

## Exercise 10.1

See example 10.1 page 169. This example contains a maximization version of UFL and  $c$  is profit. Note that in this exercise,  $c$  contains cost *not* profit. We use the formulation given on page 13:

$$\min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j \quad (1)$$

$$\sum_{j \in N} x_{ij} = 1 \text{ for } i \in M \quad (2)$$

$$x_{ij} \leq y_j \text{ for } i \in M, j \in N \quad (3)$$

$$x_{ij} \geq 0 \text{ for } i \in M \quad (4)$$

$$j \in N, y_j \in \{0, 1\} \text{ for } j \in N \quad (5)$$

We dualize the demand constraints (2) and get:

$$z(u) = \min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j - \sum_{i \in M} u_i \left( \sum_{j \in N} x_{ij} - 1 \right) \quad (6)$$

$$x_{ij} \leq y_j \text{ for } i \in M, j \in N \quad (7)$$

$$x_{ij} \geq 0 \text{ for } i \in M \quad (8)$$

$$j \in N, y_j \in \{0, 1\} \text{ for } j \in N \quad (9)$$

Rewriting  $z(u)$  we get:

$$z(u) = \min \sum_{i \in M} \sum_{j \in N} (c_{ij} - u_i) x_{ij} + \sum_{j \in N} f_j y_j + \sum_{i \in M} u_i \quad (10)$$

$$= \min \sum_{j \in N} z_j(u) + \sum_{i \in M} u_i \quad (11)$$

with  $z_j(u) = \sum_{i \in M} (c_{ij} - u_i) x_{ij} + f_j y_j$ . This can be decomposed into  $N$  subproblems (i.e. solved independently for each  $N$ ).

Each of these  $N$  subproblems can be solved by inspection using:  $\min z_j(u) = \min\{0, \sum_{i \in M} \min[c_{ij} - u_i, 0] + f_j\}$ .

$f = (4, 8, 11, 7, 5)$  and  $u = (5, 6, 3, 2, 6, 4)$  and we need the revised cost-matrix:

$$c_{ij} - u_i = \begin{pmatrix} 1 & -3 & -4 & -2 & 0 \\ -2 & 4 & -4 & 0 & -5 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & -2 & 2 & -1 & 2 \\ -5 & 2 & 0 & -4 & -1 \\ -1 & -2 & 0 & 4 & -3 \end{pmatrix}$$

For the first column  $j = 1$ :  $z_1(u) = \min\{0, 0 - 2 + 0 + 0 - 5 - 1 + f_1\} = \min\{0, -8 + 4\} = -4$ . This corresponds to setting  $y_1 = 1$  and  $x_{21} = x_{51} = x_{61} = 1$ ,  $x_{11} = 0$  and  $x_{31}$  and  $x_{41}$  can be selected arbitrarily. This is done for all columns obtaining the following values:  $z_2(u) = 0$ ,  $z_3(u) = 0$ ,  $z_4(u) = -2$  and  $z_5(u) = -4$ . The values of the decision variables are (variables not mentioned are zero)  $y_1 = y_4 = y_5 = x_{21} = x_{51} = x_{61} = x_{14} = x_{34} = x_{44} = x_{54} = x_{25} = x_{55} = x_{65} = 1$ .

The lower bound is then obtained:  $z(u) = \sum_{j \in N} z_j(u) + \sum_{i \in M} u_i = (-4 + 0 + 0 - 2 - 4) + (5 + 6 + 3 + 2 + 6 + 4) = 16$ .

Modifying the solution it can be seen that opening facility 1 and 4 can cover all clients with negative revised costs. Thus there is probably no gain from opening facility 5. Now it is a matter of, for each client, select the lowest cost facility to serve client. The solution obtained is:  $y_1 = y_4 = 1$  and  $x_{14} = x_{21} = x_{34} = x_{44} = x_{51} = x_{61} = 1$ . The solution value is  $4 + 7 + 3 + 4 + 1 + 1 + 1 + 3 = 24$  and since it is a feasible solution it is an upper bound and it is at most  $24 - 16 = 8$  units from optimal.

