

# Min Cost Flow - Definition

The Min Cost Flow problem consists in supplying the sinks from the sources by a flow in the cheapest possible way:

$$\begin{aligned} \min \quad & \sum_{(i,j) \in E} w_{ij} x_{ij} \\ \text{s.t.} \quad & f_x(i) = b_i \quad i \in V \\ & 0 \leq x_{ij} \leq u_{ij} \quad (i,j) \in E \end{aligned}$$

where  $f_x(i) = \sum_{(j,i) \in E} x_{ji} - \sum_{(i,j) \in E} x_{ij}$ .

- Flow balance constraints
- Capacity constraints

Often integral flow:  $x_{ij} \in \mathbb{Z}$

# The Min Cost Flow Problem

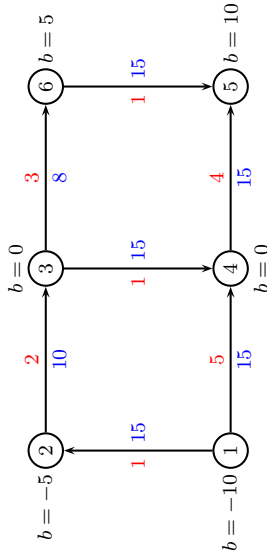
– *augmenting cycle method*

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# Min Cost Flow - Terminology

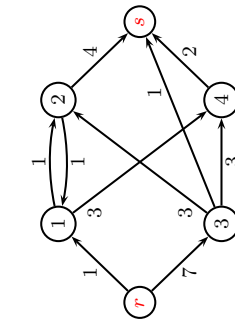
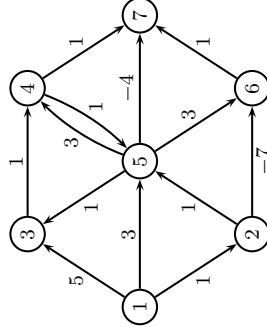
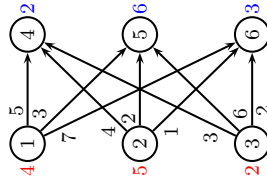
- We consider a directed graph  $G = (V, E)$  in which each edge  $e$  has a capacity  $u_e \in \mathbb{R}_+$  and a unit transportation cost  $w_e \in \mathbb{R}$ .
- Each vertex  $i$  furthermore has a demand  $b_i \in \mathbb{R}$ . If  $b_i \geq 0$  then  $i$  is called a **sink**, and if  $b_i < 0$  then  $i$  is called a **source**.
- We assume that  $b_V = \sum_{i \in V} b_i = 0$ .



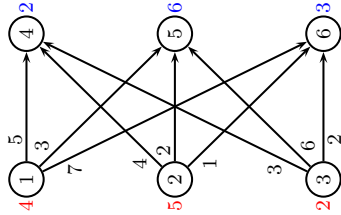
# Special cases

Numerous flow problems can be stated as a Min Cost Flow problem:

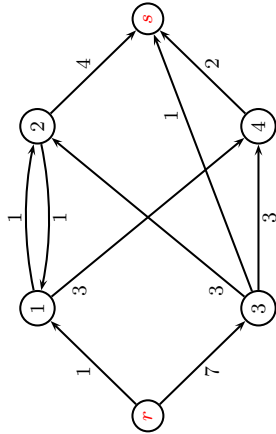
- The Transportation Problem
- The Shortest Path Problem
- The Max Flow Problem



