

Total Unimodular matrices

Jesper Larsen & Jens Clausen & David Pisinger
jesla, jenc1, pisinger@man.dtu.dk

DTU Management
Technical University of Denmark

Effective methods

Four properties are often seen together for $\max\{cx : x \in X \subset R^n\}$:

1. Efficient Optimization Property
2. Strong Dual Property
3. Efficient Separation Property
4. Explicit Convex Hull Property

Separation Problem

Definition: The **Separation problem** for a COP $\max\{cx : x \in X\}$ is: Given an $x^* \in R^n$: Is x^* in $\text{conv}(X)$?

- yes : ok
- no : find an inequality $\pi x \leq \pi_0$ s.t. $\forall x \in X$ **but** $\pi x^* > \pi_0$

Total Unimodularity I

- Consider

$$\max\{cx : Ax \leq b, x \in Z_+^n\}$$

Let us assume that A and b only contains integers.
When is an optimal integer solution to the LP-relaxation (LP) guaranteed?

- From LP theory we have $(x_B, x_N) = (B^{-1}b, 0)$, where B is an $m \times m$ non-singular submatrix of (A, I) and I is a square identity matrix of size m .

Total Unimodularity II

- **Sufficient condition:** If the optimal basis B has $\det(B) = \pm 1$, then the LP-relaxation solves the IP.
- **Cramer's Rule:** $B^{-1} = B^* / \det(B)$
- **Definition:** A matrix A is **totally unimodular (TU)** if every square submatrix of A has determinant $+1, -1, 0$.
- **Quick observation:** If A is TU, $a_{ij} \in \{+1, -1, 0\}$ for all i, j .

Properties of TU II

- Proposition:** A is TU if,
1. $a_{ij} \in \{0, 1, -1\}$ for all i, j .
 2. each column contains at most two non-zeros.
 3. A partition of the rows exists, $M_1 \cup M_2 = M, M_1 \cap M_2 = \emptyset$ s.t.
 - if column j has two nonzeros of different sign these both belong to either M_1 or M_2 .
 - if column j has two nonzeros of same sign these belong to one each of M_1 and M_2 .

Properties of TU I

- Proposition:**
- A is TU $\Updownarrow$
 - A^T is TU $\Updownarrow$
 - (A, I) is TU.

Examples

- $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is not TU
- $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ is not TU
- $\begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is TU

Examples

Interval matrices are TU

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

TU vs. Properties

If the IP $\max\{cx : Ax \leq b, x \in Z_+^n\}$ has an A that is TU then we have

- Strong Dual Property
- Explicit Convex Hull Property
- Efficient Separation Property