# Alternative Benders Decomposition Explanation

## – *explaining Benders without projections*

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## Outline

- Comments on Benders
- Feasibility ?
- Optimality ?





## Primal-Dual relations

- Primal feasible $\Rightarrow$ Dual feasible
- Primal unbounded $\Rightarrow$ Dual infeasible
- primal infeasible:
  - In general: $\Rightarrow$ unbounded **or** infeasible
  - In Benders: $\Rightarrow$ unbounded





## Benders Alorithm

Initialize RBMP
$Z_{LO} = -\infty, Z_{UP} = \infty, \overline{y}, \epsilon$
**while** $Z_{UP} - Z_{LO} > \epsilon$
  Solve BPSP($\overline{y}$)
  **if** feasible **then** add optimality constraint
  **if** unbounded **then** add feasibility constraint
  **if** infeasible **then** original problem unbounded
  update $Z_{UP}$
  Solve RBMP getting $\overline{y}$
  update $Z_{LO}$
**endwhile**

## The standard MIP Problem

**Min**

$$c^T x + f^T y$$

**s.t.**

$$\begin{aligned} Ax + By &\geq b \\ y &\in Y \\ x &\geq 0 \end{aligned}$$

Notice: No assumption about $y$, but we will only consider MIP problems, i.e. $y \in Z$





## How can we get around the difficult variables ?

If we **fix** the $y$ variables (written as: $\overline{y}$):

**Min**

$$f^T\overline{y} \; + \; \textit{Min}\; c^T x$$
$$Ax \geq b - B\overline{y}$$
$$x \geq 0$$





## How can we select the $\overline{y}$'s ???

If we somehow magically select $\overline{y}$ values, we need to answer two critical questions:

- **Is** there a feasible solution $(y, x)$ ? (Notice that we can easily introduce in-feasibility into the solution by selecting wrong $\overline{y}$ values ....)
- Is the chosen $\overline{y}$ values together with the best possible $x$ values the best solution $(y^*, x^*)$ ?





## How can we avoid in-feasibility ?

I.e. select the given $\overline{y} \in \{y | \exists x \geq 0 : Ax \geq b - B\overline{y}\}$:

**Min**

$$0 \cdot x$$

**s.t.**

$$\begin{aligned} Ax &\geq b - B\overline{y} \\ x &\geq 0 \end{aligned}$$

We need to know what to do if this system is infeasible. This correspond to avoiding unbounded dual system ...

## The dual system

Then the corresponding **dual** system also has to have a feasible system, i.e. **not** being unbounded (notice: **it cannot be infeasible, the trivial solution** $u = 0$ **is always feasible**)

**Max**

$$u(b - B\overline{y})$$

**s.t.**

$$\begin{aligned} uA &\leq 0 \\ u &\geq 0 \end{aligned}$$





## The optimal value of the dual system

The dual system cannot be infeasible, hence it must be either feasible or unbounded. If it is feasible, it will have the optimal value $0$ (because that is the optimal value of the primal system). Hence the requirement regarding the election of the $\overline{y}$ is: Select $\overline{y}$ such that $u(b - B\overline{y}) \leq 0$ for all $u$ which satisfies:

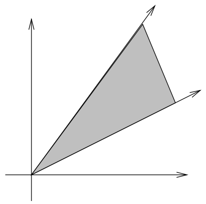
$$\begin{aligned} uA &\leq 0 \\ -u &\leq 0 \end{aligned}$$

This a so called **homogeneous** system because the right hand sides are equal to zero. This also means that all the closed half-spaces go through origo.





## The homogeneous system is always a cone !



But such a system we can represent with extreme rays !





## Represented by just extreme rays

**Max**

$$u(b - B\overline{y})$$

**s.t.**

$$\begin{aligned} u &= \sum_r \gamma_r \cdot \tilde{u}_r \\ u &\geq 0 \\ \gamma_r &\geq 0 \end{aligned}$$

## Reformulation

Now we can reformulate the system using the extreme rays $\tilde{u}$: $\overline{y} = \{y | \tilde{u}_r(b - By) \leq 0, r = 1, ..., R\}$.





## Including extreme rays into the problem

**Min**

$$
\begin{array}{rl}
f^T\overline{y} + & \text{Min } c^T x \\
 & Ax \geq b - B\overline{y} \\
\tilde{u}_r(b - By) \leq 0 & \quad , r = 1, ..., R \\
y \in Y & \\
 & x \geq 0
\end{array}
$$





## Sofar so good ...

This clearly does not solve all our problems:

- We need to know **all** the extreme rays ...
- We still have not gotten rid of the $x$ variables ...

But, we have made one crucial improvement: If we satisfy all the **feasibility** constraints, the interior ($x$ variables) program is guaranteed to contain a solution.





## Dual (sub)-problem ($x$ problem)

**Min**

$$c^T x$$

**s.t.**

$$
\begin{array}{rcl}
Ax & \geq & b - B\overline{y} \\
x & \geq & 0
\end{array}
$$

## Primal problem ($u$ problem)

**Max**

$$u(b - B\overline{y})$$

**s.t.**

$$\begin{array}{rcl} uA & \leq c & \\ u & \geq & 0 \end{array}$$





## Primal-Dual relationship

There are three possibilities:

- Primal sub-problem unbounded ($\infty$) $\leftrightarrow$ we need extreme ray constraints in the master problem
- Primal sub-problem infeasible $\leftrightarrow$ dual problem unbounded, meaning that the original problem is unbounded
- Finite primal optimum $\leftrightarrow$ finite (equal) dual optimum, hence:

$min\{c^T x | Ax \geq b - B\overline{y}, x \geq 0\}$ =
$max\{u(b - B\overline{y}) | uA \leq c, u \geq 0\}$





## Dual reformulation

We can hence assume:

- The primal sub-problem contains a feasible solution and is not un-bounded.
- Then we can change representation: Instead of representing our feasible space using constraints, we can represent it using extreme points !





## $u$ problem constraint formulation

**Max**

$$u(b - B\overline{y})$$

**s.t.**

$$\begin{array}{rcl} uA & \leq c & \\ u & \geq & 0 \end{array}$$

## $u$ extreme point formulation

**Max**

$$u^p(b - B\overline{y})$$

**s.t.**

$$u^p \in \textit{set of extreme points}$$





## Including extreme points into the problem

**Min**

$$\begin{aligned} f^T y \quad + \quad &\textit{Max } u^p(b - B\overline{y}) \quad p = 1, ..., P \\ \tilde{u}_r(b - By) \leq 0 \quad & \quad r = 1, ..., R \\ y \in Y \\ & x \geq 0 \end{aligned}$$





## Final version (Master Program)

**Min**

$$z$$

**St.**

$$\begin{aligned} z &\geq f^T y + u^p(b - By) \quad \forall p \\ 0 &\geq \tilde{u}_r(b - By) \quad \forall r \\ y &\in Y \end{aligned}$$





## How can we use this ?

- The complete program contains an exponential number of extreme rays and extreme points
- ... but we **may** be lucky that we only need to generate a few of these points ...
- If our master program does **not** contain all constraints, our program is a **relaxed** master program ...
- ... hence we may get an optimal solution, but we are only guaranteed a (lower) bound.

## How can we use this ? II

- Based on the lower bound, we get a set of fixed $\overline{y}$ variables ...
- ... these are then used to generate either: A new extreme ray or a new extreme point **and** a solution (upper bound) to the original problem: $(x, y)$
- An so we continue until the upper bound and the lower bound are sufficiently close