSP: Benders **P**rimal Sub-Problem: The direct formulation for finding the u values.
- BDSP: Benders **D**ual Sub-Problem: The original problem with fixed $\bar{y}$ values
- RGP: Ray Generation Problem



But ...

When we solve the BPSP (the u problem) what if:

- Problem solved: No problem (we add optimality cut)
- Problem infeasible: We have a solution which is infeasible because of the y values. Hence we need a *feasibility cut* based on an extreme ray.
- Problem un-bounded: Original problem un-bounded !



Benders Alorithm

Initialize RBMP, RGP, BDSP/BPSP

$Z_{LO} = -\infty, Z_{UP} = \infty, \bar{y}, \epsilon$

while $Z_{UP} - Z_{LO} > \epsilon$

 Solve BDSP($\bar{y}$) or BPSP($\bar{y}$) getting u
 if feasible

 update Z_{UP}

 Add Optimality constraint

else

 Solve RGP($\bar{y}$) getting u

 Add Feasibility constraint

 Solve RBMP getting $\bar{y}$

 update Z_{LO}

endwhile



Example

Min:

$$z_0 = 5x - 3y$$

s.t.:

$$x + 2y \geq 4$$

$$2x - y \geq 0$$

$$x - 3y \geq -13$$

$$x \geq 0$$

$$y \in \{0, \dots, 10\}$$

Optimal value: $(x=2.5, y=5), z=-2.5$



Example 10.6: The B matrix

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

$$b^T = [4, 0, -13]$$

$$c^T = [5]$$

$$f^T = [-3]$$



The Optimality cuts

The optimality cuts we are looking for are:

$$z_0 \geq f^T y + (\bar{u}^i)^T (b - By) \quad i = 1, \dots, q$$

Which becomes:

$$z_0 \geq -3y + (\bar{u}^i)^T ([4, 0, -13]^T - [2, -1, -3]^T y)$$

Hence, every time we find new optimality cut values, i.e. u values, we only have to plug them in, in the above equation.



Example

Min:

$$z_0 = 5x - 3y$$

s.t.:

$$x \geq 4 - 2y$$

$$2x \geq 0 + y$$

$$x \geq -13 + 3y$$

$$x \geq 0$$

$$y \in \{0, \dots, 10\}$$

Optimal value: $(x=2.5, y=5), z=-2.5$



1. Iteration: Subproblem

Max:

$$z_1 = (4 - 2y)u_1 + y \cdot u_2 + (-13 + 3y)u_3$$

s.t.:

$$\begin{aligned} u_1 + 2u_2 + u_3 &\leq 5 \\ u_i &\geq 0 \end{aligned}$$



Initialisation

Initial solution

$$\bar{y}_0 = 0.$$

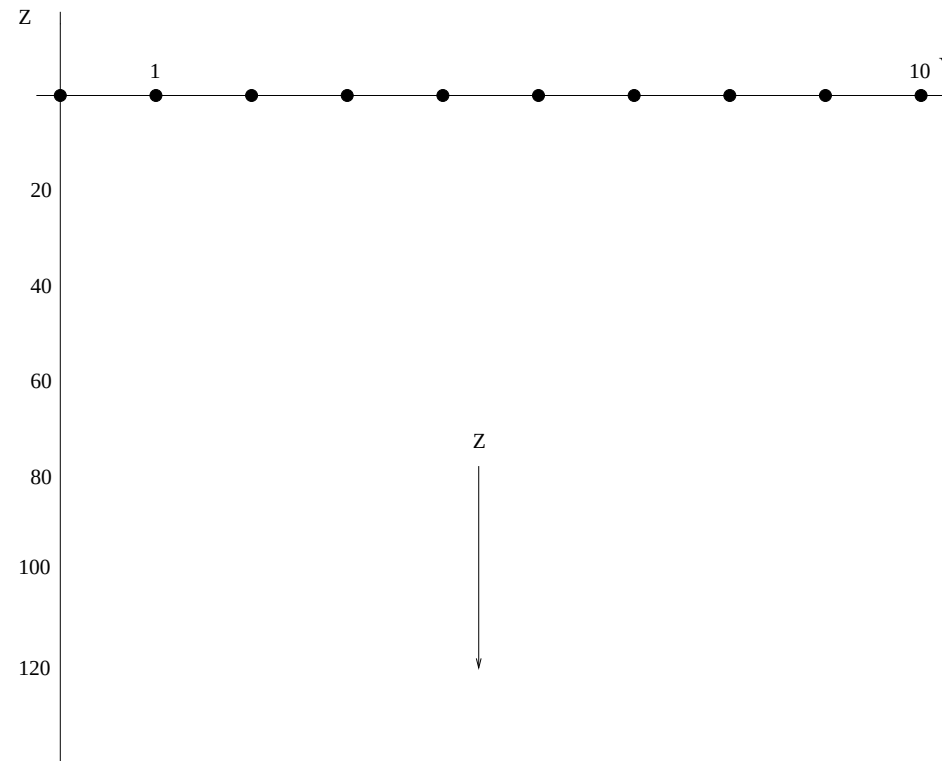
Upper bound:

$$UP_0 = \infty.$$

Lower bound:

$$LO_0 = -\infty.$$

Graphical BMP, initialisation





1. Iteration: Subproblem

Max:

$$z_1 = 4u_1 - 13u_3$$

s.t.:

$$\begin{aligned} u_1 + 2u_2 + u_3 &\leq 5 \\ u_i &\geq 0 \end{aligned}$$



1. Iteration: Subproblem

Optimal solution:

$$\begin{aligned}\bar{z}_1 &= 20 \\ \bar{u} &= (5, 0, 0)\end{aligned}$$

Giving a new upper bound:

$UP_1 = \min\{\bar{z}_1, UP_0(= \infty)\} = 20$. And arriving at the cut:

$$\begin{aligned}z_0 &\geq -3y + (5, 0, 0)^T ([4, 0, -13]^T - [2, -1, -3]^T y) \\ &= \\ z_0 &\geq -13y + 20\end{aligned}$$



1. Iteration: Benders Master Problem

Min:

$$z_1$$

s.t.:

$$\begin{aligned} z_1 &\geq 20 - 13y \\ y &\in \{0, \dots, 10\} \end{aligned}$$



1. Iteration: Benders Master Problem

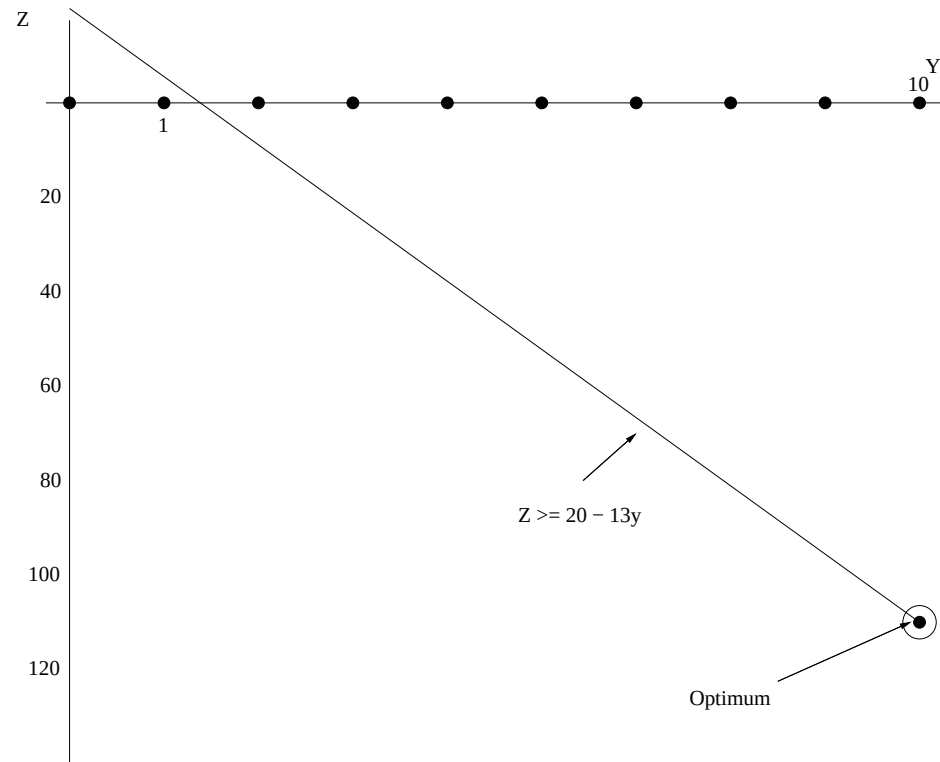
Optimal solution

$$\begin{aligned}\bar{z}_1 &= -110 \\ \bar{y} &= 10\end{aligned}$$

New lower bound:

