



A General Cut

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A general cut

In solving an integer programming problem

$$\begin{aligned} \max \quad & cx \\ Ax \quad &= b \\ x \quad &\geq 0, x \in \mathbb{Z}^n \end{aligned}$$

one often starts by solving the LP-relaxation. Let us denote the optimal basis of the LP-relaxation B .



- The i 'th row in the optimal Simplex tableau now expresses the following equation:

$$x_{B_i} + \sum_{j \notin B} \alpha_{ij} x_j = \beta_i$$

in which α_{ij} and β_i are the coefficients in the i 'th row of the tableau.

- NOW, if all RHS in the Simplex tableau are integer the optimal solution has been found.



If β_i fractional then...

If not, we construct an inequality, which is satisfied by all solutions to the IP as follows:

- consider a row i , in which β_i is non-integer.
- separate α_{ij} and β_i into their **integral** parts $\lfloor \alpha_{ij} \rfloor$ and $\lfloor \beta_i \rfloor$ and their **fractional** parts $r\alpha_{ij}$ and $r\beta_i$:

$$\alpha_{ij} = \lfloor \alpha_{ij} \rfloor + r\alpha_{ij}, \beta_i = \lfloor \beta_i \rfloor + r\beta_i$$

where $\lfloor \alpha_{ij} \rfloor, \lfloor \beta_i \rfloor \in \mathbb{Z}, r\alpha_{ij}, r\beta_i \in [0, 1[$.



$$x_{B_i} + \sum_{j \notin B} \alpha_{ij} x_j = \beta_i \Leftrightarrow$$

$$x_{B_i} + \sum_{j \notin B} (\lfloor \alpha_{ij} \rfloor + r \alpha_{ij}) x_j = \lfloor \beta_i \rfloor + r \beta_i$$



Since all x are non-negative, LHS becomes smaller if the fractional parts are discarded in the coefficients of x :

$$x_{B_i} + \sum_{j \notin B} \lfloor \alpha_{ij} \rfloor x_j \leq \lfloor \beta_i \rfloor + r \beta_i$$



Now LHS is integral for all integral x . If $r \beta_i$ is discarded in the inequality, it is thus valid for all integral solutions (but not necessarily for non-integral solutions):

$$x_{B_i} + \sum_{j \notin B} \lfloor \alpha_{ij} \rfloor x_j \leq \lfloor \beta_i \rfloor$$



The constructed inequality is now subtracted from the considered “row equation” giving an inequality, which is valid for all integer solutions, but not necessarily for non-integral solutions:

$$\frac{x_{B_i} + \sum_{j \notin B} (\lfloor \alpha_{ij} \rfloor + r \alpha_{ij}) x_j = \lfloor \beta_i \rfloor + r \beta_i}{- (x_{B_i} + \sum_{j \notin B} \lfloor \alpha_{ij} \rfloor x_j \leq \lfloor \beta_i \rfloor)} \geq r \beta_i$$



- Since $x_j = 0$ for $j \notin B$ the above inequality does not hold for the basic LP-solution.
- So it is a **valid inequality**, which when added to the problem *cuts off* the current LP-solution still leaving all integer solutions in the feasible set.
- This inequality is called a **Gomory-cut**.