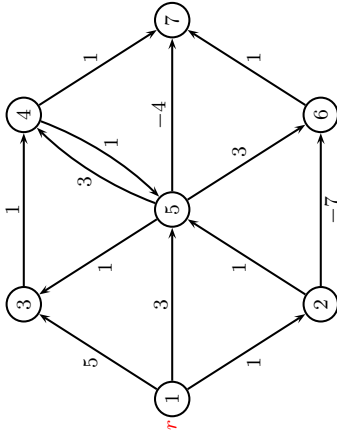
# Transportation Problem



# Max Flow Problem



# Shortest Path Problem



# TU and Min Cost Flow

- (Prop. 3.1): A matrix  $A$  is TU if and only if (i) the transpose matrix  $A^T$  is TU if and only if (ii) the matrix  $(A, I)$  is TU.
- (Prop. 3.2): (Sufficient condition). A matrix  $A$  is TU if
  1.  $a_{ij} \in \{+1, -1, 0\}$  for all  $i, j$ .
  2. Each column contains at most two nonzero coefficients  $(\sum_{i=1}^m |a_{ij}| \leq 2)$ .
  3. There exists a partition  $(M_1, M_2)$  of the set  $M$  of rows such that each column  $j$  containing two nonzeros satisfies  $\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0$ .

# TU and Min Cost Flow

|         | $x_{e1}$ | $x_{e2}$ | ... | $x_{ij}$ | ... | $x_{em}$  |
|---------|----------|----------|-----|----------|-----|-----------|
|         | $w_{e1}$ | $w_{e2}$ | ... | $w_{ij}$ | ... | $w_{em}$  |
| 1       | -1       | .        | ... | ...      |     | $b_1$     |
| 2       | .        | .        | ... | .        |     | $b_2$     |
| .       | .        | .        | ... | ...      |     | .         |
| $i$     | 1        | .        | ... | -1       |     | $b_i$     |
| .       | .        | .        | ... | ...      |     | .         |
| $j$     | .        | .        | ... | 1        |     | $b_j$     |
| .       | .        | .        | ... | ...      |     | .         |
| $n$     | .        | .        | ... | .        |     | $b_n$     |
| $e_1$   | -1       |          |     |          |     | $-u_1$    |
| $e_2$   |          | -1       |     |          |     | $-u_2$    |
| .       | .        | .        | .   | .        |     | .         |
| $(i,j)$ |          |          |     | -1       |     | $-u_{ij}$ |
| .       | .        | .        | .   | .        |     | .         |
| $e_m$   |          |          |     | .        | -1  | $-u_m$    |

# Min Cost Flow - Dual LP

$$\begin{aligned} \min \quad & \sum_{(i,j) \in E} w_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{(j,i) \in E} x_{ji} - \sum_{(i,j) \in E} x_{ij} = b_i \quad (y_i) \quad i \in V \\ & x_{ij} \leq u_{ij} \quad (z_{ij}) \quad (i,j) \in E \\ & x_{ij} \geq 0 \quad (x_{ij}) \quad (i,j) \in E \end{aligned}$$

Dual problem:

$$\begin{aligned} \max \quad & \sum_{i \in V} b_i y_i - \sum_{(i,j) \in E} u_{ij} z_{ij} \\ \text{s.t.} \quad & -w_{ij} - y_i + y_j \leq z_{ij} \quad (i,j) \in E \\ & z_{ij} \geq 0 \quad (i,j) \in E \end{aligned}$$

Reduced cost  $\bar{w}_{ij} = w_{ij} + y_i - y_j$ . Last constraint

$$-\bar{w}_{ij} \leq z_{ij} \quad (i,j) \in E \quad (x_{ij})$$

# Optimality Conditions I

- If  $u_e = +\infty$  (i.e. no capacity constraints for  $e$ ) then  $z_e$  must be 0 and hence  $\bar{w}_e \geq 0$  just has to hold (primal optimality condition for the LP).
- If  $u_e \neq +\infty$  then  $z_e \geq 0$  and  $z_e \geq -\bar{w}_e$  must hold.  $z$  has **negative** coefficient in the objective function – hence the best choice for  $z$  is as small as possible:  
 $z_e = \max\{0, -\bar{w}_e\}$ .
- Therefore, the optimal value of  $z_e$  is uniquely determined from the other variables, and  $z_e$  is “unnecessary” in the dual problem.

