Solution to 11.16 from "Network Flows" by Ahuja et al. Firs we consider the transshipment problem. Initial tree is $T = \{(1,3),(3,2),(2,4),(4,5),(5,6)\}$. We select vertex 1 as the root vertex. We get the situation in Figure 1.

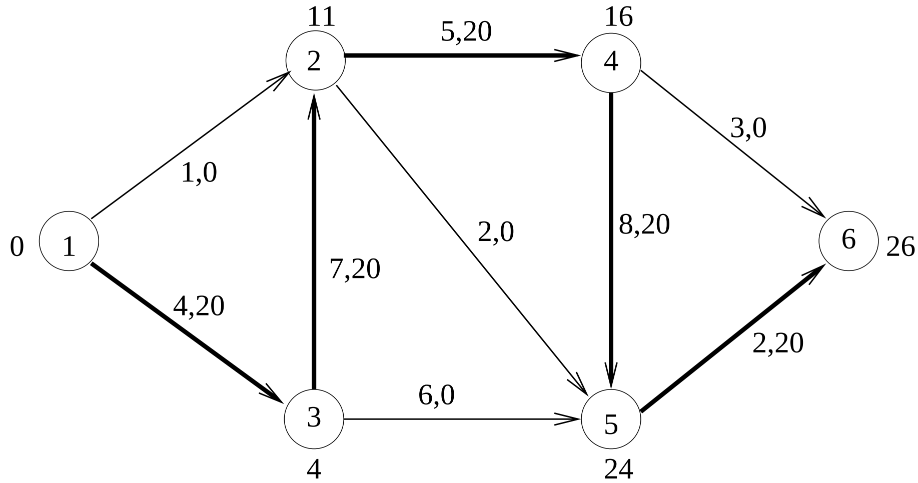

Figure 1: Graph and initial tree. Labels on the edges are u_e, x_e. Labels on the nodes are the potential

we compute $\bar{w}_{12} = w_{12} + y_1 - y_2 = 1 + 0 - 11 = -10$, so $(1,2)$ defines a negative-cost circuit $\langle 1,2,3,1\rangle$. Both $(1,3)$ and $(3,2)$ are backward edges and defines the change in the flow. So we add 20 to $(1,2)$ and decrease flow by 20 on $(1,3)$ and $(3,2)$. We then get the situation in Figure 2.

Note that we choose $(3,2)$ to remain in the tree although we are in this case free to choose any of the two backward edges.

Now we compute $\bar{w}_{13} = w_{13} + y_1 - y_3 = 4 + 0 - (-6) = 10$, so we continue to $\bar{w}_{25} = w_{25} + y_2 - y_5 = 2 + 1 - 14 = -11$. $(2,5)$ defines a negative-cost circuit. We push 20 unit in the corresponding circuit and get the situation in Figure 3.

Again there are two candidates as the "leaving edge", and we select arbitrarily $(4,5)$. Then we compute $\bar{w}_{13} = w_{13} + y_1 - y_3 = 4 + 0 - (-6) = 10$ and $\bar{w}_{35} = w_{35} + y_3 - y_5 = 6 + (-6) - 3 = -3$. So $(3,5)$ defines the negative-cost circuit $\langle 3,5,2,3\rangle$. Note that this circuit actually does not change any flow, but due to a different potential it will result in a different tree (see Figure 4.

With the update let us check out reduced cost: $\bar{w}_{13} = w_{13} + y_1 - y_3 = 4 + 1 - (-3) = 8$, $\bar{w}_{32} = w_{32} + y_3 - y_2 = 7 + (-3) - 1 = 3$, $\bar{w}_{46} = w_{46} + y_4 - y_6 = 3 + 6 - 5 = 4$ and finally $\bar{w}_{45} = w_{45} + y_4 - y_5 = 8 + 6 - 3 = 11$. As all reduced cost are positive we have found the minimum cost flow for the transshipment variant of our problem.

Now we focus on the minimum cost flow problem. Using the setup described in the exercise we get the following graph (see Figure 5).

Let us compute reduced cost for the non-tree edges: $\bar{w}_{13} = 10$, $\bar{w}_{35} = -9$, $\bar{w}_{24} = 11$, and finally $\bar{w}_{56} = 7$. All edges are potential for improving the solution. Let us select edge $(2, 4)$ that defines the circuit $\langle 2, 4, 5, 2\rangle$. Flow will be decrease in the orientation given by $(2, 4)$. The bottleneck will be defined by $(2, 5)$ with a flow change of 5 units. After having updated the potentials we get the situation in Figure 6.

