



Optimality and relaxation

Jesper Larsen & Jens Clausen

jla, jc@imm.dtu.dk

Department of Management Engineering
Technical University of Denmark



Optimality and Relaxation

Basic solution approach to any IP or COP:

$$z = \max\{c(x) : x \in X \subseteq Z^n\}$$

- Find lower bound (LB) $\underline{z}$ s.t. $\underline{z} \leq z$
- Find upper bound (UB) $\bar{z}$ s.t. $\bar{z} \geq z$

Now clearly $\bar{z} \geq \underline{z}$. Furthermore, if we have $\bar{z} = \underline{z} = z$ we are done.



Upper and lower bounds

A general approach will usually be:

$$\underline{z}_1 < \underline{z}_2 < \underline{z}_3 < \dots \leq z \leq \dots \bar{z}_3 < \bar{z}_2 < \bar{z}_1$$

If $\bar{z}_t - \underline{z}_s < \epsilon$ then we may stop — if $\bar{z}_t - \underline{z}_s = 0$ we have found an optimum.



Bounds

How do we actually find (upper and lower) bounds?

- Primal bounds:** (lower bound for a max problem). Every feasible solution $x^* \in X$ is a lower bound.
- Dual bounds:** (upper bound for a max problem). Most important approach is by “relaxation”, that is, replace the original problem by a simpler optimization problem whose value is at least as large as z .



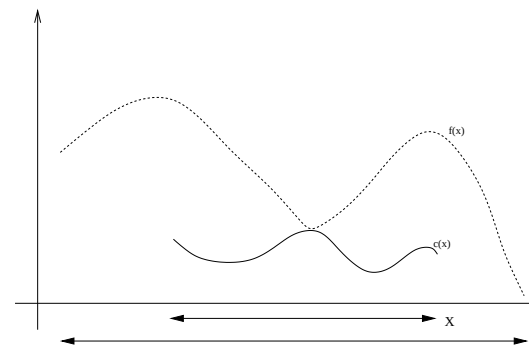


Relaxation

- A problem (RP) $z^R = \max\{f(x) : x \in T \subseteq R^n\}$ is a **relaxation** of (IP) $z = \max\{c(x) : x \in X \subseteq R^n\}$ if:
 - ▶ (i) $X \subseteq T$
 - ▶ (ii) for all $x \in X$: $c(x) \leq f(x)$



Relaxation



Relaxation

1. Linear Programming relaxation
2. Combinatorial relaxation
3. Lagrangian relaxation



Linear Programming relaxation

- For the integer program $\max\{cx : x \in P \cap Z^n\}$ with the formulation $P = \{x \in R_+^n : Ax \leq b\}$ the **linear programming relaxation** is $\max\{cx : x \in P\}$.
- **Proposition 2.3:**
 - ▶ (i) If a relaxation RP is infeasible, the original problem is infeasible.
 - ▶ (ii) Let x^* be an optimal solution to RP. If $x^* \in X$ and $f(x^*) = c(x^*)$ then x^* is an optimal solution to IP.





LP relaxation example

Consider the following integer program:

$$\begin{aligned} z = \max \quad & 4x_1 - x_2 \\ & 7x_1 - 2x_2 \leq 14 \\ & x_2 \leq 3 \\ & 2x_1 - 2x_2 \leq 3 \\ & x \in \mathbb{Z}_+^2 \end{aligned}$$

- Lower bound: $(2, 1)$ is a feasible solution.
- Upper bound: LP relaxation with LP optimum being $x^* = (\frac{20}{7}, 3)$



LP relaxation and formulations

The following proposition shows an interesting relationship between the Linear Programming relaxation and the formulation for an integer programming problem.

Proposition 2.2: P_1 and P_2 are two formulations for the same integer programming problem. Let P_1 be a better formulation than P_2 . Let $z_i^{\text{LP}} = \max\{cx : x \in P_i\}$. Then $z_1^{\text{LP}} \leq z_2^{\text{LP}}$.



Revisit the TSP formulations

- It can be shown that the “cut set formulation” (C) is better than the “sequence variable formulation” (S).
- We have constructed a small 10-city TSP. The optimal solution is 881.

Type	# Var	# Constr	LP opt	Gap (%)
C	90	502	878	0.34
S	99	92	773.6	12.2

- This is of course just one example. The only thing we know is that $z_C^{\text{LP}} \geq z_S^{\text{LP}}$ (it is a lower bound for a min problem).



Lagrangian Relaxation

Consider an integer programming problem:

$$\begin{aligned} \max \quad & cx \\ & Ax \leq b \\ & x \in X \end{aligned}$$

Now assume that if we dropped $Ax \leq b$ the problem

$$\begin{aligned} \max \quad & cx \\ & x \in X \end{aligned}$$

would be “easy”.





Lagrangian relaxation

Now go one step further and add a penalty term (to the objective function) that is “active” when $Ax \leq b$ is violated, that is,

$$\max_{x \in X} cx + u(b - Ax)$$

$z(u) = \max\{cx + u(b - Ax) : x \in X\}$ is called the **Lagrangian relaxation of**
 $z = \max\{cx : Ax \leq b, x \in X\}$.



Duality

- The two problems $z = \max\{c(x) : x \in X\}$ and $w = \min\{w(u) : u \in U\}$ form a **(weak-)dual pair** if $c(x) \leq w(u)$ for all $x \in X, u \in U$.
- If $z = w$, that is, there exists $x^* \in X$ and $u^* \in U$ s.t. $c(x^*) = w(u^*)$ they form a **(strong-)dual pair**.



Linear programming relaxations immediately leads to a weak dual.

- The integer program $z = \max\{c(x) : Ax \leq b, x \in Z_+^n\}$ and the linear program $w^{LP} = \min\{ub : uA \geq c, u \in R_+^m\}$ form a weak dual pair.
- Suppose that IP and D are a weak-dual pair.
 1. If D is unbounded, IP is infeasible.
 2. If $x^* \in X, u^* \in U$ satisfy $c(x^*) = w(u^*)$ then x^* is optimal for IP and u^* is optimal for D.



Matchings and Coverings

- **Matching:** Given a graph, a matching is a subgraph with the property that no two edges are incident with the same node.
- **Covering:** Given a graph, a cover is a subset of nodes such that at least one endpoint for all edges belongs to the cover.





Greedy algorithms

- A greedy algorithm obtains an feasible solution to a problem by making a sequence of choices. For each decision point in the algorithm, the choice that seems best at the moment is chosen.
- This strategy does *not* always produce an optimal solution, but sometimes it does.