



Formulations of Integer Programs

– Are some better than others?

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Uncapacitated Facility Location (UFL)

- Given a set of *potential depots* $N = \{1, 2, \dots, n\}$ and a set *clients* (or customers), $M = \{1, 2, \dots, m\}$, suppose there is a fixed cost f_j associated with the use of depot j , and a transportation cost c_{ij} if all of client i 's order is delivered from depot j .
- The problem is to decide which depots to open, and which depots serves each client so as to minimize the sum of fixed and transportation cost.

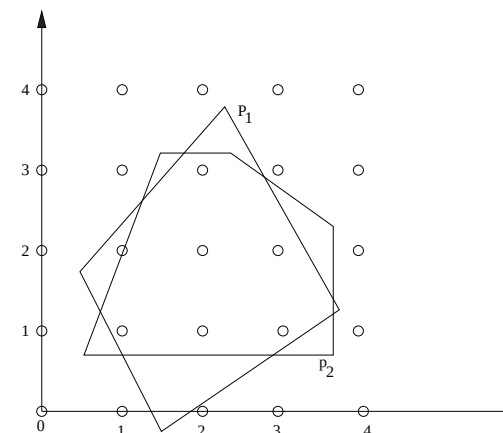


Polyhedron and Formulation

- A subset of R^n described by a finite set of linear constraints $P = \{x \in R^n : Ax \leq b\}$ is a **polyhedron**.
- A polyhedron $P \subset R^{n+p}$ is a **formulation** for a set $X \subset Z^n \times R^p$ if and only if $X = (Z^n \times R^p) \cap P$.
- Integer Program:** A polyhedron $P \subset R^n$ is a **formulation** for a set $X \subset Z^n$ if and only if $X = Z^n \cap P$.



Example of formulations





Convex Hull

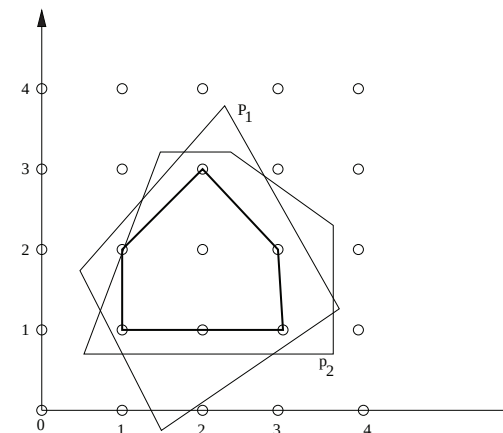
- Given a set $x \subset R^n$ the **convex hull** of X , denoted $\text{conv}(X)$ is defined as:

$$\text{conv}(X) = \left\{ x : x = \sum_{i=1}^t \lambda_i x^i, \sum_{i=1}^t \lambda_i = 1, \lambda_i \geq 0 \text{ for } i = 1, \dots, t \text{ over all finite subsets } \{x^1, x^2, \dots, x^t\} \text{ of } X \right\}$$

- Proposition:** $\text{conv}(X)$ is a polyhedron
- Proposition:** The extreme points of $\text{conv}(X)$ all lie in X .



Ideal Formulation



Ideal Formulation

- In the ideal formulation our integer programming problem can be solved by solving a linear programming problem over the formulation.
- The **ideal formulation** in most cases consists of an enormous (exponential) number of inequalities needed to describe $\text{conv}(X)$, and there is no simple characterization of them.



Better Formulations

Instead we could rather ask:

- Given two formulations P_1 and P_2 for X when can we say that one is better than the other?
- Given a set $X \subset R^n$ and two formulations P_1 and P_2 for X , P_1 is a **better formulation** than P_2 if $P_1 \subset P_2$.



Formulations for a Knapsack Set

Look at the set $X =$

$$\{(0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1)\}$$

and the three formulations

$$P_1 = \{x \in \mathbb{R}^4 : 0 \leq x \leq 1, 83x_1 + 61x_2 + 49x_3 + 20x_4 \leq 100\}$$

$$P_2 = \{x \in \mathbb{R}^4 : 0 \leq x \leq 1, 4x_1 + 3x_2 + 2x_3 + 1x_4 \leq 4\}$$

$$P_3 = \{x \in \mathbb{R}^4 : 0 \leq x \leq 1, \begin{array}{rrrr} 4x_1 & +3x_2 & +2x_3 & +1x_4 & \leq 4 \\ 1x_1 & +1x_2 & +1x_3 & & \leq 1 \\ & & & +1x_4 & \leq 1 \end{array}\}$$

Wolsey writes: "It is easily seen that $P_3 \subset P_2 \subset P_1$, and it can be checked that $P_3 = \text{conv}(X)$ ".



Comparing formulations of UFL

Let P_1 be the formulation with the constraints

$$\sum_{i \in M} x_{ij} \leq my_j$$

Let P_2 be the formulation with the constraints

$$x_{ij} \leq y_j$$

Now we want to show that P_2 is a better formulation than P_1 .



Projection

- first formulation: $\min\{cx : x \in P \cap Z^n\}$ with $P \subset \mathbb{R}^n$.
- second formulation:
 $\min\{cx : (x, w) \in Q \cap (Z^n \times \mathbb{R}^p)\}$ with $Q \subset \mathbb{R}^n \times \mathbb{R}^p$.
- Given a polyhedron $Q \subset \mathbb{R}^n \times \mathbb{R}^p$ the **projection of Q onto the subspace $\mathbb{R}^n$** , denoted $\text{proj}_x Q$ is defined as:

$$\text{proj}_x Q = \{x \in \mathbb{R}^n : (x, w) \in Q \text{ for some } w \in \mathbb{R}^p\}$$