Now we again compute the reduced cost for the non-tree edges: $\bar{w}_{13} = 10$, $\bar{w}_{25} = -11$, $\bar{w}_{35} = -14$, and now $\bar{w}_{56} = 7$. Of several possible improving circuits we choose the one defined by $(3, 5)$, which is $\langle 3, 5, 4, 2, 3\rangle$. The edge $(4, 5)$ defines with its flow of 5 the bottleneck. We then get to the setup in Figure 7.

We compute the reduced cost for the non-tree edges (again!): $\bar{w}_{13} = 10$, $\bar{w}_{25} = 3$, $\bar{w}_{45} = 14$, and $\bar{w}_{56} = -7$. Here we select the circuit defined by $(1, 3)$. This results in the situation in Figure 8.

Yet another time we compute the reduced cost on the non-tree edges: $\bar{w}_{12} = -10$, $\bar{w}_{25} = 3$, $\bar{w}_{45} = 14$, and $\bar{w}_{56} = -7$. Here we use the circuit defined by $(2, 5)$ and augment the flow along $\langle 2, 5, 3, 2\rangle$. This gets us to Figure 9.

As we compute the reduced cost for the non-tree arcs we get $\bar{w}_{12} = -7$, $\bar{w}_{32} = 3$, $\bar{w}_{45} = 11$ and $\bar{w}_{56} = -4$. Now for the edges in U the reduced cost is negative and for the edges in L the reduced cost is positive. This means that we have found an optimal solution.

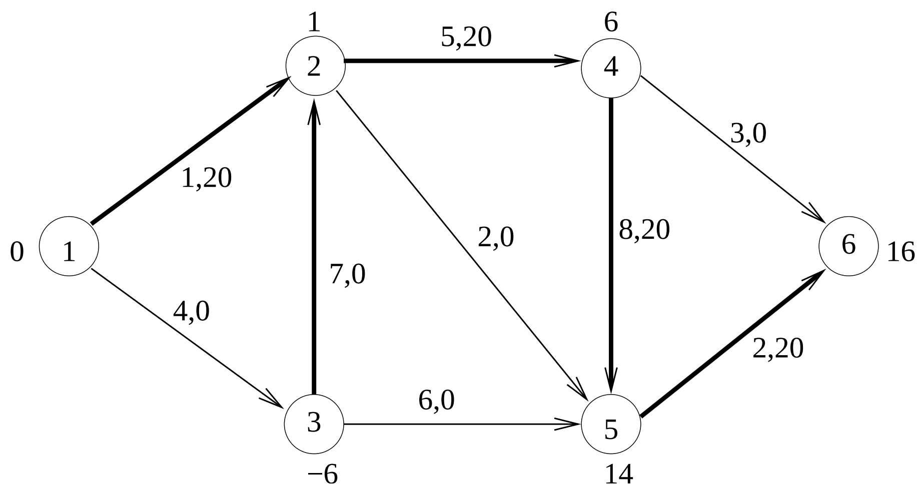

Figure 2: Graph after first iteration. Labels on the edges are u_e, x_e. Labels on the nodes are the potential

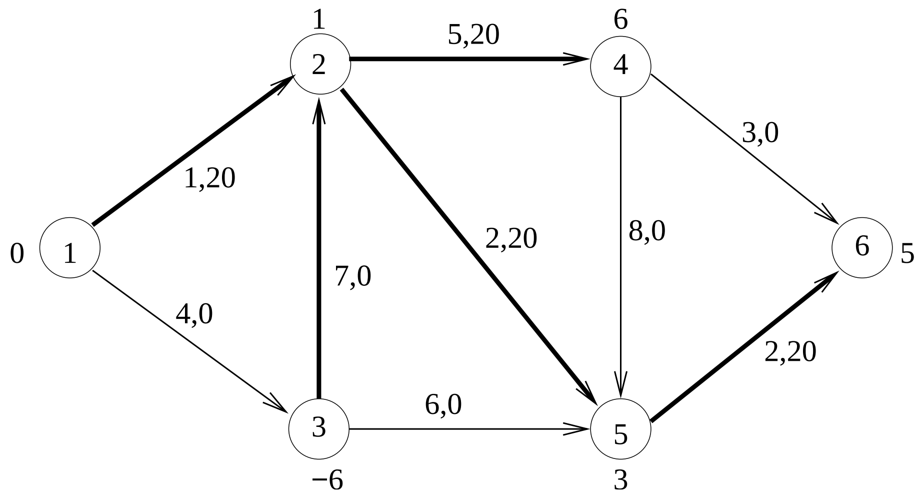

Figure 3: Graph after the second iteration. Labels on the edges are u_e, x_e. Labels on the nodes are the potential

