Benders (once more ...)

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Outline

- The algorithm
- The assignment from last week
- GAMS Implementation

Definitions

- RBMP: Relaxed Benders Master Problem
- BPSP: Benders **Primal** Sub-Problem: The direct formulation for finding the u values.
- BDSP: Benders **Dual** Sub-Problem: The original problem with fixed $\overline{y}$ values
- RGP: Ray Generation Problem

But ...

When we solve the BPSP (the u problem) what if:

- Problem solved: No problem (we add optimality cut)
- Problem infeasible: We have a solution which is infeasible because of the y values. Hence we need a *feasibility cut* based on an extreme ray.
- Problem un-bounded: Original problem un-bounded !

Benders Alorithm

Initialize RBMP, RGP, BDSP/BPSP
$Z_{LO} = -\infty, Z_{UP} = \infty, \overline{y}, \epsilon$
while $Z_{UP} - Z_{LO} > \epsilon$
  Solve BDSP($\overline{y}$) or BPSP($\overline{y}$)getting u
  if feasible
    update Z_{UP}
    Add Optimality constraint
  else
    Solve RGP($\overline{y}$) getting u
    Add Feasibility constraint
  Solve RBMP getting $\overline{y}$
  update Z_{LO}
endwhile

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Example

Min:

$$z_0 = 5x - 3y$$

s.t.:

$$\begin{aligned} x + 2y &\geq 4 \\ 2x - y &\geq 0 \\ x - 3y &\geq -13 \\ x &\geq 0 \\ y &\in \{0, \ldots, 10\} \end{aligned}$$

Optimal value: (x=2.5, y=5), z=-2.5

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Example 10.6: The B matrix

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

$b^T = [4, 0, -13]$
$c^T = [5]$
$f^T = [-3]$

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The Optimality cuts

The optimality cuts we are looking for are:

$$z_0 \geq f^T y + (\overline{u}^i)^T (b - By) \qquad i = 1, \ldots, q$$

Which becomes:

$$z_0 \geq -3y + (\overline{u}^i)^T ([4, 0, -13]^T - [2, -1, -3]^T y)$$

Hence, every time we find new optimality cut values, i.e. u values, we only have to plug them in, in the above equation.

Example

Min:

$$z_0 = 5x - 3y$$

s.t.:

$$\begin{aligned} x &\geq 4 - 2y \\ 2x &\geq 0 + y \\ x &\geq -13 + 3y \\ x &\geq 0 \\ y &\in \{0, \ldots, 10\} \end{aligned}$$

Optimal value: (x=2.5, y=5), z=-2.5

1. Iteration: Subproblem

Max:

$$z_1 = (4 - 2y)u_1 + y \cdot u_2 + (-13 + 3y)u_3$$

s.t.:

$$\begin{aligned} u_1 + 2u_2 + u_3 &\leq 5 \\ u_i &\geq 0 \end{aligned}$$

Inititialisation

Initial solution

$$\bar{y}_0 = 0.$$

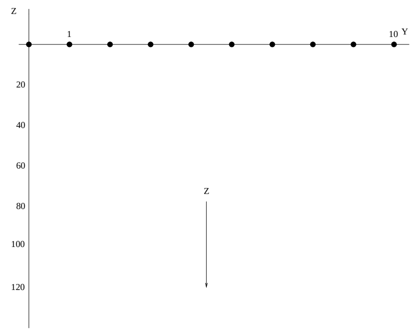
Upper bound:

$$UP_0 = \infty.$$

Lower bound:

$$LO_0 = -\infty.$$

Graphical BMP, initialisation



1. Iteration: Subproblem

Max:

$$z_1 = 4u_1 - 13u_3$$

s.t.:

$$\begin{aligned} u_1 + 2u_2 + u_3 &\leq 5 \\ u_i &\geq 0 \end{aligned}$$

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1. Iteration: Subproblem

Optimal solution:

$$\begin{aligned} \bar{z}_1 &= 20 \\ \bar{u} &= (5, 0, 0) \end{aligned}$$

Giving a new upper bound:
$UP_1 = \min\{\bar{z}_1, UP_0(= \infty)\} = 20$. And arriving at the cut:

$$\begin{aligned} z_0 &\geq -3y + (5,0,0)^T([4,0,-13]^T - [2,-1,-3]^T y) \\ &= \\ z_0 &\geq -13y + 20 \end{aligned}$$

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1. Iteration: Benders Master Problem

Min:

$$z_1$$

s.t.:

$$\begin{aligned} z_1 &\geq 20 - 13y \\ y &\in \{0, \ldots, 10\} \end{aligned}$$

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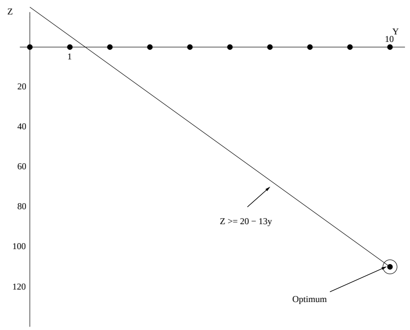
1. Iteration: Benders Master Problem