## Exercise 10.5

Dualizing the budget constraint gives the following problem:

$$\begin{aligned} z(u) &= \max 10y_1 + 4y_2 + 14y_3 + u(4 - 3y_1 - y_2 - 4y_3) \\ &= \max(10 - 3u)y_1 + (4 - u)y_2 + (14 - 4u)y_3 + 4u \\ \text{Subject to: } &u \geq 0, y \in B^3 \end{aligned}$$

Or stated in another way:

$$z(u) = \max\{0, 10 - 3u\} + \max\{0, 4 - u\} + \max\{0, 14 - 4u\} + 4u$$

This is a piecewise linear function, so its minimum can be found by considering all breaks and endpoints of the function, i.e.  $u = 0$ ,  $u = 3\frac{1}{3}$ ,  $u = 4$ ,  $u = 3\frac{1}{2}$  and  $u \rightarrow \infty$ . The values are:  $z(0) = 28$ ,  $z(3\frac{1}{3}) = 14\frac{2}{3}$ ,  $z(4) = 16$ ,  $z(3\frac{1}{2}) = 14\frac{1}{2}$  and  $\lim_{u \rightarrow \infty} z(u) = \infty$ . Thus the optimal(minimum) value of  $u$  is  $3\frac{1}{2}$  with the upper bound value  $14\frac{1}{2}$ .

The step of the subgradient algorithm is given by:

$$u^{k+1} = \max\{u_k - \mu_k(4 - 3y_1^* - y_2^* - 4y_3^*), 0\}$$

Initial parameters are:  $u^0 = 0$ ,  $\mu_0 = 1$  and  $\rho = 1/2$ .

The following table shows the first iterations of a run of the subgradient algorithm.

| $k$ | $u$                                                                                  | $\mu$ |
|-----|--------------------------------------------------------------------------------------|-------|
| 0   | 0                                                                                    | 1     |
| 1   | $\max\{0 - 1(4 - 3 - 1 - 4), 0\} = 4$                                                | 1/2   |
| 2   | $\max\{4 - 1/2(4 - 0 - 0 - 0), 0\} = 2$                                              | 1/4   |
| 3   | $\max\{2 - 1/4(4 - 3 - 1 - 4), 0\} = 3$                                              | 1/8   |
| 4   | $\max\{3 - 1/8(4 - 3 - 1 - 4), 0\} = 3\frac{1}{2}$                                   | 1/16  |
| 5   | $\max\{3\frac{1}{2} - 1/16(4 - 0 - 1 - 4), 0\} = 3\frac{9}{16}$                      | 1/32  |
| 6   | $\max\{3\frac{9}{16} - 1/32(4 - 0 - 1 - 0), 0\} = 3\frac{15}{32} \approx 3.46875$    | 1/64  |
| 7   | $\max\{3\frac{15}{32} - 1/64(4 - 0 - 1 - 4), 0\} = 3\frac{33}{64} \approx 3.516$     | 1/128 |
| 8   | $\max\{3\frac{33}{64} - 1/128(4 - 0 - 1 - 0), 0\} = 3\frac{69}{128} \approx 3.539$   | 1/256 |
| 9   | $\max\{3\frac{69}{128} - 1/256(4 - 0 - 1 - 0), 0\} = 3\frac{141}{256} \approx 3.551$ | 1/512 |

Note that  $u$  will continue increasing and the same solution will be optimal until  $u$  reach the value 4 or above. However the increase is at most  $3(\frac{1}{512} + \frac{1}{1024} + \dots) < 0.024$  and thus  $u$  will remain less than 4 and specifically does not approach  $3\frac{1}{2}$ . To see that  $3(\frac{1}{512} + \frac{1}{1024} + \dots) < 0.024$  note that  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \rightarrow 1$ . Subtracting  $\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{128} = 0.9921875$  we get  $(\frac{1}{512} + \frac{1}{1024} + \dots) \rightarrow 0.0078125$ , and the result follows.

## Exercise 10.8

### (i) Dualize budget-constraint

**Strength:** LP-relaxation.

**Subproblem:** Assignment problem - efficient even compared with LP-relaxation.

**Lagrangian dual:** Only one variable (relatively easy).

### (ii) Dualize budget- and assignment-constraints

**Strength:** LP-relaxation.

**Subproblem:** Inspection, easier than (i).

**Lagrangian dual:**  $1+|N|+|M|$  variables, harder than (i).

### (iii) Dualize assignment-constraints

**Strength:** Potentially tighter than LP-relaxation.

**Subproblem:** One 0-1 knapsack problem NP-hard (easy NP-hard though).

**Lagrangian dual:**  $|N|+|M|$  variables, same as (ii).