



Benders Decomposition

– *Using projections to solve problems*

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Outline

- Introduction
- Using projections
- Benders decomposition
- Simple plant location example
- Problems with Benders decomposition algorithm



Important readings

Good news: The only important readings from this lecture is:

- Section 10.1
- Section 10.2
- Section 10.3



What is Benders ?

Benders algorithm:

- Was invented by J.F. Benders, 1962
- A **decomposition algorithm** for solution of hard optimization problems
- Requires iterative solution of a MIP master problem and LP subproblem(s)



Applying Projections

$$\begin{aligned} z_0 &\geq d_k u_k^i & k \in 1, \dots, q \\ 0 &\geq d_k u_k^i & k \in q+1, \dots, r \end{aligned}$$

- $b^T u_k = d_k \quad \forall k$, Lemma 2.10
- $A^T u_k - c = 0 \quad k \in 1, \dots, q$, Lemma 2.10
- $A^T u_k = 0 \quad k \in q+1, \dots, r$, Lemma 2.10



The standard MIP Problem

Min

$$c^T x + f^T y$$

s.t.

$$\begin{aligned} Ax + By &\geq b \\ y &\in Y \\ x &\geq 0 \end{aligned}$$

Notice: No assumption about y , but we will only consider MIP problems, i.e. $y \in Z$



Optimizing using projections

Remember how we could optimize using projections ?

$$\begin{aligned} z_0 - c^T x &\geq 0 \\ Ax &\geq b \\ x_j &\geq 0 \end{aligned}$$



Re-arranging

Min

$$z_0$$

s.t.

$$\begin{aligned} z_0 - c^T x &\geq f^T y \\ Ax &\geq b - By \\ x \geq 0 \quad z \in R \quad y \in Y \end{aligned}$$

We make this rearrangement to enhance projection.



Now we can project the x variables out !

Min

$$z_0$$

s.t.

$$u_0^i z_0 \geq u_0^i f^T y + (u^i)^T (b - By) \quad i = 1, \dots, r$$

$$z \in R \quad y \in Y$$

Notice that we only have one type of constraints (the two types are hidden in the possible values of u_0^i , equal or not equal to zero).



Rearranging projection results

Now we re-scale with u_0^i (though not if $u_0^i = 0$):

Min

$$z_0$$

s.t.

$$z_0 \geq f^T y + (u^i)^T (b - By) \quad i = 1, \dots, q$$

$$0 \geq (u^i)^T (b - By) \quad i = q + 1, \dots, r$$

$$z \in R \quad y \in Y$$

This is called **Benders Master Program BMP**

(Notice: only y variables).



But ...

What did I tell you previously ? That the worstcase number of constraints is: $(\frac{1}{2})^{(2^{n+1}-2)} m^{2^n}$. This was exactly the reason why we stated that the projection method was not usefull in practice, so why bother ? **Because we can generate the constraints on the fly.** Big deal you may think, what if we still have to generate all of them ?



So ...

We start with a problem with only a *sub-set* of the two types of constraints:

- A subset of the *optimization* constraints
- A subset of the *feasibility* constraints

Hence our *restricted* Benders Master Program is a relaxation of the *full* Benders Master Program.



Hence ...

If we optimize our *restricted* Benders Master Program and find the solution $\bar{y}$, we may not have the overall optimal solution because:

- The solution may be too low (not optimal):
 $z_{opt} > \bar{z}$, because we need at least one more optimization constraint
- The solution may be infeasible, because one or more of the feasibility constraints in the full Benders Master Program are missing.



Benders Algorithm (Intuitively) (RKM 353)

- Start with a relaxed BMP with no constraints or just a few of these. Solve the problem to optimality getting the values $\bar{y}$ (or set initial values).
- Given the values $\bar{y}$ we are also getting a lowerbound z_{LO} of the original problem.
- Solve subproblem to get u , it may be:
 - ▶ Infeasible (unbounded), generate ray u to find the violated feasibility constraint ...
 - ▶ Subproblem has a solution, find the constraint such that $f^T \bar{y} + (b - B\bar{y})^T u > z_{LO}$
- If the two bounds are sufficiently close, i.e.
 $z^{UP} - z_{LO} \leq \epsilon$, stop, otherwise iterate.



Error in book

There is an error in the algorithm on p. 353: Step 3, line 3 it says Bx in the feasibility cut, it should be By .