$$LO_1 = \max\{\bar{z}_1, LO_0(= -\infty)\} = -110.$$

Graphical BMP, 1. iteration





2. iteration: Primal Subproblem

Max:

$$z_2 = -16u_1 + 10u_2 + 17u_3 - 30$$

s.t.:

$$\begin{aligned} u_1 + 2u_2 + u_3 &\leq 5 \\ u_i &\geq 0 \end{aligned}$$



2. iteration: Primal Subproblem

Optimal solution:

$$\begin{aligned}\bar{z}_2 &= 55 \\ \bar{u} &= (0, 0, 5)\end{aligned}$$

New upper bound: $UP_2 = \min\{\bar{z}_2, UP_1(= 20)\} = 20.$



2. iteration: Benders Master Problem

Min:

$$z_2$$

s.t.:

$$z_2 \geq 20 - 13y$$

$$z_2 \geq -65 + 12y$$

$$y \in \{0, \dots, 10\}$$



2. iteration: Benders Master Problem

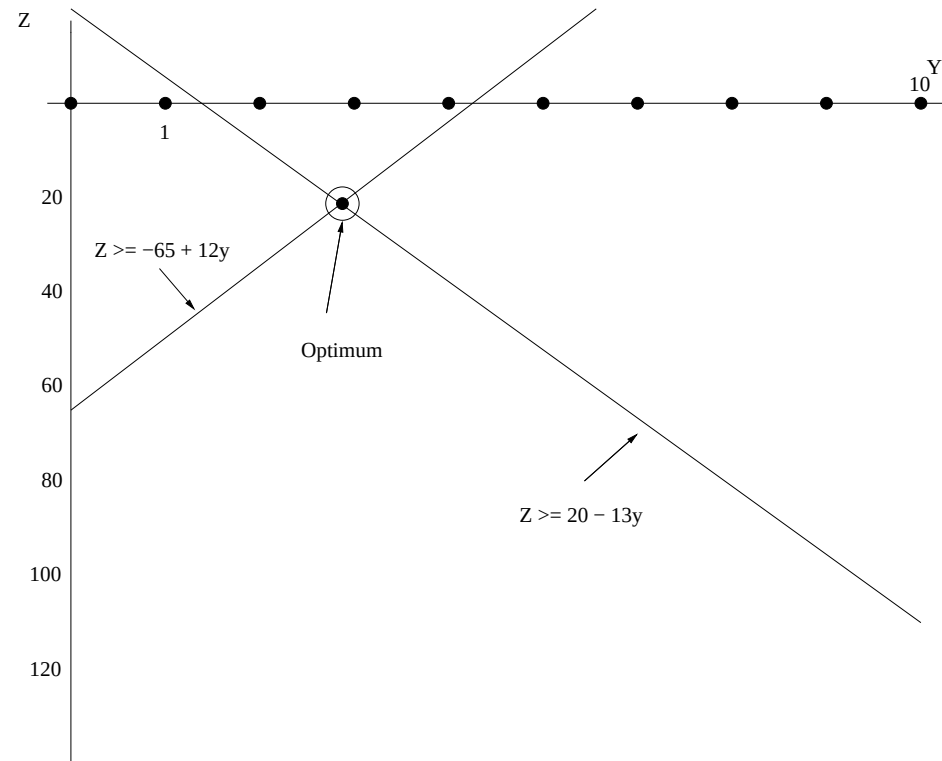
Optimal solution:

$$\begin{aligned} z_2 &= -19 \\ y &= 3 \end{aligned}$$

New upper bound:

$$LO_2 = \min\{\bar{z}_2, LO_1(-110)\} = -19.$$

Graphical BMP, 2. iteration



3. iteration: Primal Subproblem

Max:

$$z_3 = -2u_1 + 3u_2 - 4u_3 - 9$$

s.t.:

$$\begin{aligned} u_1 + 2u_2 + u_3 &\leq 5 \\ u_i &\geq 0 \end{aligned}$$



3. iteration: Primal Subproblem

Optimal solution:

$$\begin{aligned}\bar{z}_2 &= -1.5 \\ \bar{u} &= (0, 2.5, 0)\end{aligned}$$

New upper bound:

$$UP_3 = \min\{\bar{z}_3, UP_2(= 19)\} = -2.5.$$

3. iteration: Benders Master Problem

Min:

$$z_3$$

s.t.:

$$z_3 \geq 20 - 13y$$

$$z_3 \geq -65 + 12y$$

$$z_3 \geq -0.5y$$

$$y \in \{0, \dots, 10\}$$



3. iteration: Benders Master Problem

Optimal solution:

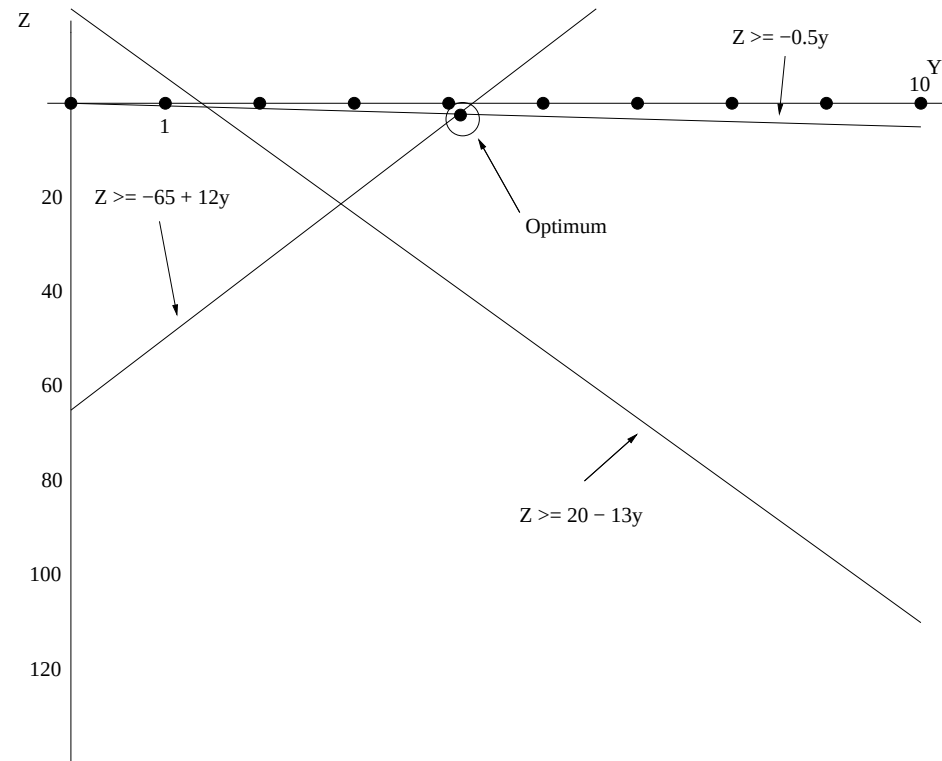
$$\begin{aligned} z_3 &= -2.5 \\ y &= 5 \end{aligned}$$

New

lowerbound: $LO_3 = \max\{\bar{z}_1, LO_3(= -19)\} = -2.5.$

Now: $LO_3 = UP_3$

Graphical BMP, 3. iteration





Exercise 10.1 (RKM)

Several constraints are added without improving bound (in case of minimization, non-increasing) why: Primal degeneracy: Possible to include new variables into the simplex tableau without changing the objective value (the new variables may stay at value zero). **Dual degeneracy:** If the current “solution” has **several** active constraints.