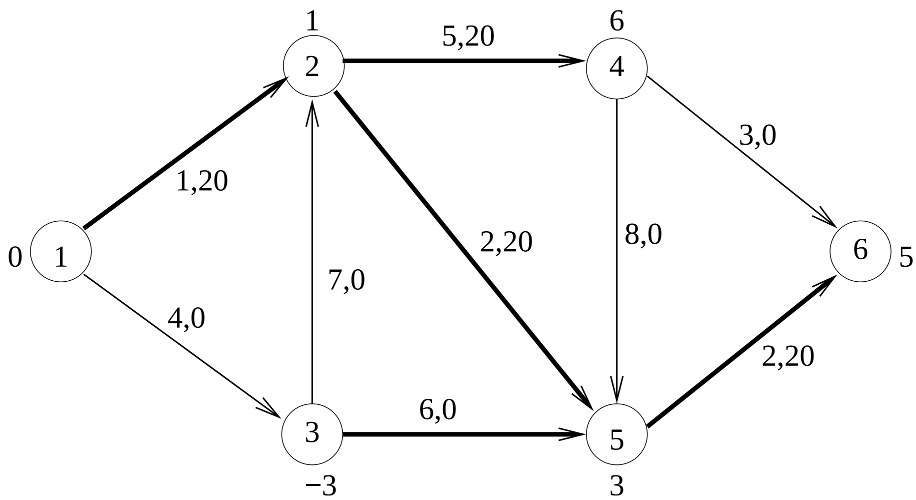

Figure 4: Graph after the third iteration. Labels on the edges are u_e, x_e. Labels on the nodes are the potential

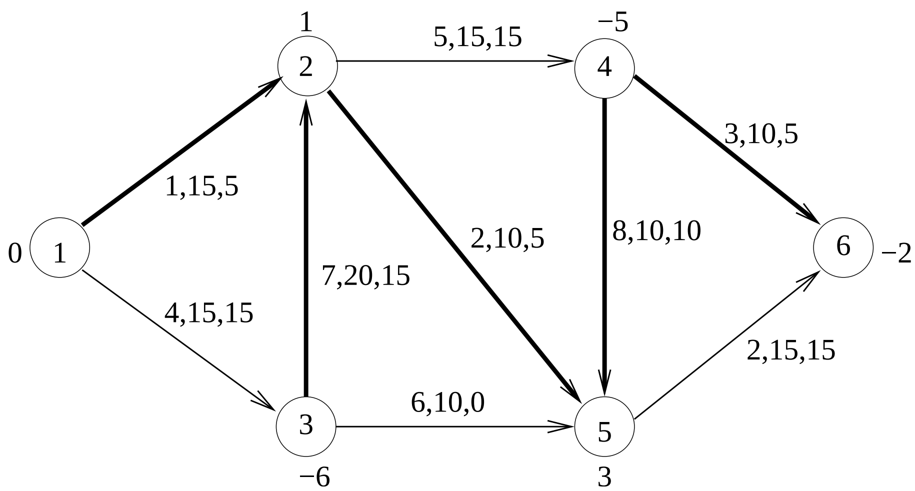

Figure 5: Example for exercise. Labels on the edges are w_e, u_e, x_e. Labels on the nodes are the potential

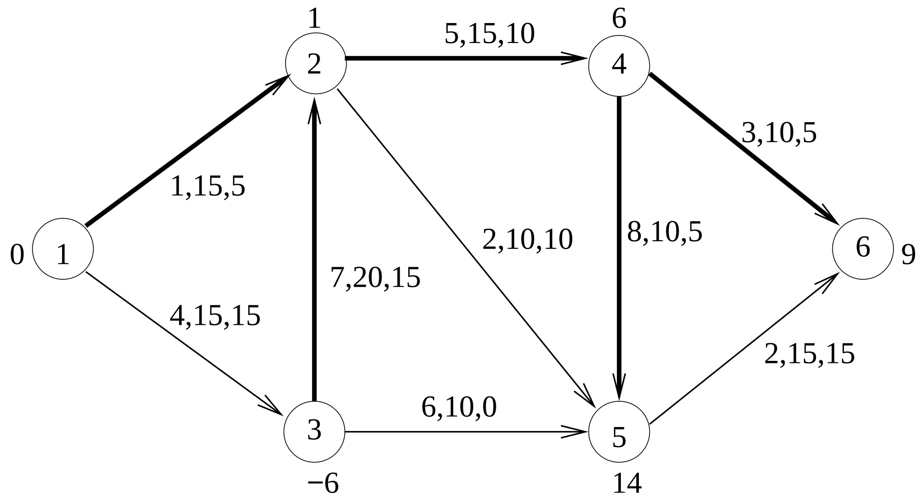

Figure 6: Example for exercise. Labels on the edges are w_e, u_e, x_e. Labels on the nodes are the potential