How to find the u^i

We need to find the u^i values to generate the feasibility constraints and the optimality constraints. Given our problem:

Min

$$z_0$$

s.t.

$$\begin{aligned} z_0 &\geq f^T y + (\bar{u}^i)^T (b - By) & i = 1, \dots, q \\ 0 &\geq (\bar{u}^i)^T (b - By) & i = q + 1, \dots, r \\ z &\in R & y \in Y \end{aligned}$$





How to find the u^i II

Prop. 10.3 gives the $\bar{u}_i$ as:

- Extreme rays of $\{u | A^T u \leq 0, u \geq 0\}$.
- Extreme point of $\{u | A^T u \leq c, u \geq 0\}$ i.e.
 $\max\{f^T \bar{y} + (b - B\bar{y})^T u | A^T u \leq c, u \geq 0\}$.

We will deal with the problem of extreme rays to the lecture next week.



The Subproblem (called: Primal subproblem)

Thus we have to find the extreme points u given a set of fixed values $\bar{y}$:

Max

$$f^T \bar{y} + (b - B\bar{y})^T u$$

s.t.

$$\begin{aligned} A^T u &\leq c \\ u &\geq 0 \end{aligned}$$



Dual Subproblem (called: Dual subproblem)

On the other hand, it may be significantly easier to solve the dual subproblem (Lemma 10.4):

Min

$$c^T x + f^T \bar{y}$$

s.t.

$$\begin{aligned} Ax + B\bar{y} &\geq b \\ x &\geq 0 \end{aligned}$$

Notice that this problem may be in-feasible, i.e. the subproblem may be unbounded.



Dual Subproblem II

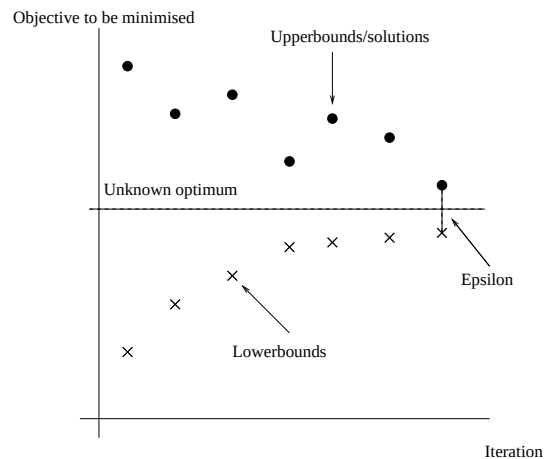
What is the dual subproblem ??? It is simply the primal subproblem where the $\bar{y}$ are fixed !

- This makes the dual subproblem an easy version of the original problem, because the “hard” variables are fixed.
- We can use a standard LP solver to solve the problem.

There is absolutely nothing wrong in solving the primal subproblem instead, but then you first have to dualize the problem !



Upper and Lower bounds



Section (10.3): Simple Facility Location

We will now look at the example in section 10.3.

This example you should study in detail !
minimise:

$$\sum_i \sum_j c_{i,j} \cdot x_{i,j} + \sum_i f_i \cdot y_i$$

s.t.

$$\begin{aligned} \sum_i x_{i,j} &\geq 1 & j = 1, \dots, m \\ -x_{i,j} + y_i &\geq 0 & i = 1, \dots, n, j = 1, \dots, m \\ x_{i,j} &\geq 0 & y_i \in \{0, 1\} \end{aligned}$$



The (dual) subproblem

Given a choice of facility locations $\bar{y}$, $O(i)$ are the open facilities and $C(i)$ are the closed facilities.

minimise:

$$\sum_i \sum_j c_{i,j} \cdot x_{i,j} + \sum_{i \in O(\bar{y})} f_i$$

s.t.

$$\begin{aligned} \sum_i x_{i,j} &\geq 1 & j = 1, \dots, m \\ x_{i,j} &\leq 1 & i = 1 \in O(\bar{y}) \\ x_{i,j} &\leq 0 & i = 1 \in C(\bar{y}) \\ x_{i,j} &\geq 0 \end{aligned}$$



Example 10.6: The data

Table 10.1, p. 355 RKM:

	1	2	3	4	5	fixed costs
Plant 1	2	3	4	5	7	2
2	4	3	1	2	6	3
3	5	4	2	1	3	3



Example 10.6: The matrixes

Unfortunately it is necessary to fully describe the matrixes for the problem in a form corresponding to the formulas (10.1) - (10.4). This is necessary in order to be able to calculate the optimality and feasibility cuts.



Example 10.6: The A matrix

A plot of the matrix A , showing a sparse matrix with 1s and -1s on a grid. The matrix is labeled $A =$ and the plot is attributed to mas Stidsen.