# Optimality Conditions I - Comp. Slackness

- Complementary slackness conditions (each primal variable times the corresponding dual slack must equal 0, and each dual variable times the corresponding primal slack must equal 0 in optimum)

$$x_e > 0 \Rightarrow -\bar{w}_e = z_e = \max(0, -\bar{w}_e)$$

$$\text{i.e. } (x_e > 0 \Rightarrow -\bar{w}_e \geq 0)$$

and

$$z_e > 0 \Rightarrow x_e = u_e$$

$$\text{i.e. } (-\bar{w}_e > 0 \Rightarrow x_e = u_e)$$

$$\begin{array}{ccc} x_e > 0 & \Rightarrow & \bar{w}_e \leq 0 \\ x_e < u_e & \Rightarrow & \bar{w}_e \geq 0 \end{array}$$

$$\begin{array}{ccc} \bar{w}_e < 0 & \Rightarrow & x_e = u_e \\ \bar{w}_e > 0 & \Rightarrow & x_e = 0 \end{array}$$

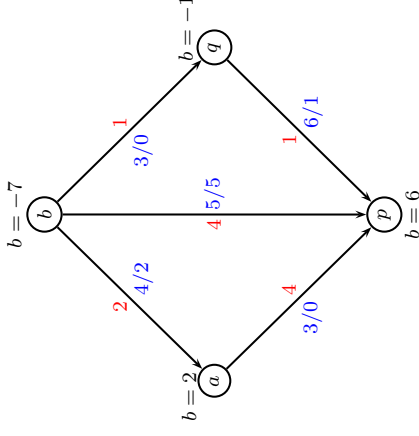
## Residual Graph I

Idea:

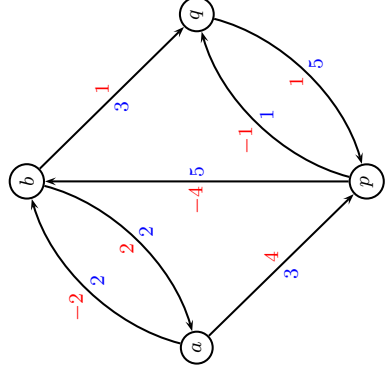
- Find initial feasible flow
- Consider residual graph
- Find augmenting path (cycle) in residual graph
- Construct new feasible flow
- For a legal flow  $x$  in  $G$ , the residual graph is (like for Max Flow) a graph, in which the paths indicate **how flow excess can be moved** in  $G$  given that the flow  $x$  already is present. The only difference is that each edge has a cost assigned.

The Min Cost Flow Problem – p. 13

## Residual Graph



Graph with flow



Residual graph with costs and capacities

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## Residual Graph II

- The residual graph  $G_x$  for  $G$  wrt.  $x$  is defined by

$$E(G_x) = E_x = \{(i, j) : (i, j) \in E \wedge x_{ij} < u_{ij}\} \cup \{(i, j) : (j, i) \in E \wedge x_{ji} > 0\}$$

$$V(G_x) = V$$

- The unit cost

$$w'_{ij} = \begin{cases} w_{ij} & \text{if } x_{ij} < u_{ij} \\ -w_{ji} & \text{if } x_{ji} > 0 \end{cases}$$

- The residual capacity

$$r'_{ij} = \begin{cases} u_{ij} - x_{ij} & \text{if } x_{ij} < u_{ij} \\ x_{ji} & \text{if } x_{ji} > 0 \end{cases}$$

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## Residual Graph III

- Note that a directed circuit with negative cost in  $G_x$  corresponds to a negative cost circuit in  $G$ , if cost are added for forward edges and subtracted for backward edges.
- Note that if a set of potentials  $y_i, i \in V$  are given, and the cost of a circuit wrt. the reduced costs for the edges ( $\bar{w}_{ij} = w_{ij} + y_i - y_j$ ) are calculated, the cost remains the same as the original costs – the potentials are “telescoped” to 0.