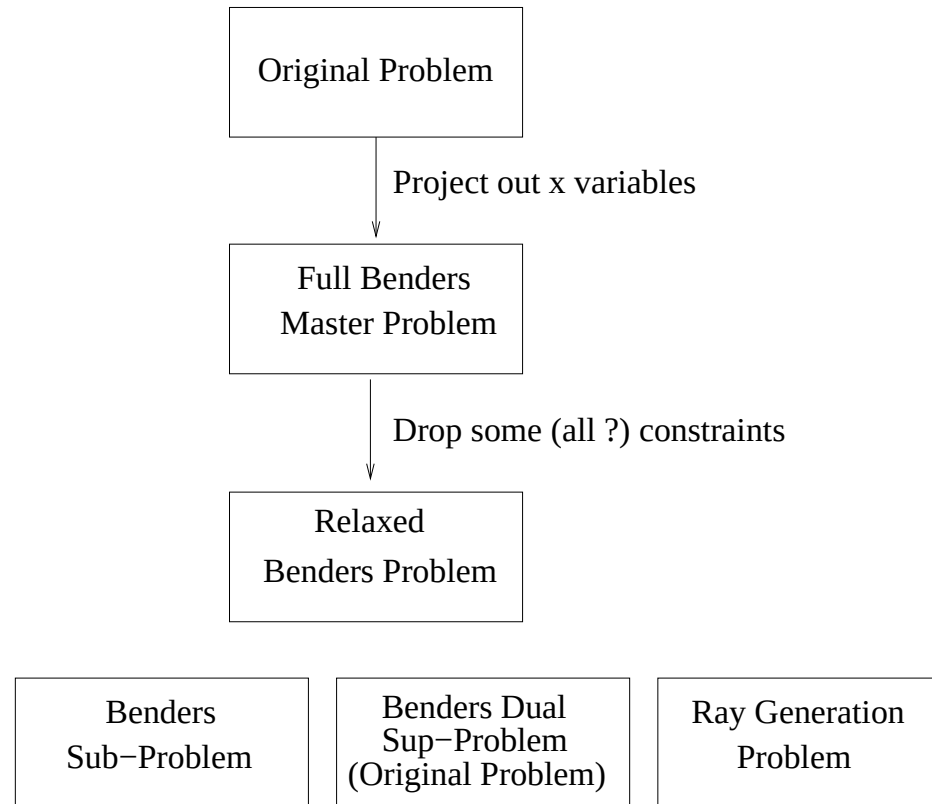
Exercise 10.2 (RKM)

If the Bender's subproblem is infeasible what does that imply about the original problem ?

- Original problem un-bounded ! Why ?
- Bender's Dual Sub-Problem is then un-bounded.
- Bender's Dual Sub-Problem is a **restricted** version of the original problem meaning, hence the original problem is also un-bounded.



How did we get here ?





Master Problem: Lowerbound

- Farka's lemma guarantees direct correspondence between original problem and the Full Benders Master Problem
- The Relaxed Benders Master Problem will hence always give a lower bound ...



Sub Problem: Upperbound

- Bender's dual problem is a restricted version of the original problem, hence a solution to the sub-problem is also a solution to the original problem



The Benders Sub-problem

Max:

$$f^T \bar{y} + (b - B\bar{y})^T u$$

s.t.:

$$\begin{aligned} A^T u &\leq c \\ u &\geq 0 \end{aligned}$$

Problem: If the objective is un-bounded, how to find rays ?



Extreme rays

We need to find the extreme rays in the cone $\{u \mid A^T u \leq 0, u \geq 0\}$, such that a cut $(u^i)^T (b - B\bar{y}) > 0$ is found, see Proposition 10.3, hence:

Max:

$$(b - B\bar{y})^T u$$

s.t.:

$$\begin{aligned} A^T u &\leq 0 \\ 0 &\leq u \leq L \end{aligned}$$



Ray generation: An example

Min:

$$2x_1 + 6x_2 + 2y_1 + 3y_2$$

s.t.:

$$-x_1 + 2x_2 + 3y_1 - y_2 \geq 5$$

$$x_1 - 3x_2 + 2y_1 + 2y_2 \geq 4$$

$$x_1, x_2 \geq 0 \quad y \in \{0, 1, 2\}$$



Primal Subproblem

Max:

$$u_1 \cdot [5 - 3y_1 + y_2] + u_2 \cdot [4 - 2y_1 - 2y_2]$$

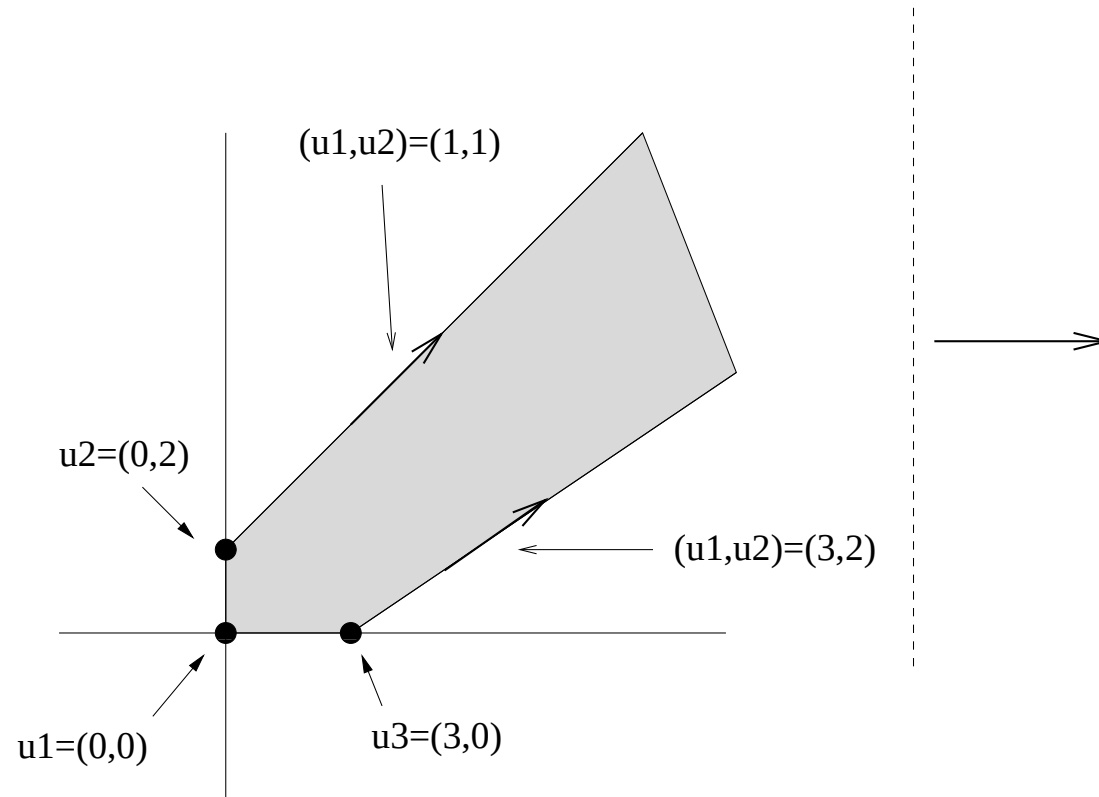
s.t.:

$$-u_1 + u_2 \leq 2$$

$$2u_1 - 3u_2 \leq 6$$

$$u_1, u_2 \geq 0$$

Visually



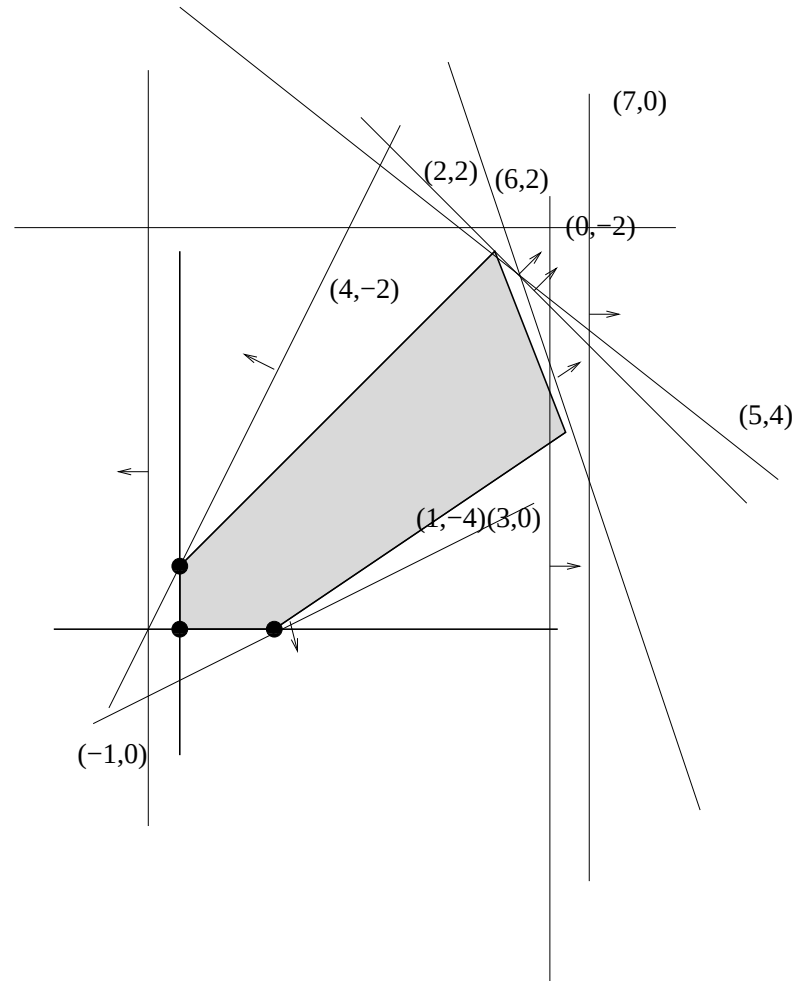


Objectives for subproblem

$(5 - 3y_1 + y_2, 4 - 2y_1 - 2y_2)$, and $y_1, y_2 \in \{0, 1, 2\}$

- $(0, 0) : (5, 4)$
- $(0, 1) : (6, 2)$
- $(0, 2) : (7, 0)$
- $(1, 0) : (2, 2)$
- $(1, 1) : (3, 0)$
- $(1, 2) : (4, -2)$
- $(2, 0) : (-1, 0)$
- $(2, 1) : (0, -2)$
- $(2, 2) : (1, -4)$

Visually with possible objectives





First iteration

Initial guess for $(y_1, y_2) = (1, 1)$:

Max:

$$u_1 \cdot [5 - 3y_1 + y_2] + u_2 \cdot [4 - 2y_1 - 2y_2] = 3u_1$$

s.t.:

$$-u_1 + u_2 \leq 2$$

$$2u_1 - 3u_2 \leq 6$$

$$u_1, u_2 \geq 0$$



Tabular LP form, first iteration

B.V.	u_1	u_2	d_1	d_2	R.S.	UB
Z	-3	0	0	0	0	
d_1	-1	1	1	0	2	no limit
d_2	2	-3	0	1	6	$u_1 \leq \frac{6}{2} = 3 \leftarrow \text{min}$

B.V.: Basic Variables

R.S.: Right hand side



Tabular LP form, first iteration

B.V.	u_1	u_2	d_1	d_2	R.S.	UB
Z	0	$-\frac{9}{2}$	0	$\frac{3}{2}$	9	
d_1	0	$-\frac{1}{2}$	1	$\frac{1}{2}$	5	no limit
u_1	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3	no limit

We can increase u_2 as much as we like !!! Ok, so what is the extreme ray ?



The simplex tableau as equations

$$\begin{aligned} -\frac{1}{2}u_2 + d_1 + \frac{1}{2}d_2 &= 5 \\ u_1 - \frac{3}{2}u_2 + \frac{1}{2}d_2 &= 3 \end{aligned}$$

But when we use the simplex algorithm, we only want to increase **one** variable at the time, and we hence freeze the slack variable $d_2 = 0$