Example 10.6: The B matrix

[illegible]



Example 10.6: The b, c and d vectors

$$\begin{aligned} b^T &= [1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \\ c^T &= [2, 3, 4, 5, 7, 4, 3, 1, 2, 6, 5, 4, 2, 1, 3] \\ f^T &= [2, 3, 3] \end{aligned}$$



Example 10.6

1. iteration, subproblem, with:
 $\bar{y}_1 = 1, \bar{y}_2 = 0, \bar{y}_3 = 0$, gives the solution: $x_{1j} = 1$ and
 an upper bound $z^{UP} = 23$. Since the lower bound is
 $z_{LO} = -\infty$ we cannot terminate.



Example 10.6

How to get the bounding constraints. Observe that:

- If $\sum_i y_i \geq 1$ the solution to the dual subproblem is always feasible, i.e. the primal subproblem is never unbounded (extreme rays). Hence, given this restriction we will never have to add feasibility cuts.
- We need to get a way to construct the optimality constraints for the BMP



Example 10.6

Finally, lets get to the optimality cuts, in general:

$$z_0 \geq \bar{u}^T(b - By) + f^T y$$

We need the u values. We can get these, either using an LP solver or by hand calculations ...



Example 10.6: “Hand” calculation of duals v_j

$u = [v, w]$ where v are the demand duals and w are the open/closed facility duals.

- The dual variables v_j for the demand constraints have the value: $v_j = \min_{i \in O(\bar{y})} \{c_{ij}\}$, i.e. equal to the cost from the “closest” open facility.



Example 10.6: “Hand” calculation of duals w_{ij}

- The dual variables w_{ij} corresponding to the open and closed facilities:
 - ▶ $w_{ij} = 0, i \in O(\bar{y})$, because adding more capacity to an already open plant will not change the cost of a solution.
 - ▶ $w_{ij} = \max_{i \in C(\bar{y})} \{(v_j - c_{ij}), 0\}$. It can never cost, besides the fixed costs f_i to open a facility, hence it is always greater or equal to zero. The most we can gain by opening a facility is the difference between the current cost v_j and the cost if this i'th was available c_{ij} .



The u values

$$v = [2, 3, 4, 5, 7]$$

and

$$w = [0, 0, 0, 0, 0, 0, 0, 3, 3, 1, 0, 0, 2, 4, 4]$$



The u values

but given the above dual variables v_j and w_{ij} :

$$z_0 \geq \sum_{j=1}^5 \bar{v}_j + \sum_{i=1}^3 (f_i - \rho_i) y_i$$

where:

$$\rho_i = \sum_{j=1}^5 \bar{w}_{ij}$$



The u values

Hence:

$$\rho_1 = 0 + 0 + 0 + 0 + 0 = 0, \text{ i.e. } (f_1 - \rho_1) = 2 - 0 = 2$$

$$\rho_2 = 0, 0, 3, 3, 1 = 7, \text{ i.e. } (f_2 - \rho_2) = 3 - 7 = -4$$

$$\rho_3 = 0, 0, 2, 4, 4 = 10, \text{ i.e. } (f_3 - \rho_3) = 3 - 10 = -7$$

Hence:

$$z_0 \geq 21 + 2y_1 - 4y_2 - 7y_3$$



Example 10.6

Given the solution to the first dual subproblem:

$x_{11} = x_{12} = x_{13} = x_{14} = x_{15} = 1$
with the optimal value $UB = 23$.

But what we really need is the u variables which are the dual variables for the dual subproblem !



Example 10.6

The first BMP becomes:

minimise:

z_0

s.t.

$$\begin{aligned} z_0 &\geq 21 + 2y_1 - 4y_2 - 7y_3 \\ y_1 + y_2 + y_3 &\geq 1 \\ y_i &\in \{0, 1\} \end{aligned}$$



Example 10.6

Giving the optimal solution:

$y_1 = 0, y_2 = 1, y_3 = 1$ and $z_0 = 10$ and hence
 $LO = \max(-\infty, 10)$. Because we have $UB = 23$
(from the dual subproblem) we decide that
 $UB - LO = 13$ is too much, and we continue ...



Questions:

There are a number of interesting questions which may be raised:

- How many iterations needs to be performed ?
- A related question: How many constraints are binding for the optimal solution ?
- The quote: "For the Benders decomposition algorithm to be effective it is essential that the linear programming subproblem have special structure so that it is easily optimized", p. 357. This is in my view wrong. The real problem is solving the BMP !



Number of iterations

The number of iterations is critical and we cannot give any guarantees, why ? We may have to generate all extreme rays, and this number may be exponential (corner points $A^T u \leq c$).