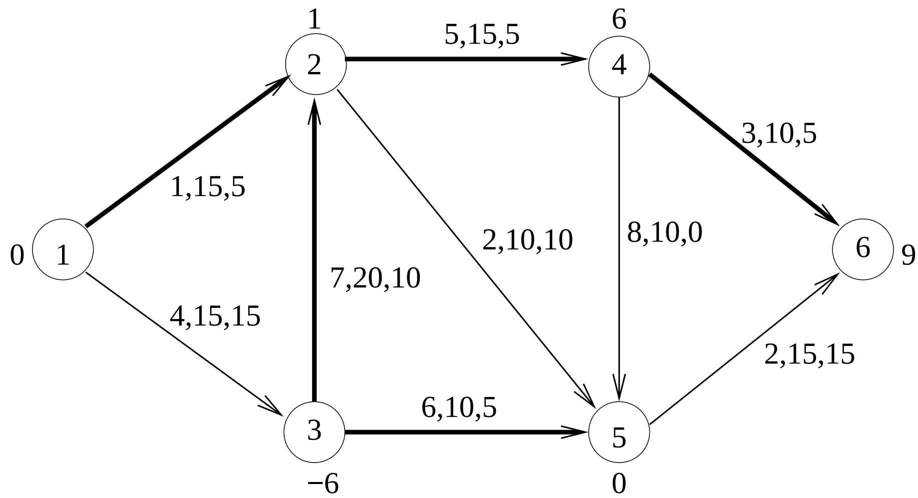

Figure 7: Example for exercise. Labels on the edges are w_e, u_e, x_e. Labels on the nodes are the potential

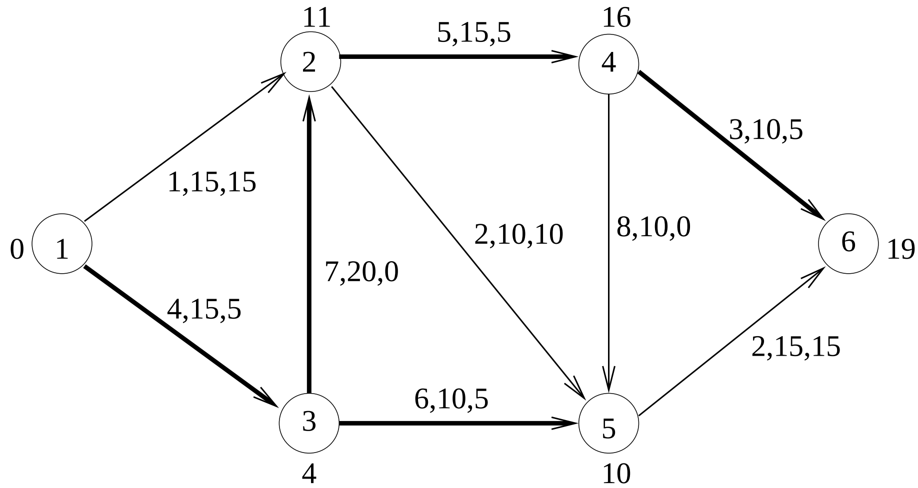

Figure 8: Example for exercise. Labels on the edges are w_e, u_e, x_e. Labels on the nodes are the potential

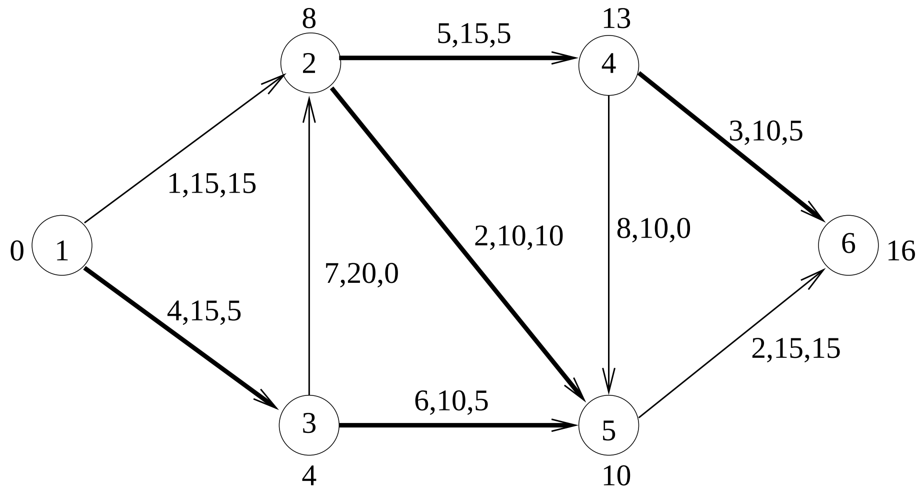

Figure 9: Example for exercise. Labels on the edges are w_e, u_e, x_e. Labels on the nodes are the potential