After the freeze

$$u_1 = 3 + \frac{3}{2}u_2$$

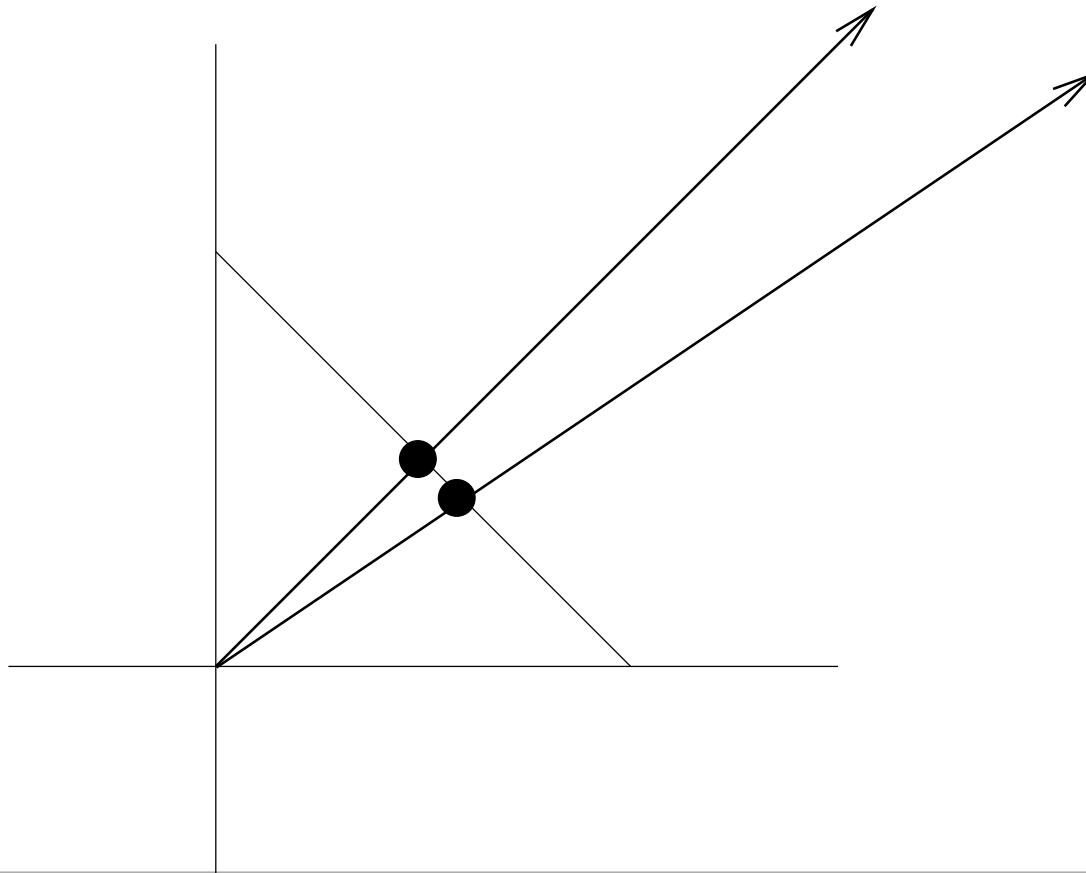
$$d_1 = 5 + \frac{1}{2}u_2$$

$\Downarrow$

$$u_2 = \frac{2}{3}u_1 - 2$$

To find extreme rays by problem modifications

Ok, that's all very nice, but what about using computer codes for finding the rays ?





Extreme Ray

The extreme rays arises when our dual subproblem is unbounded:

Max:

$$\text{dummy}_z$$

s.t.:

$$\begin{aligned}\text{dummy}_z &= 0 \\ f^T \bar{y} + (b - B\bar{y})^T u &= 1 \\ A^T u &\leq 0 \\ u &\geq 0\end{aligned}$$

What about the objective ???



Modifications

- Zero right hand sides
- Dummy objective (the objective does not matter !)
- Fix value of objective function to 1

What solution will the simplex or dual-simplex solve find ?



Example

A good example with the fixed charge
Transportation Problem ... see the Erwin
Kalvelagen note.

The Fixed Charge Transportation problem

Min:

$$\sum_{i,j} (f_{i,j} y_{i,j} + c_{i,j} x_{i,j})$$

s.t.:

$$\sum_j x_{i,j} = s_i \quad \forall i$$

$$\sum_i x_{i,j} = d_j \quad \forall j$$

$$x_{i,j} \leq M_i y_{i,j} \quad \forall i, j$$

$$x_{i,j} \geq 0 \quad y_{i,j} \in \{0, 1\}$$



Benders (primal) subproblem

Max:

$$\sum_i (-s_i)u_i + \sum_j d_j v_j + \sum_{i,j} (-M_{i,j} \overline{y_{i,j}})w_{i,j}$$

s.t.:

$$\begin{aligned} u_i + v_j - w_{i,j} &\leq c_{i,j} \quad \forall i, j \\ u_i, v_j, w_{i,j} &\geq 0 \end{aligned}$$

Benders Master Problem

Min:

z

s.t.:

$$\begin{aligned}
 z &\geq \sum_{i,j} f_{i,j} y_{i,j} + \sum_i (-s_i) \bar{u}_i^k + \sum_j d_j \bar{v}_j^k \\
 &\quad + \sum_{i,j} (-M_{i,j} \bar{w}_{i,j}^k) y_{i,j} \quad \forall k \\
 0 &\geq \sum_i (-s_i) \bar{u}_i^{k'} + \sum_j d_j \bar{v}_j^{k'} \\
 &\quad + \sum_{i,j} (-M_{i,j} \bar{w}_{i,j}^{k'}) y_{i,j} \quad \forall k'
 \end{aligned}$$



Extreme rays

What about the extreme rays ? How does Kalvelagen handle these ?

- How does he detect an extreme ray ? He adds an artificial upper bound on the objective !
- If the optimal objective value is significantly less than the upper bound, he assumes it to be correct ...
- If there is an extreme ray, he adds the dummy objective adds the objective constraint and fix the original objective value to 1 and solve the problem ...



Changed Benders Subproblem

Max:

dummy

s.t.:

$$dummy = 0$$

$$\sum_i (-s_i)u_i + \sum_j d_j v_j + \sum_{i,j} (-M_{i,j} \overline{y_{i,j}})w_{i,j} = 1$$

$$u_i + v_j - w_{i,j} \leq c_{i,j} \quad \forall i, j$$
$$u_i, v_j, w_{i,j} \geq 0$$