Optimal solution

$$\begin{aligned} \bar{z}_1 &= -110 \\ \bar{y} &= 10 \end{aligned}$$

New lower bound:
$LO_1 = \max\{\bar{z}_1, LO_0(= -\infty)\} = -110.$

Graphical BMP, 1. iteration



2. iteration: Primal Subproblem

Max:

$$z_2 = -16u_1 + 10u_2 + 17u_3 - 30$$

s.t.:

$$\begin{aligned} u_1 + 2u_2 + u_3 &\leq 5 \\ u_i &\geq 0 \end{aligned}$$

2. iteration: Primal Subproblem

Optimal solution:

$$\begin{aligned} \bar{z}_2 &= 55 \\ \bar{u} &= (0, 0, 5) \end{aligned}$$

New upper bound: $UP_2 = \min\{\bar{z}_2, UP_1(=20)\} = 20.$

2. iteration: Benders Master Problem

Min:

$$z_2$$

s.t.:

$$\begin{aligned} z_2 &\geq 20 - 13y \\ z_2 &\geq -65 + 12y \\ y &\in \{0, \ldots, 10\} \end{aligned}$$

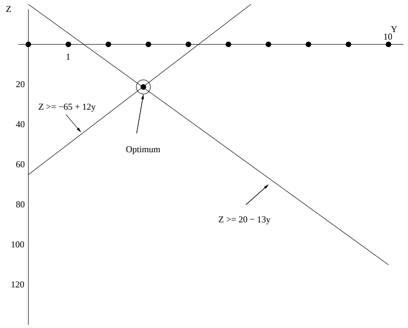
2. iteration: Benders Master Problem

Optimal solution:

$$\begin{aligned} z_2 &= -19 \\ y &= 3 \end{aligned}$$

New upper bound:
$LO_2 = \min\{\bar{z}_2, LO_1(-110)\} = -19.$

Graphical BMP, 2. iteration



3. iteration: Primal Subproblem

Max:

$$z_3 = -2u_1 + 3u_2 - 4u_3 - 9$$

s.t.:

$$\begin{aligned} u_1 + 2u_2 + u_3 &\leq 5 \\ u_i &\geq 0 \end{aligned}$$

3. iteration: Primal Subproblem

Optimal solution:

$$\begin{aligned} \bar{z}_2 &= -1.5 \\ \bar{u} &= (0, 2.5, 0) \end{aligned}$$

New upper bound:
$UP_3 = \min\{\bar{z}_3, UP_2(=19)\} = -2.5.$

3. iteration: Benders Master Problem

Min:

$$z_3$$

s.t.:

$$\begin{aligned} z_3 &\geq 20 - 13y \\ z_3 &\geq -65 + 12y \\ z_3 &\geq -0.5y \\ y &\in \{0, \ldots, 10\} \end{aligned}$$

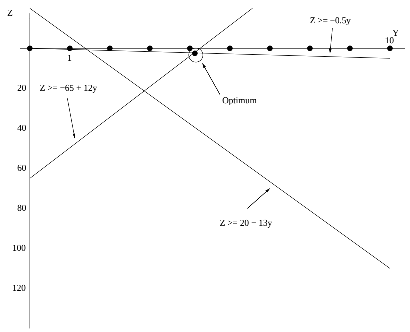
3. iteration: Benders Master Problem

Optimal solution:

$$\begin{aligned} z_3 &= -2.5 \\ y &= 5 \end{aligned}$$

New lowerbound:$LO_3 = \max\{\bar{z}_1, LO_3(= -19)\} = -2.5$.
Now: $LO_3 = UP_3$

Graphical BMP, 3. iteration



Exercise 10.1 (RKM)

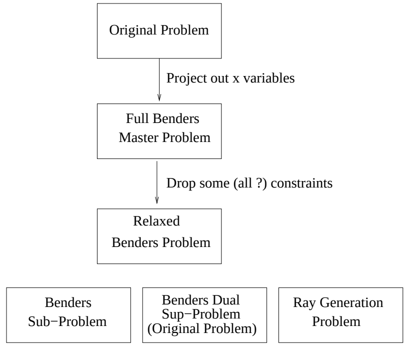
Several constraints are added without improving bound (in case of minimization, non-increasing) why: Primal degeneracy: Possible to include new variables into the simplex tableau without changing the objective value (the new variables may stay at value zero). **Dual degeneracy**: If the current “solution” has **several** active constraints.

Exercise 10.2 (RKM)

If the Bender's subproblem is infeasible what does that imply about the original problem ?

