Solving Real-Life Problems with Integer Programming

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42113 Network and Integer Programming

Linear Programming

- Linear Programming is a strong tool for many real-life optimization problems.
- We can solve large problems (thousands of constraints and millions of variables).
- We can solve problems fast (even big problems with hundreds of constraints and thousands of variables solve in seconds or fractions hereof).
- We can model a lot of problem type and using a modelling language (CPLEX, GAMS, OPL, MPL, etc.) it is easy to make the computer solve the problem.

Integer Programming

- Integer variables extends the possibilities of problem solving.
- Basically all modeling languages incorporates integer variables.

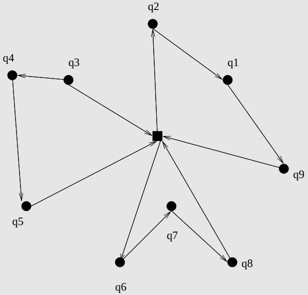
- Problem is that integer programs are (in general) much more difficult to solve than linear programs.
- It is therefore important to know:
  - How does an integer programming solver work.
  - How can we exploit structure to implement efficient methods.

Vehicle Routing

Vehicle Routing

Given is a set of vehicles K and a set of nodes N. Each node i has to be supplied from a depot with some quantity q_i. Each vehicle has a capacity of Q.

A Solution



Integer Program

- Objective: minimize number of vehicles or minimize to distance driven.
- Constraints: Routes start and finish at depots, must respect capacity, and go from customer to customer.

A scheduling problem

We have 6 assignments A, B, C, D, E and F, that needs to be carried out. For every assignment we have a start time and a duration (in hours).

Assignment	A	B	C	D	E	F
Start	0.0	1.0	2.0	2.5	3.5	5.0
Duration	1.5	2.0	2.0	2.0	2.0	1.5

Objective

A Workplan

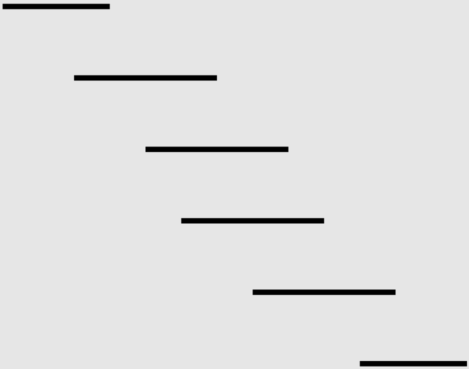
A workplan is a set of assignments. Now we want to formulate a mathematical model that finds the cheapest set of workplans that fullfills all the assignments.

Workplan rules

- A workplan can not consist of assignments that overlap each other.
- The length L of a workplan is equal to the finish time of the last assignment minus start of the first assignments plus 30 minutes for checking in and checking out.
- the cost of a workplan is $\max(4.0, L)$.

A View of the Assignments

	Start	Duration
A	0.0	1.5
B	1.0	2.0
C	2.0	2.0
D	2.5	2.0
E	3.5	2.0
F	5.0	1.5



Workplans starting with assignment A

	c_1	c_2	c_3	c_4	c_5	c_6	c_7
A	1	1	1	1	1	1	1
B	0	0	0	0	0	0	0
C	0	1	1	0	0	0	0
D	0	0	0	1	1	0	0
E	0	0	0	0	0	1	0
F	0	0	1	0	1	0	1

	4	$4\frac{1}{2}$	7	5	7	6	7
A	1	1	1	1	1	1	1
B	0	0	0	0	0	0	0
C	0	1	1	0	0	0	0
D	0	0	0	1	1	0	0
E	0	0	0	0	0	1	0
F	0	0	1	0	1	0	1

Completing the model

	4	$4\frac{1}{2}$	7	5	7	6	7	4	5	6	4	5	4	$4\frac{1}{2}$	4	4
A	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
B	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0
C	0	1	1	0	0	0	0	0	0	0	1	1	0	0	0	0
D	0	0	0	1	1	0	0	0	0	0	0	0	1	1	0	0
E	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0
F	0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1

The Mathematical Model

We get the following model:

$$\begin{array}{ll} \min & \sum_{j=1}^{N} c_j x_j \\ \text{s.t.} & \sum_{j=1}^{N} a_{ij} x_j = 1 \quad \forall i \\ & x_j \in \{0,1\} \end{array}$$

- $x_j = 1$ if we use workplan j and 0 otherwise (variable).
- c_j is equal to the cost of the plan (parameter).
- $a_{ij} = 1$ if assignment i is included in workplan j (parameter).

Solving the problem

- A feasible solution is: $x_3 = 1, x_9 = 1, x_{13} = 1$ with a solution value of 16.
- An optimal solution is $x_2 = 1, x_9 = 1, x_{14} = 1$ with a solution value of 13.
- In general the integrality constraint makes the problem much more difficult to solve than if it was a linear program.
- Another challenge is the number of variables/workplans. On this small example we only have 16 with is manageable, but real-life problems does not only have 6 assignments.

Airline Crew Scheduling

- In major airlines it is possible to generate millions if not billions. There are techniques to eliminate many of the "unproductive" workplans. Still the problem remains a challenge....
- Air New Zealand is one of the smaller major airlines.
  - Estimated an annual savings of US$ 8 millions.
  - Development costs since 1984: US$ 1 millions.
  - Total crewing costs in 1999: US$ 125 millions (6.4%).
  - Air NZ op. profit in 1999: US$ 67 millions (12%).