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# Min Cost Flow - Negative cost circuits

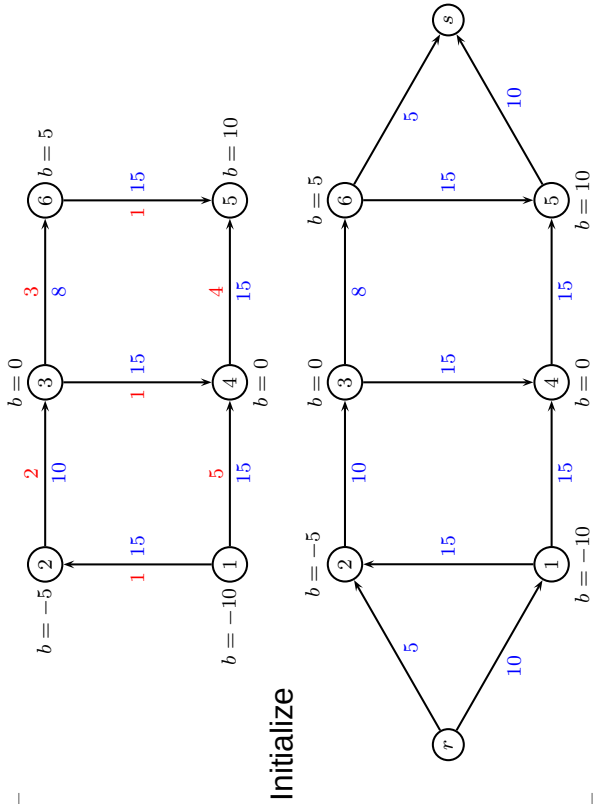
- A primal feasible flow satisfying sink demands from sources and respecting the capacity constraints is **optimal if and only if** an  $x$ -augmenting circuit with negative  $w$ -cost (or negative  $\bar{w}$ -cost – there is no difference) does not exist.
- This is the idea behind the identification of optimal solutions in the **network simplex algorithm**.

# Augmenting Circuit Algorithm

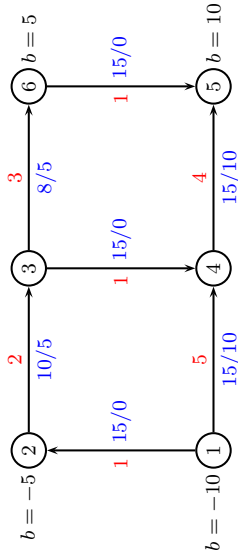
- 1 find a feasible solution  $x$  ;
- 2 **while** *there exists an augmenting circuit* **do**
- 3    find an augmenting circuit  $C$  ;
- 4    find  $\delta = \min_{(i,j) \in C} \{r_{ij}\}$  ;
- 5    augment  $x$  on  $C$  with  $\delta$  units ;

- initial flow is found by solving max-flow problem
- negative circuit is found by Bellman-Ford in  $O(nm)$

# Example

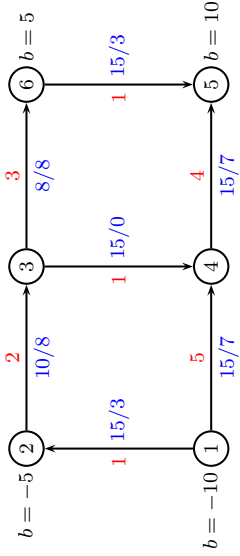


# Example



cost:  $z = 2 \cdot 5 + 3 \cdot 5 + 5 \cdot 10 + 4 \cdot 10 = 115$   
cycle:  $C = \{2, 3, 6, 5, 4, 1\}$   
 $w(C) = 2 + 3 + 1 - 4 - 5 + 1 = -2$   
 $\delta = \min\{5, 3, 15, 10, 10, 15\} = 3$