- Original problem un-bounded ! Why ?
- Bender's Dual Sub-Problem is then un-bounded.
- Bender's Dual Sub-Problem is a **restricted** version of the original problem meaning, hence the original problem is also un-bounded.

How did we get here ?



Master Problem: Lowerbound

- Farka's lemma guarantees direct correspondence between original problem and the Full Benders Master Problem
- The Relaxed Benders Master Problem will hence always give a lower bound ...

Sub Problem: Upperbound

- Bender's dual problem is a restricted version of the original problem, hence a solution to the sub-problem is also a solution to the original problem

The Benders Sub-problem

Max:

$$f^T\overline{y} + (b - B\overline{y})^T u$$

s.t.:

$$\begin{aligned} A^T u &\leq c \\ u \geq 0 & \end{aligned}$$

Problem: If the objective is un-bounded, how to find rays ?

Extreme rays

We need to find the extreme rays in the cone $\{u | A^T u \leq 0, u \geq 0\}$, such that a cut $(u^i)^T(b - B\overline{y}) > 0$ is found, see Proposition 10.3, hence:
Max:

$$(b - B\overline{y})^T u$$

s.t.:

$$\begin{aligned} A^T u &\leq 0 \\ 0 \leq u &\leq L \end{aligned}$$

Ray generation: An example

Min:

$$2x_1 + 6x_2 + 2y_1 + 3y_2$$

s.t.:

$$\begin{aligned} -x_1 + 2x_2 + 3y_1 - y_2 &\geq 5 \\ x_1 - 3x_2 + 2y_1 + 2y_2 &\geq 4 \\ x_1, x_2 \geq 0 & \quad y \in \{0, 1, 2\} \end{aligned}$$

Primal Subproblem

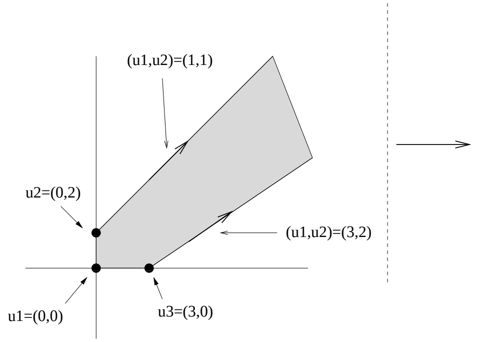
Max:

$$u_1 \cdot [5 - 3y_1 + y2] + u_2 \cdot [4 - 2y_1 - 2y_2]$$

s.t.:

$$\begin{aligned} -u1 + u_2 &\leq 2 \\ 2u_1 - 3u_2 &\leq 6 \\ u_1, u_2 \geq 0 & \end{aligned}$$

Visually

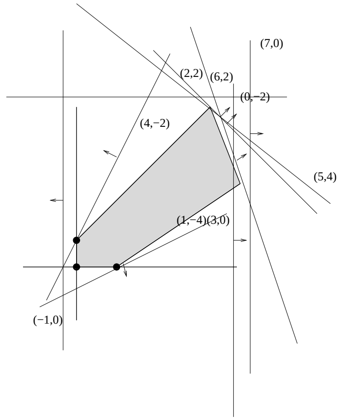


Objectives for subproblem

$(5 - 3y_1 + y_2, 4 - 2y_1 - 2y_2)$, and $y_1, y_2 \in \{0, 1, 2\}$

- $(0,0) : (5,4)$
- $(0,1) : (6,2)$
- $(0,2) : (7,0)$
- $(1,0) : (2,2)$
- $(1,1) : (3,0)$
- $(1,2) : (4,-2)$
- $(2,0) : (-1,0)$
- $(2,1) : (0,-2)$
- $(2,2) : (1,-4)$

Visually with possible objectives



First iteration

Initial guess for $(y_1, y_2) = (1, 1)$:
Max:

$$u_1 \cdot [5 - 3y_1 + y2] + u_2 \cdot [4 - 2y_1 - 2y_2] = 3u_1$$

s.t.:

$$\begin{aligned} -u1 + u_2 &\leq 2 \\ 2u_1 - 3u_2 &\leq 6 \\ u_1, u_2 \geq 0 & \end{aligned}$$

Tabular LP form, first iteration

B.V.	u_1	u_2	d_1	d_2	R.S.	UB
Z	-3	0	0	0	0	
d_1	-1	1	1	0	2	no limit
d_2	$\boxed{2}$	-3	0	1	6	$u_1 \leq \frac{6}{2} = 3$ $\leftarrow$ min

B.V.: Basic Variables
R.S.: Right hand side

Tabular LP form, first iteration

B.V.	u_1	u_2	d_1	d_2	R.S.	UB
Z	0	$-\frac{9}{2}$	0	$\frac{3}{2}$	9	
d_1	0	$\boxed{-\frac{1}{2}}$	1	$\frac{1}{2}$	5	no limit
u_1	1	$\boxed{-\frac{3}{2}}$	0	$\frac{1}{2}$	3	no limit

We can increase u_2 as much as we like !!! Ok, so what is the extreme ray ?

