

Exercise 12.1

One by one, we apply the heuristics to the problem instance. The nodes are referred to as 1-7. It is important to note that the triangular inequality is not satisfied for this distance matrix and hence the lower bounds created by the heuristics are no longer valid.

Nearest Neighbour Heuristic

Each time a node is added, it is placed after the current last node in the tour. The tour is built as follows:

1
1 – 2
1 – 2 – 3
1 – 2 – 3 – 5
1 – 2 – 3 – 5 – 4
1 – 2 – 3 – 5 – 4 – 6
1 – 2 – 3 – 5 – 4 – 6 – 7

Total cost of the tour: $c = 232$.

Cheapest Insertion Heuristic

We have to calculate the cheapest point of insertion in the tour for each node. The minimum of these costs determines which node is inserted. The procedure looks like this:

1
1 – 2
1 – 2 – 3
1 – 2 – 3 – 5
1 – 2 – 3 – 5 – 4
1 – 2 – 3 – 5 – 6 – 4
1 – 2 – 3 – 5 – 6 – 4 – 7

In this case the only difference to the Nearest Neighbour Heuristic is in the second last iteration. Total cost of the tour: $c = 253$.

Furtherst Insertion Heuristic

Again, we have to calculate the cheapest point of insertion in the tour for each node. However, this time the maximum of these costs determines which node is inserted. The procedure looks like this:

1
 $1 - 6$
 $1 - 6 - 7$
 $1 - 5 - 6 - 7$
 $1 - 2 - 5 - 6 - 7$
 $1 - 2 - 4 - 5 - 6 - 7$
 $1 - 2 - 3 - 4 - 5 - 6 - 7$

Total cost of the tour: $c = 251$.

Savings Heuristic

We add the edges one by one in the following order:

$3 - 5$
 $4 - 5$
 $4 - 6$
 $1 - 2$
 $2 - 3$
 $6 - 7$
 $1 - 7$

This leads to the tour: $1 - 2 - 3 - 5 - 4 - 6 - 7$ with total cost $c = 232$.

Tree Heuristic

We form the following minimum spanning tree by Kruskal's algorithm:

$3 - 5$
 $4 - 5$
 $4 - 6$
 $1 - 2$
 $2 - 3$
 $3 - 7$

Doubling all edges we are able to create the Euler tour: $1 - 2 - 3 - 5 - 4 - 6 - 4 - 5 - 3 - 7 - 3 - 2 - 1$, which leads to the TSP-tour: $1 - 2 - 3 - 5 - 4 - 6 - 7$ with total cost $c = 232$.

Christofides' Heuristic

We use the same MST where only edges $1 - 7$ and $3 - 6$ are added. This leads to the Euler tour: $1 - 2 - 3 - 5 - 4 - 6 - 3 - 7 - 1$, which again leads to the TSP-tour: $1 - 2 - 3 - 5 - 4 - 6 - 7$ with total cost $c = 232$.

Exercise 12.4

Remember that the “maximum cardinality matching problem” and the “minimum cardinality covering of edges by nodes problem” forms a weak-dual problem pair (page 29).

Given a maximal matching $M \subseteq E$ let V' be the endpoint nodes of all edges in M . V' is a cover of edges by nodes, since no edge $e \in E \setminus M$ exists with both endpoint nodes in $V \setminus V'$. If such an edge existed M was not maximal. Note that $|V'| = 2|M| = 2z^H$, since matchings consist of node-disjoint edges.

The size of the cover is an upper bound on the maximum matching, i.e. $z \leq |V'| = 2|M| = 2z^H$. Rewriting we get: $\frac{1}{2}z \leq z^H$.

Exercise 12.5

Assume that the elements are ordered such that $\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \dots \geq \frac{c_N}{a_N}$.

Solving the LP relaxation gives an upper bound - a fractional solution of the form $x_i = 1$ for $i = 1 \dots r-1$, $0 \leq x_r < 1$ and $x_i = 0$ for $i = r+1 \dots N$. The value of the solution is $z_{LP} = \sum_{i=1}^{r-1} c_i + c_r \cdot x_r$ and it follows that either $\frac{1}{2}z_{LP} \leq \sum_{i=1}^{r-1} c_i$ or $\frac{1}{2}z_{LP} \leq c_r \cdot x_r$. Furthermore $\frac{1}{2}z_{LP} \leq c_r \cdot x_r \leq \max_r \{c_r\}$.

The heuristic picks the best of the rounded down value of the LP relaxation, i.e. a value of $\sum_{i=1}^{r-1} c_i$ and $\max_r \{c_r\}$, i.e. $z^G = \max\{\sum_{i=1}^{r-1} c_i, \max_r \{c_r\}\}$. But from the above it follows that $z^G \geq \frac{1}{2}z_{LP}$ and hence $z^G \geq \frac{1}{2}z$ since $z_{LP} \geq z$.