## Example



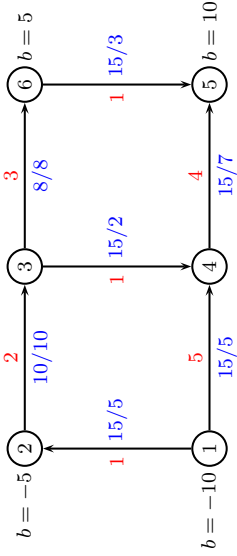
cost:  $z = 2 \cdot 8 + 3 \cdot 8 + 1 \cdot 3 + 1 \cdot 3 + 5 \cdot 7 + 4 \cdot 7 = 109$

cycle:  $C = \{2, 3, 4, 1\}$

$w(C) = 2 + 1 - 5 + 1 = -1$

$\delta = \min\{2, 15, 7, 12\} = 2$

## Example

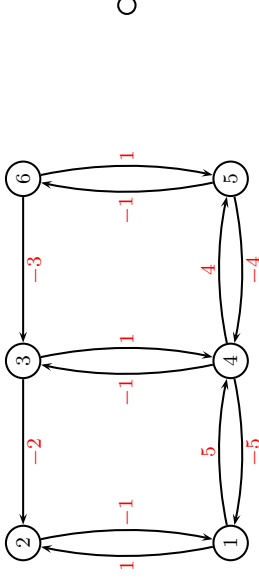


cost:  $z = 2 \cdot 10 + 3 \cdot 5 + 1 \cdot 5 + 1 \cdot 2 + 1 \cdot 3 + 5 \cdot 5 + 4 \cdot 7 = 107$

No negative cycles

## Proof of termination criteria

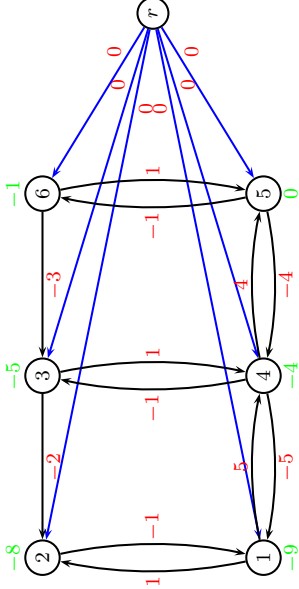
A flow  $x$  is optimal  $\Leftrightarrow$  no augmenting circuit can be found



• if negative circuit can be found then construct better solution

• now, assume no negative circuit can be found in residual graph

## Proof of termination criteria



Shortest path from  $r$  to  $i$  defines  $y_i$ . Satisfies  $y_i + w'_{ij} \geq y_j$

$y_i + w_{ij} \geq y_j$  if  $x_{ij} < u_{ij}$

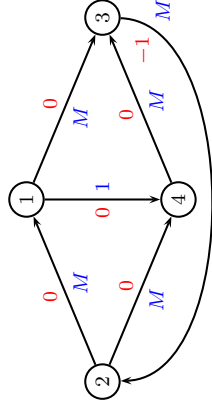
$y_i - w_{ji} \geq y_j$  if  $x_{ji} > 0$

since  $\bar{w}_{ij} = w_{ij} + y_i - y_j$  we have

$\bar{w}_{ij} \geq 0$  if  $x_{ij} < u_{ij}$

$\bar{w}_{ji} \leq 0$  if  $x_{ji} > 0$

# Time Complexity



Time complexity can be bad  
(like augmenting path for maximum flow)

# Time Complexity

- Finding negative cycle takes  $O(nm)$  with Bellman-Ford
- Objective  $z \leq mWU$  where  $w_{ij} \leq W$  and  $u_{ij} \leq U$
- Each iteration reduces  $z$  by at least 1
- At most  $mWU$  iterations
- Many strategies to limit the number of iterations
  - Most negative augmenting cycle at each step
  - Smallest average arc cost
- Expensive algorithm compared to network simplex