Production Planning in an Aluminium

Tiwai Point

This is the New Zealand Aluminium Smelter on the southern tip of the South Island.

Reduction Line

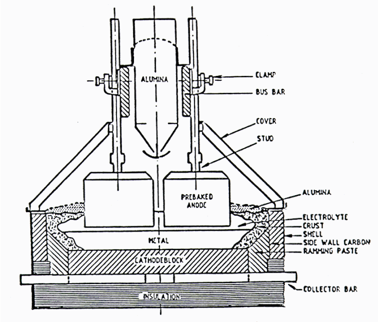
Here is a view inside one of the big reduction lines.

Production planning

This is the chemical reaction in a Hall-Heroult cell

$$(2Al_2O_3 + 3C) + \text{Energy} \rightarrow 4Al + 3CO_2$$

Every "reduction line" (600 m long) is devided into 4 tapping bays (300 m long) every consisting of 51 cells. All active cells in a tapping bay is tapped once a day.



Purity of Aluminium

The purity of aluminium in a cell depends on

- the age of the cell,
- the purity of aluminium ore, and
- the carefullness of the workers.

The purity in the metal in a cell is determined once a day. Possible impurities: Iron, Silicium, Gallium, Nickel, Vanadium

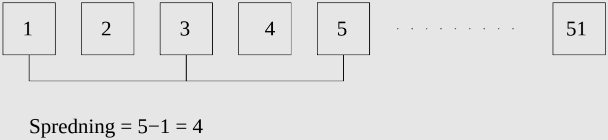
Why worry about the purity?

Metal Premium Value by Alloy Code

Code	Min Al%	Max Si%	Max Fe%	Max Ga%	Premium
AA???	0.00	1.000	1.000	1.000	-50
AA150	99.50	0.100	0.300	0.100	-40
AA160	99.60	0.100	0.300	0.100	-25
AA1709	99.70	0.100	0.200	0.100	0
AA601E	99.70	0.100	0.080	0.100	40
AA601G	99.70	0.100	0.080	0.100	40
AA185G	99.85	0.054	0.094	0.014	15
AA190A	99.90	0.054	0.074	0.014	45
AA190B	99.90	0.050	0.050	0.014	50
AA190C	99.90	0.035	0.037	0.012	110
AA190K	99.90	0.022	0.055	0.024	100
AA191P	99.91	0.030	0.045	0.010	120
AA191B	99.91	0.030	0.027	0.012	140
AA192A	99.92	0.030	0.040	0.012	140
AA194A	99.94	0.020	0.040	0.007	150
AA194B	99.94	0.034	0.034	0.010	180
AA194C	99.94	0.022	0.027	0.009	200
AA196A	99.96	0.020	0.015	0.010	260

Producing in batches

- A “batch” consists of metal from 3 cells.
- The purity of the batch is the average of the purity of contributing cells.
- Due to the long distances in the tapping bay the cells in a batch are not allowed to be to far from each other.



Batching cells

Batching conditions

- Given a metal purity for each cell in the tapping bay we want to maximize the value of the metal in our batches.
- Under the condition of:
  - all cells are tapped ones
  - there are at most three cells in a batch
  - the "spread" S in a batch is $\leq S_1$
  - except for at most C batches where $S_1 < S \leq S_2$

Example

Let $S_1 = 4$, $C = 0$. Consider 6 cells with the following contents

Cell	Al%	Si%	Fe%
652 (1)	99.87	0.050	0.058
653 (2)	99.95	0.022	0.026
654 (3)	99.94	0.020	0.029
655 (4)	99.93	0.023	0.039
656 (5)	99.78	0.024	0.019
657 (6)	99.93	0.022	0.030

Possible batches with cell 1 (652)

Batch	Cells	Al%	Si%	Fe%	Code	Premium
1	1 2 3	99.920	0.031	0.038	AA190K	100
2	1 2 4	99.917	0.032	0.041	AA190K	100
3	1 2 5	99.867	0.032	0.091	AA185G	15
4	1 3 4	99.913	0.031	0.042	AA190K	100
5	1 3 5	99.863	0.031	0.092	AA185G	15
6	1 4 5	99.860	0.032	0.096	AA1709	0

The Final Model

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
652	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
653	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0	0
654	1	0	0	1	1	0	1	1	1	0	0	0	1	1	1	0
655	0	1	0	1	0	1	1	0	0	1	1	0	1	1	0	1
656	0	0	1	0	1	1	0	1	0	1	0	1	1	0	1	1
657	0	0	0	0	0	0	0	0	1	0	1	1	0	1	1	1
	1	1		1			1		1		1			1	1	
	0	0	1	0	1		8	1	8	1	4	1	1	4	8	1
	0	0	5	0	5	0	0	5	0	5	0	5	5	0	0	5

Feasible and Optimal Solutions

- "Quick and Dirty": (652, 653, 654) and (655, 656, 657) gives a value of 115.
- Take the "Quick and Dirty" solution and interchange 654 and 656 and we get a profit of 155.
- The optimum is (652, 653, 655) and (654, 656, 657) with a profit of **280**!!

Cell Batching Optimization

Mathematical Model

$$\begin{array}{ll} \max & \sum_{i=1}^{n} p_i x_i \\ \text{st} & \sum_{i=1}^{n} a_{ji} x_i = 1 \quad j = 1, 2, \ldots, m \\ & \sum_{i=1}^{n} s_j x_j \leq C \end{array}$$

where $s_j = 1$ if the spread is larger than S_1 and 0 otherwise.