The simplex tableau as equations

$$-\frac{1}{2}u_2 + d_1 + \frac{1}{2}d_2 = 5$$
$$u_1 - \frac{3}{2}u_2 + \frac{1}{2}d_2 = 3$$

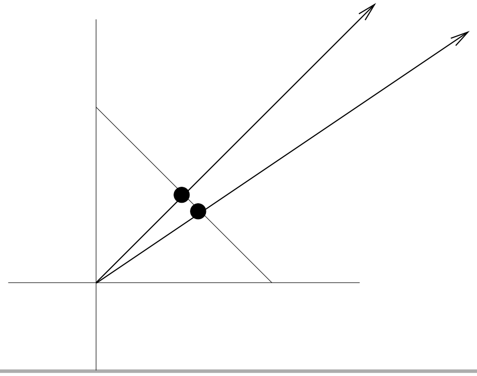
But when we use the simplex algorithm, we only want to increase **one** variable at the time, and we hence freeze the slack variable $d_2 = 0$

After the freeze

$$u_1 = 3 + \frac{3}{2}u_2$$
$$d_1 = 5 + \frac{1}{2}u_2$$
$$\Downarrow$$
$$u_2 = \frac{2}{3}u_1 - 2$$

To find extreme rays by problem modifications

Ok, that's all very nice, but what about using computer codes for finding the rays ?



Extreme Ray

The extreme rays arises when our dual subproblem is unbounded:
Max:

$$dummy_z$$

s.t.:

$$\begin{aligned} dummy_z &= 0 \\ f^T\overline{y} + (b - B\overline{y})^T u &= 1 \\ A^T u &\leq 0 \\ u \geq 0 & \end{aligned}$$

What about the objective ???

Modifications

- Zero right hand sides
- Dummy objective (the objective does not matter !)
- Fix value of objective function to 1

What solution will the simplex or dual-simplex solve find ?

Example

A good example with the fixed charge Transportation Problem ... see the Erwin Kalvelagen note.

The Fixed Charge Transportation problem

Min:

$$\sum_{i,j}(f_{i,j}y_{i,j} + c_{i,j}x_{i,j})$$

s.t.:

$$\begin{aligned}
\sum_j x_{i,j} &= s_i \quad \forall i \\
\sum_i x_{i,j} &= d_j \quad \forall j \\
x_{i,j} &\leq M_i y_{i,j} \quad \forall i,j
\end{aligned}$$

$$x_{i,j} \geq 0 \quad y_{i,j} \in \{0,1\}$$

Benders (primal) subproblem

Max:

$$\sum_i(-s_i)u_i + \sum_j d_j v_j + \sum_{i,j}(-M_{i,j}\overline{y_{i,j}})w_{i,j}$$

s.t.:

$$\begin{aligned}
u_i + v_j - w_{i,j} &\leq c_{i,j} \quad \forall i,j \\
u_i, v_j, w_{i,j} &\geq 0
\end{aligned}$$

Benders Master Problem

Min:

$$z$$

s.t.:

$$\begin{aligned}
z &\geq \sum_{i,j} f_{i,j}y_{i,j} + \sum_i(-s_i)\overline{u}_i^k + \sum_j d_j\overline{v}_j^k \\
&\quad + \sum_{i,j}(-M_{i,j}\overline{w}_{i,j}^k)y_{i,j} \quad \forall k \\
0 &\geq \sum_i(-s_i)\overline{u}_i^{k'} + \sum_j d_j\overline{v}_j^{k'} \\
&\quad + \sum_{i,j}(-M_{i,j}\overline{w}_{i,j}^{k'})y_{i,j} \quad \forall k'
\end{aligned}$$

Extreme rays

What about the extreme rays ? How does Kalvelagen handle these ?

- How does he detect an extreme ray ? He adds an artificial upper bound on the objective !
- If the optimal objective value is significantly less than the upper bound, he assumes it to be correct ...
- If there is an extreme ray, he adds the dummy objective adds the objective constraint and fix the original objective value to 1 and solve the problem ...

Changed Benders Subproblem

Max:

$$dummy$$

s.t.:

$$\begin{aligned}
dummy &= 0 \\
\sum_i (-s_i)u_i + \sum_j d_j v_j + \sum_{i,j} (-M_{i,j}\overline{y_{i,j}})w_{i,j} &= 1 \\
u_i + v_j - w_{i,j} &\leq c_{i,j} \quad \forall i,j \\
u_i, v_j, w_{i,j} &\geq 0
\end{aligned}$$