



Cutting Plane Algorithms

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Introduction

- We have an optimization problem

$$\max\{cx : x \in X\}$$

where $X = \{x : Ax \leq b, x \geq 0, x \text{ integer}\}$. From earlier we have that $\text{conv}(X)$ is a polyhedron $\{x : \tilde{A}x \leq \tilde{b}, x \geq 0\}$.

- In general $\text{conv}(X)$ is **impossible** to find if the problem at hand is NP-hard.



Valid inequality

- Find a **good approximation** – especially around the optimal solution.
- How?
- $\pi x \leq \pi_0$ is a **valid inequality** for X if and only if $\pi x \leq \pi_0$ for all $x \in X$.



Example

If $X = \{x : Ax \leq b, x \geq 0, x \text{ integer}\}$ and $\text{conv}(X) = \{x : \tilde{A}x \leq \tilde{b}\}$ then the constraints

- $a^i x \leq b$ are valid inequalities for X
- $\tilde{a}^i x \leq \tilde{b}$ are valid inequalities for X



Example – Knapsack

Consider the following 0-1-knapsack set:

$$X = \{x \in B^5 : 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leq -2\}$$

- As $x_2 = x_4 = 0$ we get lhs $3x_1 + 2x_3 + x_5$ and rhs -2 , which is impossible. So a valid inequality is $x_2 + x_4 \geq 1$.
- As $x_1 = 1$ and $x_2 = 0$ we get lhs $3 + 2x_3 - 3x_4 + x_5 \geq 3 - 3 = 0$ and rhs -2 , which is impossible. So a valid inequality is $x_1 \leq x_2$.



Example – CFL

Capacitated Facility Location has the feasible region:

$$\begin{aligned} \sum_{i \in M} x_{ij} &\leq b_j y_j & j \in N \\ \sum_{j \in N} x_{ij} &= a_i & i \in M \\ x_{ij} &\geq 0, \quad y_j \in \{0, 1\} & i \in M, j \in N \end{aligned}$$

Here we see that all feasible solutions satisfy $x_{ij} \leq b_j y_j$ and $x_{ij} \leq a_i$ with $y_j \in B^1$. Therefore $x_{ij} \leq \min\{a_i, b_j\} y_j$ is a family of valid inequalities.



Example – Integer Rounding

- Consider $P = \{x \in Z_+^4 : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\}$
- Divide by 11 and we get a valid inequality (uninteresting):

$$\frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq 6\frac{6}{11}$$

Example – Integer Rounding (Cont.)

- As $x \geq 0$ we round up the lhs and get a weaker valid inequality:

$$2x_1 + 2x_2 + x_3 + x_4 \geq 6\frac{6}{11}$$

- As x is integer the lhs must be integer, therefore we can round up:

$$2x_1 + 2x_2 + x_3 + x_4 \geq 7$$





Valid inequalities for Linear Programs

- When is the inequality $\pi x \leq \pi_0$ valid for $P = \{x : Ax \leq b, x \geq 0\}$?
- **Proposition:** $\pi x \leq \pi_0$ is valid for $P = \{x : Ax \leq b, x \geq 0\} \neq \emptyset$ if and only if
 - ▶ there exists $u, v \geq 0$ such that $uA - v = \pi$ and $ub \leq \pi_0$, or
 - ▶ there exists $u \geq 0$ such that $uA \geq \pi$ and $ub \leq \pi_0$.



A priori addition of constraints

- **Idea:** Examine the initial formulation $P = \{x : Ax \leq b, x \geq 0\}$ with $X = P \cap Z^n$. Find a set of valid inequalities $Qx \leq q$ for X , add them to the formulation giving a new formulation P' .
- Solve the new formulation eg. by LP-relaxation and branch-and-bound.



Just to give you an idea

- For a TSP of size 16
 - ▶ The number of subtour elimination constraints is 16,368.
 - ▶ The number of comb inequalities is 1,933,711,339,620.
 - ▶ These are not the only families of valid inequalities for TSP.
- For a TSP of size 120
 - ▶ The number of subtour elimination constraints is 0.6×10^{36} !
 - ▶ The number of comb inequalities is 2^{179} .





Decomposition

- Decomposition may be used when considering valid inequalities a priori.
- In many situations the feasible region X can be written as the intersection of two or more set with more structure, eg. $X = X_1 \cap X_2$. This is called a **decomposition**.
- How do we find such a decomposition? Be creative!!



The cutting plane algorithm

- Suppose $X = P \cap Z^n$ and suppose that we know a family F of valid inequalities $\pi x \leq \pi_0$, $(\pi, \pi_0) \in F$.
- Given a specific objective function one is not really interested in finding the complete convex hull, but only an approximation in the area around the optimal solution.



$t = 0, P^0 = P, f = 0$
repeat
 solve $\bar{z}^t = \max\{cx : x \in P^t\}$ (x^t is an opt solution)
 if $x^t \notin Z^n$ **then**
 Solve the separation problem between x^t and F
 (find inequality $(\pi^t, \pi_0^t) \in F$ so that $\pi^t x^t > \pi_0^t$)
 if no inequality is found **then** $f = 1$
 $P^{t+1} = P^t \cap \{x : \pi^t x \leq \pi_0^t\}$
 until $x^t \in Z^n \vee f = 1$
 if $f = 0$ **then** x^t optimal solution for IP
 else use branch-and-bound to obtain IP