

Exercise 6.1

Take a *2-PARTITION* problem specified by all a_j 's. Let $b = \sum_{j \in N} a_j$. Then the decision version of the following *0-1 KNAPSACK* problem is equivalent with the *2-PARTITION* problem.

$$\max \quad \sum_{j \in N} a_j x_j \quad (1)$$

$$\sum_{j \in N} a_j x_j \leq b/2 \quad (2)$$

$$\forall j : \quad x_j \in \{0, 1\} \quad (3)$$

The relevant decision version of this problem is: Does there exist a solution of value $\geq b/2$?

If such a solution exists, it will have value $= b/2$ and taking $S = \{j : x_j = 1\}$. Will identify a solution to the *2-PARTITION* problem.

This does *not* imply that *2-PARTITION* is $\mathcal{NP}$ -complete. The reduction “goes the other way”, so if we had a polynomial reduction of *0-1 KNAPSACK* to *2-PARTITION*, the following would be a proof, that *2-PARTITION* is $\mathcal{NP}$ -complete.

Assume *2-PARTITION* is not $\mathcal{NP}$ -complete (i.e. a polynomial algorithm that solves *2-PARTITION* exists). Given the reduction, a *0-1 KNAPSACK* problem can be converted into a *2-PARTITION* problem and solved in polynomial time - thus solving the *0-1 KNAPSACK* problem in polynomial time. This contradicts the fact that *0-1 KNAPSACK* is $\mathcal{NP}$ -complete and thus the hypothesis is wrong and *2-PARTITION* is $\mathcal{NP}$ -complete.

Exercise 6.3

(*STABLE SET* is the problem of finding a maximum weight node set of nonadjacent nodes in a graph.)

A *STABLE SET* problem is given by a graph with nodes, undirected edges and a weight on each node. Given such a problem, the reduction to a *SET PACKING* problem is as follows. There is one column for each node in the network and the weight of the node is the weight of the column. There is one row for each edge, and the only non-zero entries are the two columns corresponding to the two endpoint nodes of the edge.

Since nodes corresponds to columns, a solution for one of the problems will have the same solution value as the corresponding solution for the other problem. The set of feasible solutions is also the same, since: 1) if a subset of columns is a feasible solution to *SET PACKING*, then neither pair of columns in the subset has a 1 in the same row. But since the rows corresponds to the edges, no edge exists between any of the nodes corresponding to the subset of columns. On the other hand 2) if a subset of nodes is a feasible solution to *STABLE SET*, then no pair of nodes of the subset has an edge between them and hence it holds for the corresponding subset of columns, that no pair of columns have a 1 in the same row.

Thus solving *SET PACKING* will solve the original *STABLE SET* problem and also the

conversion is done in polynomial time, hence *SET PACKING* is polynomial time reducible to *STABLE SET*.

Exercise 6.8

(I)

Assume that $x^* \in \text{conv}(X)$, then by definition of the convex hull (see page 15), a convex combination exists, such that $x^* = \sum_{t=1}^T \lambda_t x^t$ and $\sum_{t=1}^T \lambda_t = 1$. Thus at least one set of λ 's which sum to one exists, and minimizing over all λ 's will give a solution with a value ≤ 1 , i.e. $\min\{\sum_{t=1}^T \lambda_t : x^* \leq \sum_{t=1}^T x^t \lambda_t, \lambda \in R_+^T\} \leq 1$.

On the other hand assume that $\min\{\sum_{t=1}^T \lambda_t : x^* \leq \sum_{t=1}^T x^t \lambda_t, \lambda \in R_+^T\} \leq 1$. Consider the optimal set of λ 's denoted by $\bar{\lambda}$ (elements denoted by $\bar{\lambda}_t$). Also denote $\bar{x} = \sum_{t=1}^T x^t \bar{\lambda}_t$. Note that $x^* \leq \bar{x}$ or written in more detail: $\forall j \in N : x_j^* \leq \bar{x}_j$.

In the following we will modify $\bar{\lambda}$ such that we obtain a set of λ_t 's such that $0 \leq \lambda_t \leq 1$, $\sum_{t=1}^T \lambda_t = 1$ and $\bar{x} = x^*$. Since, as we shall see, this is possible, x^* can be written as a convex combination of the points of X and hence $x^* \in \text{conv}(X)$.

By assumption $\sum_{t=1}^T \lambda_t \leq 1$. The zero vector is in Π^1 since it is in the knapsack set. Thus by increasing the λ_t corresponding to the zero vector $\sum_{t=1}^T \lambda_t = 1$ can be obtained. Consider each row in turn: $x_j^* \leq \bar{x}_j = \sum_{t=1}^T x_j^t \lambda_t$. If $x_j^* = \bar{x}_j$ do nothing. If $x_j^* < \bar{x}_j$ then each column of row j either contains a 0 or a 1 (the columns stems from the knapsack set). If they contain a 1 decreasing the corresponding λ_t will make $\bar{x}_j$ smaller, i.e. closer to x_j^* . Thus this is what we will do for each column in turn until $\bar{x}_j = x_j^*$.

Other rows that contain a 1 in the same column will be affected though, but this can be handled using the following observation: Consider column $[b_1, b_2, \dots, b_n]^T, b_i \in B$. Assume $b_i = 1$, then the same column, except that $b_i = 0$ exists as well, because the columns stems from the knapsack set. Thus assume we look at row i and have chosen to decrease λ_t , then increase the $\lambda_{t'}$ that corresponds to column t except that $b_i = 0$.

This is done for all rows until for all j , $\bar{x}_j = x_j^*$ thus completing the proof.

(II)

If $x^* \notin \text{conv}(X)$ then from (I) it follows that $\max\{x^* \pi : \pi \in \Pi^1\} > 1$. Thus at least one π exists for which $\pi x^* > 1$, namely (one of) the optimal π .